Physics Letters B 745 (2015) 48–51

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Light-cone gravity in dS_4

Sudarshan Ananth *, Mahendra Mali

Indian Institute of Science Education and Research, Pune 411008, India

ARTICLE INFO

Article history:
Received 24 March 2015
Accepted 6 April 2015
Available online 17 April 2015
Editor: M. Cvetič

ABSTRACT

We derive a closed form expression for the light-cone Lagrangian describing pure gravity on a four-dimensional de Sitter background. We provide a perturbative expansion of this Lagrangian to cubic order in the fields.

1. Introduction

Attempts to unite quantum mechanics and the general theory of relativity result in divergences which are difficult to treat. In this context, the resemblance of perturbative gravity to gauge theory is particularly striking. Specifically, the KLT relations [1] equate tree-level scattering amplitudes in pure gravity to the square of tree-level amplitudes in Yang–Mills theory. Over the past decade, our understanding of these relations and their origin has improved greatly. In particular, the Lagrangian origin of this connection is now well established [2]. At the lowest interaction order – the cubic vertex – we now possess a plethora of interesting perturbative links between interacting theories of arbitrary spin [3] (making the cubic KLT relations merely one in a family).

Almost all these perturbative ties have been derived on flat spacetime backgrounds so it is natural to ask whether these relations or their equivalents exist in curved spacetime backgrounds. While it is not clear what the stringy origin would be for a Yang–Mills $\sim$ gravity link in curved spacetime, the question itself is interesting and well posed within the framework of quantum field theory.

This paper is a companion paper to our earlier formulation of pure gravity on an AdS_4 background [4]. Although the differences in treatment from the anti-de Sitter case are not significant, we feel the closed form result for the light-cone gravity action in de Sitter will prove extremely useful in studies using perturbative quantum field theory (particularly in the context of cosmology). This closed form and the vertices that result from its expansion are also essential in the investigations of the perturbative ties described above. Thus, in this paper, we formulate pure gravity in light-cone gauge, on a dS_4 background.

On a tangential note, the surprising ultraviolet behavior of $\mathcal{N}=8$ supergravity is thought to stem from the better-than-expected behavior of pure gravity in the ultraviolet regime [5]. This motivates our study of pure gravity on various backgrounds, its nature and the detailed structure of its Lagrangian in terms of the physical degrees of freedom.

Our approach in this paper is similar in spirit to [4,6]. We make suitable gauge choices and eliminate the unphysical degrees of freedom using the light-cone constraint relations. This will result in a closed form expression for the action of pure gravity in a de Sitter background. We also provide a perturbative expansion of this action to first order in the gravitational coupling constant (the cubic interaction vertex).

2. Einstein gravity

The Einstein–Hilbert action, describing pure gravity, reads

$$S_{EH} = \int d^4x\, L = \frac{1}{2\,\kappa^2}\int d^4x\sqrt{-g}\,(\mathcal{R} - 2\,\Lambda), \tag{1}$$

where $g = \det g_{\mu\nu}$, $\mathcal{R}$ is the scalar curvature, Λ the cosmological constant of dS_4 and $\kappa^2 = 8\pi G$, the gravitational coupling constant.

This theory has been studied previously, in light-cone gauge, on both AdS_4 and flat backgrounds [4,6]. In this paper, we formulate pure gravity in dS_4 characterized by a cosmological constant, Λ. This will involve changes from both the flat spacetime and anti-de Sitter approaches referred to above and we comment on these departures as and when they occur.

2.1. De Sitter space

De Sitter spacetime is a maximally symmetric Lorentzian space with positive (constant) curvature [7]. It is a solution of the equation

* Corresponding author.
E-mail address: ananth@iiserpune.ac.in (S. Ananth).

http://dx.doi.org/10.1016/j.physletb.2015.04.006

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{2}$$

with cosmological constant Λ. De Sitter space is a hyperboloid embedded in a five dimensional Minkowski space.

Consider a five-dimensional Minkowski spacetime with metric $\eta_{MN} \equiv (-1, 1, 1, 1, 1)$ and coordinates ξ^0, ξ^1, ξ^2, ξ^3, ξ^4. The invariant interval reads

$$ds^2 = -(d\xi^0)^2 + (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2 + (d\xi^4)^2, \tag{3}$$

with $\xi^M \in (-\infty, +\infty)$, $M = 0 \dots 4$. De Sitter space is the hypersurface

$$-(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 + (\xi^4)^2 = l^2 = H^{-2}, \tag{4}$$

where we have related the radius of de Sitter space l to the Hubble constant $H = l^{-1}$. A solution of (4) is

$$\begin{aligned} -(H\xi^0)^2 + (H\xi^4)^2 &= 1 - (Hx^i)^2 e^{2Ht}, \\ (H\xi^1)^2 + (H\xi^2)^2 + (H\xi^3)^2 &= (Hx^i)^2 e^{2Ht}, \end{aligned} \tag{5}$$

with

$$\begin{aligned} H\xi^0 &= \sinh(Ht) + \frac{(Hx^i)^2}{2} e^{Ht}, \\ H\xi^i &= Hx^i e^{Ht}, \\ H\xi^4 &= -\cosh(Ht) + \frac{(Hx^i)^2}{2} e^{Ht}, \end{aligned} \tag{6}$$

where $x^i \in (-\infty, +\infty)$, $i = 1, 2, 3$ and $t \in (-\infty, +\infty)$. In terms of these new coordinates, the induced metric is

$$ds^2 = -(dt)^2 + e^{2Ht}\{(dx^1)^2 + (dx^2)^2 + (dx^3)^2\}. \tag{7}$$

Our choice of coordinates in (6) impose the following constraint

$$-\xi^0 + \xi^4 = -\frac{1}{H} e^{Ht} \le 0 \Longrightarrow \xi^0 \ge \xi^4, \tag{8}$$

implying that we are only covering one half of the de Sitter space (expanding Poincaré patch of dS). Within this patch, we define conformal time by

$$H\eta = e^{-Ht}, \tag{9}$$

which modifies the metric (7) to

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2). \tag{10}$$

Note that the conformal time runs from $\eta = +\infty$ ($t = -\infty$) to $\eta = 0$ ($t = +\infty$). We work in this expanding Poincaré patch of dS_4 but we could equally well have worked with the other patch (contracting patch of dS).

3. Light-cone formulation of pure gravity on dS_4

We start with the metric of (10) which reads

$$g^{(0)}_{\mu\nu} = \frac{1}{H^2\eta^2}\eta_{\mu\nu}\,, \tag{11}$$

where $\eta_{\mu\nu} = (-1, 1, 1, 1,)$ is the four-dimensional Minkowski metric. We now introduce light-cone coordinates, $x^\mu \equiv (x^+, x^-, x^i)$ where

$$x^\pm = \frac{\eta \pm x^3}{\sqrt{2}}, \tag{12}$$

and $i = 1, 2$ label the transverse directions. The coordinate x^+ is now the evolution parameter. In terms of these coordinates, the Minkowski metric is $\eta^{L.C.}_{\mu\nu}$ (which is off-diagonal for the $+$, $-$ coordinates and diagonal for the i directions). We also define

$$X = x^+ + x^-. \tag{13}$$

Our metric now reads

$$g^{(0)}_{\mu\nu} = \frac{2}{H^2X^2}\eta^{L.C.}_{\mu\nu}. \tag{14}$$

The cosmological constant of dS_4 is

$$\Lambda = 3H^2. \tag{15}$$

3.1. Light-cone action

We now proceed to gauge fix the Einstein–Hilbert action and derive a closed form expression for the action in terms of the physical degrees of freedom in the theory. We start with the light-cone gauge choices

$$g_{--} = g_{-i} = 0, \quad i = 1, 2. \tag{16}$$

Note that these choices are consistent with $g^{(0)}_{\mu\nu}$ since $\eta_{--} = \eta_{-i} = 0$. The fourth (and final) gauge choice will be made shortly. The other components of the metric are parametrized as follows.

$$\begin{aligned} g_{+-} &= -\frac{2}{H^2X^2}e^{\phi}, \\ g_{ij} &= \frac{2}{H^2X^2}e^{\psi}\gamma_{ij}. \end{aligned} \tag{17}$$

ϕ, ψ are real and γ_{ij} is a real, symmetric and unimodular matrix describing the two physical degrees of freedom in the theory.

In light-cone gauge, a subset of the equations of motion $\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = -\Lambda\, g_{\mu\nu}$ represent constraint relations which may be solved. The key difference in dealing with constraint relations in dS_4, as opposed to both AdS_4 and flat space [4,6] stems from the fact that X in (17) depends on ∂_-. Since constraint relations always contain ∂_- we will need integrating factors to solve them. A listing of some useful formulae used in the following is presented in Appendix A. The first constraint relation reads $\mathcal{R}_{--} = 0$ and combined with (17) implies that

$$\partial_-\phi\,\partial_-\psi - \partial_-{}^2\psi - \frac{1}{2}(\partial_-\psi)^2 - \frac{2}{X}\partial_-\phi + \frac{1}{4}\partial_-\gamma^{kl}\partial_-\gamma_{kl} = 0 \tag{18}$$

This constraint is exactly solvable if we make the following (fourth) gauge choice

$$\phi = \frac{1}{2}\psi. \tag{19}$$

Now (18) simplifies to

$$\frac{1}{4}\partial_-\gamma^{kl}\partial_-\gamma_{kl} - \partial_-{}^2\psi - \frac{2}{X}\partial_-\phi = 0, \tag{20}$$

which when multiplied by an integrating factor (X) results in

$$\psi = \frac{1}{4}\frac{1}{\partial_-}\left[\frac{1}{X}\frac{1}{\partial_-}(X\partial_-\gamma^{kl}\partial_-\gamma_{kl})\right], \tag{21}$$

where $\frac{1}{\partial_-}$ is defined following the prescription in [8]. Notice the difference in structure here (21) from the analogous results in AdS_4 and flat space [4,6].

We now move to the second constraint: $\mathcal{R}_{-i}=0$. With use of an integrating factor, $\frac{1}{X^2}$, this yields

$$g^{-i}=H^2X^2e^{-\phi}\frac{1}{\partial_-}\Bigg[X^2\gamma^{ij}e^{\phi-2\psi}\frac{1}{\partial_-}\Bigg\{\frac{1}{X^2}e^{\psi}\left(\frac{1}{4}\partial_-\gamma^{kl}\partial_j\gamma_{kl}-\frac{1}{2}\partial_-\partial_j\phi-\frac{1}{2}\partial_-\partial_j\psi+\frac{1}{2}\partial_j\phi\partial_-\psi-\frac{2}{X}\partial_j\phi\right)+\frac{1}{2X^2}\partial_l\left(e^{\psi}\gamma^{kl}\partial_-\gamma_{jk}\right)\Bigg\}\Bigg]. \quad (22)$$

Having determined these components of the metric, we turn to the action

$$S=\int d^4x\,\mathcal{L}=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(2g^{+-}R_{+-}+g^{ij}R_{ij}-2\Lambda\right), \quad (23)$$

which written out explicitly reads

$$S=\int d^4x\frac{1}{H^2X^2}e^{\psi}\left(\frac{24}{X^2}+4\partial_+\partial_-\phi-2\partial_+\psi\partial_-\psi-\partial_+\gamma^{ij}\partial_-\gamma_{ij}\right)-\frac{1}{H^2X^2}e^{\phi}\gamma^{ij}\left(2\partial_i\partial_j\phi+\partial_i\phi\partial_j\phi-2\partial_i\phi\partial_j\psi-\frac{1}{2}\partial_i\gamma^{kl}\partial_j\gamma_{kl}+\partial_i\gamma^{kl}\partial_k\gamma_{jl}\right)-\frac{4}{H^2X^2}e^{\phi-2\psi}\gamma^{ij}\frac{1}{\partial_-}R_i\frac{1}{\partial_-}R_j-\frac{8}{H^4X^4}e^{\psi}e^{\phi}\Lambda, \quad (24)$$

where

$$R_i=\frac{1}{X^2}e^{\psi}\left(\frac{1}{4}\partial_-\gamma^{kl}\partial_i\gamma_{kl}-\frac{1}{2}\partial_-\partial_i\phi-\frac{1}{2}\partial_-\partial_i\psi+\frac{1}{2}\partial_i\phi\partial_-\psi-\frac{2}{X}\partial_i\phi\right)+\frac{1}{2X^2}\partial_l\left(e^{\psi}\gamma^{kl}\partial_-\gamma_{ik}\right). \quad (25)$$

While obtaining this expression, boundary terms have been dropped. This closed form of the action (24) is valid in both patches of de Sitter.

3.2. Perturbative expansion

We now expand the action in (24) to first order in the gravitational coupling constant. We parametrize γ_{ij} as follows.

$$\gamma_{ij}=(e^H)_{ij},$$

with

$$H=\begin{pmatrix}h_{11} & h_{12}\\ h_{12} & h_{22}\end{pmatrix}. \quad (26)$$

$h_{22}=-h_{11}$ ensures that this matrix is traceless. ψ in terms of these fluctuations is

$$\psi=-\frac{1}{4}\frac{1}{\partial_-}\left[\frac{1}{X}\frac{1}{\partial_-}(X\,\partial_-h_{ij}\partial_-h_{ij})\right]+\mathcal{O}(h^4). \quad (27)$$

We re-scale the h field according to

$$h\to\frac{1}{\sqrt{2}\kappa}h. \quad (28)$$

We now present the kinetic and cubic interaction vertices in the action (24).

$$S_2=\int d^4x\,\mathcal{L}_2, \quad (29)$$

where

$$\mathcal{L}_2=\frac{1}{2H^2X^4}\frac{1}{\partial_-}(X\partial_-h_{ij}\partial_-h_{ij})-\frac{1}{2H^2X^3}\frac{\partial_+}{\partial_-}(X\partial_-h_{ij}\partial_-h_{ij})+\frac{1}{H^2X^2}\partial_+h_{ij}\partial_-h_{ij}+\frac{1}{2H^2X^2}\frac{\partial_i\partial_i}{\partial_-}\left[\frac{1}{X}\frac{1}{\partial_-}(X\partial_-h_{jk}\partial_-h_{jk})\right]-\frac{1}{2H^2X^2}\partial_ih_{jk}\partial_ih_{jk}+\frac{1}{H^2X^2}\partial_ih_{jk}\partial_jh_{ik}+\frac{3}{H^2X^4}\frac{1}{\partial_-}\left[\frac{1}{X}\frac{1}{\partial_-}(X\partial_-h_{ij}\partial_-h_{ij})\right]-\frac{1}{H^2X^2}\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_j\partial_-h_{ij}\right)\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_k\partial_-h_{ik}\right). \quad (30)$$

From (19) and the last term in (24), it is obvious that the cosmological constant Λ (15) is always accompanied by ψ. Thus, given the structure of (27), Λ only contributes to interaction vertices involving an even number of fields.

$$S_3=\int d^4x\frac{1}{\sqrt{2}}\mathcal{L}_3, \quad (31)$$

with

$$\mathcal{L}_3=\kappa\Bigg\{\frac{1}{H^2X^2}\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_-h_{jk}\partial_ih_{jk}\right)\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_l\partial_-h_{il}\right)-\frac{3}{H^2X^2}\frac{1}{\partial_-}\left[\frac{1}{X}\frac{\partial_i}{\partial_-}(X\partial_-h_{jk}\partial_-h_{jk})\right]\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_l\partial_-h_{il}\right)-\frac{1}{H^2X^2}\frac{1}{\partial_-}\left(\frac{1}{X}\frac{\partial_i}{\partial_-}\left[\frac{1}{X}\frac{1}{\partial_-}(X\partial_-h_{jk}\partial_-h_{jk})\right]\right)\times\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_l\partial_-h_{il}\right)+\frac{2}{H^2X^2}\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_jh_{jk}\partial_-h_{ik}\right)\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_l\partial_-h_{il}\right)-\frac{1}{H^2X^2}\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_j\partial_-h_{ij}^2\right)\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_l\partial_-h_{il}\right)+\frac{1}{H^2X^2}h_{ij}\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_k\partial_-h_{ik}\right)\frac{1}{\partial_-}\left(\frac{1}{X^2}\partial_l\partial_-h_{il}\right)\Bigg\}. \quad (32)$$

As expected, both the closed form and the perturbative expansions involve fields tangled with the coordinates (here x^+ and x^-) and involve conformal-like factors. From these expressions, it would be interesting to understand how to extract amplitudes and other related structures. Clearly, it would interesting to extend our work to quartic order in the fields and work out how to deal with the time derivatives (∂_+) that begin to appear then (usually handled using a suitable field redefinition). One interesting but perhaps impractical idea would be to identify a general closed form expression for the action describing light-cone gravity which may be tuned to the spacetime of our choice: whether flat, AdS or dS. Finally, such a closed form expression could prove fruitful in identifying the origin of the better than expected ultraviolet behavior seen in pure gravity [5].

Appendix A. Useful results

$$\Gamma^+_{++}=\frac{1}{2}g^{+-}[2\partial_+g_{+-}-\partial_-g_{++}]$$

$$\Gamma^+_{+-}=0$$

$$\Gamma^+_{--}=0$$

$$\Gamma^+_{i-}=0$$

$$\Gamma^+_{i+}=\frac{1}{2}g^{+-}[\partial_ig_{+-}-\partial_-g_{i+}]$$

$$\Gamma^{+}_{ij} = -\frac{1}{2} g^{+-} \partial_{-} g_{ij}$$

$$\Gamma^{-}_{--} = g^{+-} \partial_{-} g_{+-}$$

$$\Gamma^{-}_{+-} = \frac{1}{2} \{ g^{+-} \partial_{-} g_{++} + g^{-i} [\partial_{-} g_{i+} - \partial_i g_{+-}] \}$$

$$\Gamma^{-}_{++} = \frac{1}{2} \{ g^{+-} \partial_{+} g_{++} + g^{--} [2\partial_{+} g_{+-} - \partial_{-} g_{++}] + g^{-i} [2\partial_{+} g_{i+} - \partial_i g_{++}] \}$$

$$\Gamma^{-}_{+i} = \frac{1}{2} \{ g^{+-} \partial_i g_{++} + g^{--} [\partial_i g_{+-} - \partial_{-} g_{i+}] + g^{-j} [\partial_i g_{+j} + \partial_{+} g_{ij} - \partial_j g_{+i}] \}$$

$$\Gamma^{-}_{-i} = \frac{1}{2} \{ g^{+-} [\partial_i g_{+-} + \partial_{-} g_{+i}] + g^{-j} \partial_{-} g_{ij} \}$$

$$\Gamma^{-}_{ij} = \frac{1}{2} \{ g^{+-} [\partial_i g_{+j} + \partial_j g_{+i} - \partial_{+} g_{ij}] - g^{--} \partial_{-} g_{ij} + g^{-k} [\partial_i g_{kj} + \partial_j g_{ik} - \partial_k g_{ij}] \}$$

$$\Gamma^{i}_{jk} = \frac{1}{2} \{ -g^{-i} \partial_{-} g_{jk} + g^{im} [\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk}] \}$$

$$\Gamma^{i}_{-j} = \frac{1}{2} g^{ik} \partial_{-} g_{kj}$$

$$\Gamma^{i}_{+-} = \frac{1}{2} g^{ij} [\partial_{-} g_{j+} - \partial_j g_{+-}]$$

$$\Gamma^{i}_{+j} = \frac{1}{2} \{ g^{-i} [\partial_j g_{+-} - \partial_{-} g_{+j}] + g^{ik} [\partial_j g_{+k} + \partial_{+} g_{kj} - \partial_k g_{+j}] \}$$

$$\Gamma^{i}_{++} = \frac{1}{2} \{ g^{-i} [2\partial_{+} g_{+-} - \partial_{-} g_{++}] + g^{ij} [2\partial_{+} g_{+j} - \partial_j g_{++}] \}$$

$$\Gamma^{i}_{--} = 0$$

$$\Gamma^{j}_{ij} = \frac{1}{2} \{ -g^{-j} \partial_{-} g_{ij} + g^{jl} [\partial_j g_{li} + \partial_i g_{lj} - \partial_l g_{ij}] \}$$

Frequently used quantities

$$g^{+-} = -\frac{H^2 X^2}{2} e^{-\phi},$$

$$g^{ij} = \frac{H^2 X^2}{2} e^{-\psi} \gamma^{ij},$$

$$\gamma^{ij} = (e^{-H})_{ij},$$

$$g^{\mu\nu} g_{\mu\rho} = \delta^{\nu}_{\rho} \Longrightarrow g^{++} = g^{+i} = 0,$$

$$g_{+i} = -g_{+-} g_{ij} g^{-j},$$

$$g_{++} = -\frac{4}{H^4 X^4} e^{\psi} g^{--} + \frac{2}{H^2 X^2} e^{\phi} g^{-i} g_{+i},$$

$$\gamma^{ij} \gamma_{ij} = 2,$$

$$\gamma^{ij} \partial_k \gamma_{ij} = \gamma^{ij} \partial_{-} \gamma_{ij} = 0,$$

$$\sqrt{-g} = \frac{4}{H^4 X^4} e^{\psi} e^{\phi}.$$

References

[1] H. Kawai, D.C. Lewellen, S.H.H. Tye, Nucl. Phys. B 269 (1986) 001.
[2] S. Ananth, S. Theisen, Phys. Lett. B 652 (2007) 128–134;
S. Ananth, Int. J. Mod. Phys. D 19 (2010) 2379.
[3] Y.S. Akshay, S. Ananth, J. Phys. A 47 (4) (2014) 045401;
Y.S. Akshay, S. Ananth, Nucl. Phys. B 887 (2014) 168–174.
[4] Y.S. Akshay, S. Ananth, M. Mali, Nucl. Phys. B 884 (2014) 66–73.
[5] Z. Bern, L.J. Dixon, D.C. Dunbar, M. Perelstein, J.S. Rozowsky, Nucl. Phys. B 530 (1998) 401–456.
[6] Ingemar Bengtsson, Martin Cederwall, Olof Lindgren, GOTEBORG-83-55, Nov. 1983;
J. Scherk, J. Schwarz, Gen. Relativ. Gravit. 6 (1975) 537;
M. Goroff, J. Schwarz, Phys. Lett. B 127 (1983) 61;
Michio Kaku, Nucl. Phys. B 91 (1975) 99;
C. Aragone, A. Khoudeir, Class. Quantum Gravity 7 (1990) 1291;
S. Ananth, L. Brink, R. Heise, H.G. Svendsen, Nucl. Phys. B 753 (2006) 195, arXiv:hep-th/0607019.
[7] W. de Sitter, On the relativity of inertia. Remarks concerning Einstein's latest hypothesis, in: KNAW Proceedings, vol. 19 II, 1917, pp. 1217–1225.
[8] S. Mandelstam, Nucl. Phys. B 213 (1983) 149.