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Weighted inequalities for negative powers of Schrödinger operators ☆

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A B S T R A C T

In this article we obtain boundedness of the operator $(-\Delta + V)^{-\alpha/2}$ from $L^{p,\infty}(w)$ into weighted bounded mean oscillation type spaces $BMO^{\beta}_{\mathcal{L}}(w)$ under appropriate conditions on the weight w. We also show that these weighted spaces also have a point-wise description for $0 < \beta < 1$. Finally, we study the behaviour of the operator $(-\Delta + V)^{-\alpha/2}$ when acting on $BMO^{\beta}_{\mathcal{L}}(w)$.

1. Introduction

Let us consider the Schrödinger operator on $\mathbb{R}^d$ with $d \geqslant 3$,

$$\mathcal{L} = -\Delta + V$$

where $V \geqslant 0$ is a function satisfying, for some $q > \frac{d}{2}$, the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(y)^q\,dy\right)^{1/q} \leqslant \frac{C}{|B|}\int_B V(y)\,dy$$

for every ball $B \subset \mathbb{R}^d$. The set of functions with the last property is usually denoted by RH_q.

It is well known that negative powers of the Schrödinger operator can be expressed in terms of the heat diffusion semigroup generated by $\mathcal{L}$ as

$$\mathcal{I}_\alpha f(x) = \mathcal{L}^{-\alpha/2} f(x) = \int_0^\infty e^{-t\mathcal{L}} f(x) t^{\alpha/2}\frac{dt}{t}, \quad \alpha > 0.$$

For each $t > 0$ the operator $e^{-t\mathcal{L}}$ is an integral operator with kernel $k_t(x, y)$ having a better behaviour far away form the diagonal than the classical heat kernel. Some useful properties of k_t where obtained in [3,4,6]. As a consequence $\mathcal{I}_\alpha f$ turns out to be finite a.e. even if f belongs to L^p with p greater than the critical index d/α. Particularly, in [1] the authors proved that $\mathcal{I}_\alpha$ maps $L^{d/\alpha}$ into an appropriate substitute of L^∞ denoted by $BMO_{\mathcal{L}}$ which in fact is smaller than the classical BMO space of John–Nirenberg.

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In this work we extend and improve their result by analysing the behaviour of $\mathcal{I}_\alpha$ on weighted weak L^p spaces with $p \geqslant d/\alpha$ for a suitable class of weights. In order to do that we introduce a family of spaces $BMO^\beta_{\mathcal{L}}(w)$ that includes, as a particular case, the space $BMO_{\mathcal{L}}$. We point out that in the case of $w \equiv 1$ and $p = d/\alpha$, we obtain a better result than that in [1] since L^p is strictly contained in weak L^p.

It is worth mentioning that for $w \equiv 1$ the spaces $BMO^\beta_{\mathcal{L}}$ are the duals of the H^p-spaces introduced in [2] and [4], as it can be easily checked from the atomic decomposition given there. For $\beta = 0$ such representation was already pointed out in [1].

We also study the behaviour of $\mathcal{I}_\alpha$ on $BMO^\beta_{\mathcal{L}}(w)$ proving that, under appropriate conditions on the weight, they are transformed into $BMO^{\beta+\alpha}_{\mathcal{L}}(w)$. In proving such result we give a point-wise characterization of our spaces $BMO^\beta_{\mathcal{L}}(w)$ when $0 < \beta < 1$, which we believe to be of independent interest.

Finally, we remark that when the potential V belongs to $RH_{d/2}$, as it is the case of the Hermite differential operator, the classes of weights for which we prove our boundedness results coincide with those obtained in [5] for $V = 0$.

This article is organized as follows. In Section 2 we introduce the family of spaces $BMO^\beta_{\mathcal{L}}(w)$ and we prove some basic properties. In particular the aforementioned point-wise description is given in Proposition 4. The remaining two sections contain the main results: Section 3 is devoted to the analysis of $\mathcal{I}_\alpha$ acting on $L^{p,\infty}(w)$ while Section 4 deals with the boundedness on $BMO^\beta_{\mathcal{L}}(w)$.

2. $\mathbf{BMO}^\beta_{\mathcal{L}}(w)$ spaces

For a given potential $V \in RH_q$, with $q > \frac{d}{2}$, we introduce the function

$$\rho(x) = \sup\left\{r > 0\colon\ \frac{1}{r^{d-2}} \int\limits_{B(x,r)} V \leqslant 1\right\}, \quad x \in \mathbb{R}^d.$$

Due to the above assumptions $\rho(x)$ is finite for all $x \in \mathbb{R}^d$. This auxiliary function plays an important role in the estimates of the operators and in the description of the spaces associated to $\mathcal{L}$ (see [1,3,4,7]).

The following propositions contain some properties of ρ that will be useful in the sequel.

Proposition 1. *(See [7, Lemma 1.4].) There exist C and $k_0 \geqslant 1$ such that*

$$C^{-1}\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \leqslant \rho(y) \leqslant C\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}} \tag{1}$$

for all $x, y \in \mathbb{R}^d$.

Throughout this work, we denote $w(E) = \int_E w$ for every measurable subset $E \subset \mathbb{R}^d$, and $CB = B(x, Cr)$, for $x \in \mathbb{R}^d$, $r > 0$ and $C > 0$.

Proposition 2. *(See [2].) There exists a sequence of points $\{x_k\}_{k=1}^\infty$ in $\mathbb{R}^d$, so that the family $B_k = B(x_k, \rho(x_k))$, $k \geqslant 1$, satisfies*

(1) $\bigcup_k B_k = \mathbb{R}^d$.
(2) *There exists N such that, for every $k \in \mathbb{N}$,* $\operatorname{card}\{j\colon\ 4B_j \cap 4B_k \neq \emptyset\} \leqslant N$.

We denote by L^1_{loc} the set of locally integrable functions of $\mathbb{R}^d$. For $\eta \geqslant 1$ and w a weight, i.e. $w \geqslant 0$ and $w \in L^1_{\mathrm{loc}}$, we say that $w \in D_\eta$ if there exists a constant C such that

$$w(tB) \leqslant Ct^{d\eta} w(B),$$

for every ball $B \subset \mathbb{R}^d$ and $t \geqslant 1$.

It is easy to see that a weight w belongs to $D = \bigcup_{\eta \geqslant 1} D_\eta$ if and only if it satisfies the doubling condition

$$w(2B) \leqslant Cw(B).$$

For $\beta \geqslant 0$ we define the space $BMO^\beta_{\mathcal{L}}(w)$ as the set of functions f in L^1_{loc} satisfying for every ball $B = B(x, R)$, with $x \in \mathbb{R}^d$ and $R > 0$,

$$\int\limits_B |f - f_B| \leqslant Cw(B)|B|^{\beta/d}, \quad \text{with } f_B = \frac{1}{|B|}\int\limits_B f, \tag{2}$$

and

$$\int_B |f| \leqslant C w(B)|B|^{\beta/d}, \quad \text{if } R \geqslant \rho(x). \tag{3}$$

Let us note that if (3) is true for some ball B then (2) holds for the same ball, so we might ask to (2) only for balls with radii lower than $\rho(x)$.

The constants in (2) and (3) are independent of the choice of B but may depend on f. A norm in the space $BMO^{\beta}_{\mathcal{L}}(w)$ can be given by the maximum of the two infima of the constants that satisfy (2) and (3) respectively.

The case $\beta = 0$ and $w \equiv 1$ was introduced in [1] as a natural substitute of L^∞ in the context of the semigroup generated by the operator $\mathcal{L}$. As in that case we can replace condition (3) by the following weaker condition (4) that only takes into account critical balls.

Proposition 3. *Let $\beta \geqslant 0$ and $w \in D_\eta$. If $\{x_k\}_{k=1}^\infty$ is a sequence as in Proposition 2, then a function f belongs to $BMO^{\beta}_{\mathcal{L}}(w)$ if and only if f satisfies (2) for every ball, and*

$$\int_{B(x_k,\rho(x_k))} |f| \leqslant C w\big(B\big(x_k, \rho(x_k)\big)\big)\big|\rho(x_k)\big|^{\beta}, \quad \textit{for all } k \geqslant 1. \tag{4}$$

Proof. Let f satisfy (4), and let $B = B(x, R)$ be a ball with radius $R > \rho(x)$. From Proposition 2 the set

$$F = \{k\colon\ B \cap B_k \neq \emptyset\}$$

is finite and

$$\sum_{k\in F} \int_{B_k} w \leqslant (N+1) \int_{\bigcup_{k\in F} B_k} w, \tag{5}$$

where N is the constant controlling the overlapping (see Proposition 2).

It is easy to see that for some constant C, $B_k \subset CB$ for every $k \in F$. In fact, if $z \in B_k \cap B$, from (1),

$$\begin{aligned}
\rho(x_k) &\leqslant C\rho(z)\left(1 + \frac{|x_k - z|}{\rho(x_k)}\right)^{k_0} \leqslant C2^{k_0}\rho(z) \\
&\leqslant C2^{k_0}\rho(x)\left(1 + \frac{|x - z|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}} \leqslant C2^{k_0}\rho(x)\left(1 + \frac{R}{\rho(x)}\right) \\
&\leqslant C2^{k_0+1}R,
\end{aligned}$$

then

$$\begin{aligned}
\int_B |f| &\leqslant \sum_{k\in F} \int_{B\cap B_k} |f| \leqslant \sum_{k\in F} \int_{B_k} |f| \\
&\leqslant C \sum_{k\in F} w(B_k)\,|B_k|^{\beta/d} \leqslant C\,|B|^{\beta/d} \sum_{k\in F} \int_{B_k} w \\
&\leqslant C|B|^{\beta/d} \int_{\bigcup_{k\in F} B_k} w \leqslant C|B|^{\beta/d} \int_{CB} w.
\end{aligned}$$

Since we assumed that w is doubling the last expression is bounded up to a constant by $w(B)|B|^{\beta/d}$. □

Corollary 1. *A function f belongs to $BMO^{\beta}_{\mathcal{L}}(w)$ if and only if condition (2) is satisfied for every ball $B = B(x, R)$ with $x \in \mathbb{R}^d$ and $R < \rho(x)$, and*

$$\int_{B(x,\rho(x))} |f| \leqslant C w\big(B\big(x, \rho(x)\big)\big)\big|\rho(x)\big|^{\beta}, \quad \textit{for all } x \in \mathbb{R}^d. \tag{6}$$

For $\beta > 0$ and $w \in L^1_{\text{loc}}$, we define

$$W_\beta(x, r) = \int_{B(x,r)} \frac{w(z)}{|z - x|^{d-\beta}}\, dz.$$

for all $x \in \mathbb{R}^d$ and $r > 0$.

We introduce a kind of Lipschitz space $\Lambda_{\mathcal{L}}^{\beta}(w)$ as the set of functions f such that

$$\left|f(x)-f(y)\right| \leqslant C\left[W_\beta\left(x,|x-y|\right)+W_\beta\left(y,|x-y|\right)\right] \tag{7}$$

and

$$\left|f(x)\right| \leqslant CW_\beta\left(x,\rho(x)\right) \tag{8}$$

for almost all x and y in $\mathbb{R}^d$.

It is possible to define a norm in these spaces by taking the maximum of the two infima of the constants that satisfy Eqs. (7) and (8) respectively.

Remark 1. For almost every $x \in \mathbb{R}^d$, $W_\beta(x,r)$ is finite for all $r>0$, and it is always increasing as a function of r. Also, if w satisfies the doubling condition, then we have

$$W_\beta(x,2r) \leqslant CW_\beta(x,r), \tag{9}$$

for almost every $x \in \mathbb{R}^d$ and $r>0$, where the constant C does not depend on r or x.

Proposition 4. *If $0<\beta<1$ and w satisfies the doubling condition, then*

$$\Lambda_{\mathcal{L}}^{\beta}(w)=BMO_{\mathcal{L}}^{\beta}(w),$$

and the norms are equivalent.

Proof. Let f be in $BMO_{\mathcal{L}}^{\beta}(w)$ with $\|f\|_{BMO_{\mathcal{L}}^{\beta}(w)}=1$, x and y in $\mathbb{R}^d$. Since f satisfies (2), from [5, Proposition 1.3] we obtain

$$\left|f(x)-f(y)\right| \leqslant C\left[W_\beta\left(x,2|x-y|\right)+W_\beta\left(y,2|x-y|\right)\right]$$

for all x and y Lebesgue points of f. Hence, Remark 1 implies that f satisfies (7).

To verify (8), if $x \in \mathbb{R}^d$ is a Lebesgue point of f, and $B=B(x,\rho(x))$, from (7) and condition (3), we have

$$\begin{aligned}\left|f(x)\right| &\leqslant \frac{1}{|B|}\int_B \left|f(x)-f(y)\right|dy+\frac{1}{|B|}\int_B\left|f(y)\right|dy\\ &\leqslant \frac{C}{|B|}\left(\int_B W_\beta\left(x,|x-y|\right)dy+\int_B W_\beta\left(y,|x-y|\right)dy+w(B)|B|^{\frac{\beta}{d}}\right).\end{aligned} \tag{10}$$

In the last sum, by Remark 1, the first term is

$$\int_B W_\beta\left(x,|x-y|\right)dy \leqslant |B|W_\beta\left(x,\rho(x)\right).$$

For the second term of (10), if $y \in B$, we have $B(y,|x-y|) \subset B(x,2\rho(x))$, then

$$\begin{aligned}\int_B W_\beta\left(y,|x-y|\right)dy &\leqslant \int_{B(x,2\rho(x))} w(z)\left(\int_B \frac{1}{|z-y|^{d-\beta}}\,dy\right)dz\\ &\leqslant C|B|^{\beta/d}w(B) \leqslant C|B|W_\beta\left(x,\rho(x)\right).\end{aligned}$$

Finally, the last term of (10) is bounded by

$$|B|^{\frac{\beta}{d}-1}w(B) \leqslant W_\beta\left(x,\rho(x)\right),$$

and we have shown that (8) is satisfied.

In order to prove the other inclusion, consider $\|f\|_{\Lambda_{\mathcal{L}}^{\beta}(w)}=1$. From [5, Proposition 1.3] we have that (7) implies (2). To see condition (3), let $x \in \mathbb{R}^d$ and $R \geqslant \rho(x)$. If $y \in B(x,R)$, from Proposition 1,

$$\rho(y) \leqslant C\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}} \leqslant C\rho(x)\left(\frac{R}{\rho(x)}\right)^{\frac{k_0}{k_0+1}} \leqslant CR$$

and thus by (8)

$$\begin{aligned}\int\limits_{B(x,R)} \left|f(y)\right| dy &\leqslant \int\limits_{B(x,R)} \int\limits_{B(y,\rho(y))} \frac{w(z)}{|z-y|^{d-\beta}}\,dz\,dy \\ &\leqslant \int\limits_{B(x,CR)} w(z) \int\limits_{B(x,R)} \frac{1}{|z-y|^{d-\beta}}\,dy\,dz \\ &\leqslant CR^{\beta} w\big(B(x,CR)\big) \\ &\leqslant C\left|B(x,R)\right|^{\beta/d} w\big(B(x,R)\big),\end{aligned}$$

where in the last inequality we have used the fact that w is doubling. □

Remark 2. Observe that in the last proof (see inequality (10)) we have shown that (6) and (7) with $|x-y| < \rho(x)$ implies (8), and thus conditions (6) and (7) imply that f belongs to $\Lambda^{\beta}_{\mathcal{L}}(w)$.

3. $\mathcal{I}_\alpha$ on $L^{p,\infty}(w)$ spaces

We begin by stating a series of lemmas that will be useful in proving the main results. We omit the proofs though we provide references where they can be found.

For $L^{p,\infty}(w)$, $p > 1$, we mean the space of measurable functions f such that

$$[f]_{p,w} = \left(\sup_{t>0} t^p \left|\left\{x\colon \frac{|f(x)|}{w(x)} > t\right\}\right|\right)^{1/p} \tag{11}$$

is finite. The quantity (11) is not a norm (triangular inequality fails) but it turns to be equivalent to a norm. Clearly, the Lebesgue spaces $L^p(w) = \{f\colon \int_{\mathbb{R}^d} |f/w|^p < \infty\}$ are continuously embedded in $L^{p,\infty}(w)$.

As usual p' denotes the Hölder conjugate exponent of p.

Lemma 1. *(See [5].) Let $p > 1$ and w a weight in $RH_{p'}$. If f is a locally integrable function and B is a ball in $\mathbb{R}^d$ then, there exists a constant C such that*

$$\int\limits_B |f| \leqslant C w(B)|B|^{-\frac{1}{p}} [f]_{p,w}.$$

For $t > 0$, let k_t be the kernel of $e^{-t\mathcal{L}}$. Then, the kernel of $\mathcal{I}_\alpha$ is given by the formula

$$K_\alpha(x,y) = \int\limits_0^\infty k_t(x,y) t^{\alpha/2} \frac{dt}{t}. \tag{12}$$

Some estimates of k_t are presented below.

Lemma 2. *(See [6].) Given $N > 0$, there exists a constant C such that for all x and y in $\mathbb{R}^d$,*

$$k_t(x,y) \leqslant C t^{-d/2} e^{-\frac{|x-y|^2}{Ct}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

As a consequence of the previous lemma we have

$$K_\alpha(x,y) \leqslant \frac{C}{|x-y|^{d-\alpha}} \tag{13}$$

for all x and y in $\mathbb{R}^d$.

Lemma 3. *(See [4, Proposition 4.11].) Given $N > 0$ and $0 < \delta < \min(1, 2 - \frac{d}{q})$, there exists a constant C such that*

$$\left|k_t(x,y) - k_t(x_0,y)\right| \leqslant C\left(\frac{|x-x_0|}{\sqrt{t}}\right)^{\delta} t^{-d/2} e^{-\frac{|x-y|^2}{Ct}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N},$$

for all x, y and x_0 in $\mathbb{R}^d$ with $|x-x_0| < \sqrt{t}$.

A function ψ is rapidly decaying (see [3]) if for every $N>0$ there exists a constant C_N such that

$$\left|\psi(x)\right| \leqslant C_N\left(1+|x|\right)^{-N}.$$

If ψ is a real function on $\mathbb{R}^d$ and $t>0$, we define

$$\psi_t(x)=\frac{1}{t^{d/2}}\psi\left(\frac{x}{\sqrt{t}}\right).$$

We will also need some estimates for the kernel

$$q_t(x,y)=k_t(x,y)-\tilde{k}_t(x,y),$$

for all $x,y\in\mathbb{R}^d$ and $t>0$, where $\tilde{k}_t$ is the kernel of the classical heat operator $e^{-t\Delta}$.

Lemma 4. *(See [3].) There exists a rapidly decaying function $\psi\geqslant 0$ such that*

$$q_t(x,y)\leqslant C\left(\frac{\sqrt{t}}{\rho(x)}\right)^{2-\frac{d}{q}}\psi_t(x-y),$$

for all x, y in $\mathbb{R}^d$ and $t>0$.

Lemma 5. *(See [3].) For every $0<\delta<\min(1,2-\frac{d}{q})$ and C, there exists a rapidly decaying function ψ such that*

$$\left|q_t(x,y+h)-q_t(x,y)\right|\leqslant C'\left(\frac{|h|}{\rho(x)}\right)^{\delta}\psi_t(x-y)$$

for all x, y and in $\mathbb{R}^d$ and $t>0$, with $|h|<C\rho(y)$ and $|h|<\frac{|x-y|}{4}$.

In [1] the authors obtain boundedness of $\mathcal{I}_\alpha$ from $L^{d/\alpha}$ into $BMO_{\mathcal{L}}=BMO^0_{\mathcal{L}}$. The next theorem presents an extension of this result to L^p spaces with p greater than d/α. Moreover, $L^{p,\infty}$ spaces are considered instead of L^p.

Theorem 1. *Let us assume that the potential V belongs to RH_q with $q\geqslant d/2$ and set $\delta_0=\min\{1,2-\frac{d}{q}\}$. Let $0<\alpha<d$, $\frac{d}{\alpha}\leqslant p<\frac{d}{(\alpha-\delta_0)^+}$ and $w\in RH_{p'}\cap D_\eta$, where $1\leqslant\eta<1-\frac{\alpha}{d}+\frac{\delta_0}{d}+\frac{1}{p}$, then the operator $\mathcal{I}_\alpha$ is bounded from $L^{p,\infty}(w)$ into $BMO^{\alpha-d/p}_{\mathcal{L}}(w)$.*

Proof. We need the following fact: if f is a locally integrable function and B a ball in $\mathbb{R}^d$, then

$$\frac{1}{w(B)}\int_B \mathcal{I}_\alpha\left(|f\chi_{2B}|\right)\leqslant C|B|^{\frac{\alpha}{d}-\frac{1}{p}}[f]_{p,w}. \tag{14}$$

To get this estimate, from (13), we have

$$\frac{1}{w(B)}\int_B \mathcal{I}_\alpha\left(|f\chi_{2B}|\right)\leqslant C\frac{1}{w(B)}\int_B\int_{2B}\frac{|f(y)|}{|x-y|^{d-\alpha}}\,dy\,dx.$$

Let x_0 be the center of B and r its radius. Applying Tonelli's theorem, the last integral is

$$\int_{2B}\left|f(y)\right|\int_B\frac{dx}{|x-y|^{d-\alpha}}\,dy\leqslant Cr^\alpha\int_{2B}\left|f(y)\right|dy\leqslant Cw(B)|B|^{\frac{\alpha}{d}-\frac{1}{p}}[f]_{p,w},$$

where the last inequality is due to Lemma 1, finishing the proof of (14).

In order to see that $\mathcal{I}_\alpha f$ is in $BMO^{\alpha-d/p}_{\mathcal{L}}(w)$, in view of Corollary 1, it is enough to check that there exists a constant C such that the two following conditions hold:

(i) For any $x_0\in\mathbb{R}^d$

$$\frac{1}{w(B(x_0,\rho(x_0)))}\int_{B(x_0,\rho(x_0))}|\mathcal{I}_\alpha f|\leqslant C\left|B\left(x_0,\rho(x_0)\right)\right|^{\frac{\alpha}{d}-\frac{1}{p}}[f]_{p,w}.$$

(ii) For every ball $B=B(x_0,r)$ with $r<\rho(x_0)$ and some constant c_B

$$\frac{1}{w(B)}\int_B\left|\mathcal{I}_\alpha f(x)-c_B\right|dx\leqslant C|B|^{\frac{\alpha}{d}-\frac{1}{p}}[f]_{p,w}.$$

We first prove (i). Let $B = B(x_0, R)$ with $R = \rho(x_0)$. Splitting $f = f_1 + f_2$, with $f_1 = f\chi_{2B}$, by the claim (14), we have

$$\frac{1}{w(B)}\int_B |\mathcal{I}_\alpha f_1| \leqslant C|B|^{\frac{\alpha}{d}-\frac{1}{p}}[f]_{p,w}.$$

To deal with f_2, we split the integral representation of $\mathcal{I}_\alpha$ as follows. Let $x \in B$,

$$\mathcal{I}_\alpha f_2(x) = \int_0^{R^2} e^{-t\mathcal{L}} f_2(x) t^{\alpha/2-1}\,dt + \int_{R^2}^{\infty} e^{-t\mathcal{L}} f_2(x) t^{\frac{\alpha}{2}-1}\,dt. \tag{15}$$

For $x \in B$ and $y \in \mathbb{R}^d \setminus 2B$, we have $|x_0 - y| \leqslant C\,|x-y|$, then for the first term of (15), we have

$$\begin{aligned}
\left|\int_0^{R^2} e^{-t\mathcal{L}} f_2(x) t^{\alpha/2-1}\,dt\right| &= \left|\int_0^{R^2}\int_{\mathbb{R}^d\setminus 2B} k_t(x,y) f(y)\,dy\, t^{\frac{\alpha}{2}-1}\,dt\right| \\
&\leqslant C\int_0^{R^2}\int_{\mathbb{R}^d\setminus 2B} \frac{1}{t^{d/2}} e^{-\frac{|x-y|^2}{t}} \left|f(y)\right| dy\, t^{\frac{\alpha}{2}-1}\,dt \\
&\leqslant C\int_0^{R^2} t^{\frac{-d+\alpha}{2}-1} \int_{\mathbb{R}^d\setminus 2B} \left(\frac{t}{|x-y|^2}\right)^{M/2} \left|f(y)\right| dy\,dt \\
&\leqslant C\int_0^{R^2} t^{\frac{M-d+\alpha}{2}-1}\,dt \int_{\mathbb{R}^d\setminus 2B} \frac{|f(y)|}{|x_0-y|^M}\,dy,
\end{aligned}$$

where M is a constant to be determined later and C depends on M.

Splitting the domain of the second integral into dyadic annuli $2^{k+1}B \setminus 2^k B$, and applying Lemma 1 we get

$$\begin{aligned}
\int_{(2B)^c} \frac{|f(y)|}{|x_0-y|^M}\,dy &= \sum_{k=1}^{\infty} \int_{2^{k+1}B\setminus 2^k B} \frac{|f(y)|}{|x_0-y|^M}\,dy \\
&\leqslant \frac{1}{R^M}\sum_{k=1}^{\infty}\frac{1}{2^{kM}}\int_{2^{k+1}B} \left|f(y)\right| dy \\
&\leqslant CR^{-\frac{d}{p}-M}[f]_{p,w}\sum_{k=1}^{\infty} w\big(2^{k+1}B\big)2^{-k(\frac{d}{p}+M)} \\
&\leqslant Cw(B)R^{-\frac{d}{p}-M}[f]_{p,w}\sum_{k=1}^{\infty} 2^{-k(\frac{d}{p}+M-d\eta)},
\end{aligned} \tag{16}$$

where the last inequality follows from the fact that $w \in D_\eta$.

The last series converges if $M > d\eta - \frac{d}{p}$. Therefore, for such M,

$$\begin{aligned}
\left|\int_0^{R^2} e^{-t\mathcal{L}} f_2(x) t^{\alpha/2-1}\,dt\right| &\leqslant Cw(B)R^{-\frac{d}{p}-M}[f]_{p,w}\int_0^{R^2} t^{(M-d+\alpha)/2-1}\,dt \\
&= Cw(B)|B|^{\frac{\alpha}{d}-\frac{1}{p}-1}[f]_{p,w}.
\end{aligned}$$

For the second term of (15), we use the extra decay of the kernel $k_t(x,y)$ given by Lemma 2. Thus, we can choose M as above and $N \geqslant M$ so that,

$$\begin{aligned}
\left|\int_{R^2}^{\infty} e^{-t\mathcal{L}} f_2(x) t^{\alpha/2-1}\,dt\right| &= \int_{R^2}^{\infty}\int_{\mathbb{R}^d\setminus 2B} k_t(x,y)\left|f(y)\right| dy\, t^{\alpha/2-1}\,dt \\
&\leqslant C\int_{R^2}^{\infty}\int_{\mathbb{R}^d\setminus 2B} t^{(\alpha-d-N)/2-1}\rho(x)^N e^{-\frac{|x-y|^2}{t}}\left|f(y)\right| dy\,dt
\end{aligned}$$

and the last expression is bounded by

$$C\rho(x)^N \int_{R^2}^{\infty} t^{(\alpha-d-N)/2-1} \int_{\mathbb{R}^d\setminus 2B} \left(\frac{t}{|x-y|^2}\right)^{M/2} \left|f(y)\right| dy\, dt.$$

As $x \in B$, we have $\rho(x) \sim \rho(x_0) = R$. Then, the last expression is bounded by a constant times

$$R^N \int_{R^2}^{\infty} t^{(M+\alpha-d-N)/2-1}\, dt \int_{\mathbb{R}^d\setminus 2B} \frac{|f(y)|}{|x_0-y|^M}\, dy.$$

Since $M+\alpha-d-N<0$, the integral in t converges. Then, splitting the second integral in the same way as before, the last term is bounded by

$$Cw(B)R^{\alpha-\frac{d}{p}-d}[f]_{p,w} = Cw(B)|B|^{\frac{\alpha}{d}-\frac{1}{p}-1}[f]_{p,w}$$

and we have proved (i).

Now we will see (ii). Let $B=\{x\in\mathbb{R}^d\colon\ |x-x_0|<r\}$, with $r<\rho(x_0)$. We set $f=f_1+f_2$ with $f_1=f\chi_{2B}$ and

$$c_B = \int_{r^2}^{\infty} e^{-t\mathcal{L}} f_2(x_0) t^{\alpha/2-1}\, dt.$$

By the claim (14) we have

$$\begin{aligned}\frac{1}{w(B)}\int_B \left|\mathcal{I}_\alpha(f)-c_B\right| &\leqslant \frac{1}{w(B)}\int_B \mathcal{I}_\alpha\left(|f_1|\right) + \frac{1}{w(B)}\int_B \left|\mathcal{I}_\alpha(f_2)-c_B\right| \\ &\leqslant C|B|^{\alpha/d-1/p}[f]_{p,w} + \frac{1}{w(B)}\int_B \left|\mathcal{I}_\alpha(f_2)-c_B\right|.\end{aligned}$$

For the second term, we will show that

$$\left|\mathcal{I}_\alpha f_2(x)-c_B\right| \leqslant Cw(B)|B|^{\frac{\alpha}{d}-\frac{1}{p}-1}[f]_{p,w}. \tag{17}$$

Let x be in B and split $\mathcal{I}_\alpha f_2(x)$ as in (15). For the first term we can proceed as before to obtain that

$$\left|\int_0^{r^2} e^{-t\mathcal{L}} f_2(x) t^{\alpha/2-1}\, dt\right| \leqslant Cw(B)|B|^{\frac{\alpha}{d}-\frac{1}{p}-1}[f]_{p,w}.$$

The remaining part, by the definition of c_B, is bounded by

$$\left|\int_{r^2}^{\infty} e^{-t\mathcal{L}} f_2(x) t^{\alpha/2-1}\, dt - c_B\right| \leqslant \int_{r^2}^{\infty} \int_{\mathbb{R}^d\setminus 2B} \left|k_t(x,y)-k_t(x_0,y)\right|\left|f(y)\right| dy\, t^{\alpha/2-1}\, dt$$

and by Lemma 3, for any $0<\delta<\delta_0$ the last integral is majorised by

$$C_\delta \int_{r^2}^{\infty} \int_{\mathbb{R}^d\setminus 2B} \left(\frac{|x-x_0|}{\sqrt{t}}\right)^{\delta} t^{-d/2} e^{-\frac{|x-y|^2}{Ct}} \left|f(y)\right| dy\, t^{\alpha/2-1}\, dt.$$

Since $|x_0-x|<r$, applying Fubini's theorem the last integral is bounded by

$$r^\delta \int_{\mathbb{R}^d\setminus 2B} \left|f(y)\right| \int_{r^2}^{\infty} t^{-(d-\alpha+\delta)/2} e^{-\frac{|x-y|^2}{Ct}} \frac{dt}{t}\, dy.$$

Now, changing variables $s=\frac{|x-y|^2}{t}$ we obtain the bound

$$r^\delta \int_{\mathbb{R}^d\setminus 2B} \frac{|f(y)|}{|x-y|^{d-\alpha+\delta}}\, dy \int_0^{\infty} s^{(d-\alpha+\delta)/2} e^{-s/C} \frac{ds}{s}.$$

Since the integral in s is finite, we only need to estimate the integral in y. We perform the same calculation as in (16) with $M = d - \alpha + \delta$. But now, to make the series convergent we need $\eta < 1 - \alpha/d + \delta/d + 1/p$ which holds true by our assumption on η, and taking δ close enough to δ_0. Notice this is the only place where we have used the condition on the size of η. In this way the above expression can be controlled by $w(B)r^{\alpha-\frac{d}{p}-d}[f]_{p,w}$ and so (17) is proved. □

4. $\mathcal{I}_\alpha$ on $\mathrm{BMO}^\beta_{\mathcal{L}}(w)$ spaces

The definition of $BMO^\beta_{\mathcal{L}}(w)$ only establishes a control for the averages over balls with radii greater than ρ at their centres (see (3)). However, for lower radii some kind of estimate can be proved.

Lemma 6. *Let $w \in D_\eta$ with $\eta \geqslant 1$ and $f \in BMO^\beta_{\mathcal{L}}(w)$. Then, for every ball $B = B(x,r)$, we have*

$$\int_B |f| \leqslant C\|f\|_{BMO^\beta_L(w)} w(B)|B|^{\beta/d} \max\left\{1, \left(\frac{\rho(x)}{r}\right)^{d\eta-d+\beta}\right\},$$

if $\eta > 1$ or $\beta > 0$, and

$$\int_B |f| \leqslant C\|f\|_{BMO_{\mathcal{L}}(w)} w(B) \max\left\{1, 1+\log\left(\frac{\rho(x)}{r}\right)\right\},$$

if $\eta = 1$ and $\beta = 0$.

Proof. Let $f \in BMO^\beta_{\mathcal{L}}(w)$. If $r \geqslant \rho(x)$ the conclusion follows from condition (3). If $r < \rho(x)$, let $j_0 = \lfloor \log_2(\frac{\rho(x)}{r})\rfloor + 1$, where $\lfloor\cdot\rfloor$ denotes the greatest integer function. Then

$$\begin{aligned}\frac{1}{|B|}\int_B |f| &\leqslant 2^d \sum_{j=0}^{j_0-1} \frac{1}{|2^jB|}\int_{2^jB} \left|f(z) - f_{2^jB}\right| dz + \frac{1}{|2^{j_0}B|}\int_{2^{j_0}B} |f| \\ &\leqslant C\|f\|_{BMO^\beta_{\mathcal{L}}(w)} \sum_{j=0}^{j_0} w\left(2^jB\right)\left|2^jB\right|^{\frac{\beta}{d}-1},\end{aligned}$$

since $r2^{j_0} \geqslant \rho(x)$. Using now that $w \in D_\eta$, we get

$$\int_B |f| \leqslant C\|f\|_{BMO^\beta_{\mathcal{L}}(w)} w(B)|B|^{\beta/d} \sum_{j=0}^{j_0} 2^{j(d\eta-d+\beta)} \leqslant C\|f\|_{BMO^\beta_{\mathcal{L}}(w)} w(B)|B|^{\beta/d}\left(\frac{\rho(x)}{r}\right)^{d\eta-d+\beta},$$

in the case $\eta > 1$ or $\beta > 0$. If $\eta = 1$ and $\beta = 0$, we have

$$\sum_{j=1}^{j_0} 2^{j(d\eta-d+\beta)} = j_0 \leqslant 1 + \log_2\left(\frac{\rho(x)}{r}\right),$$

and the proof is finished. □

Theorem 2. *Let us assume that the potential V belongs to RH_q with $q \geqslant d/2$ and set $\delta_0 = \min\{1, 2 - \frac{d}{q}\}$. Let $0 < \alpha < 1$, $\beta \geqslant 0$, $\alpha + \beta < \delta_0$ and, $w \in D_\eta$ with $1 \leqslant \eta < 1 + \frac{\delta_0 - \alpha - \beta}{d}$, then the operator $\mathcal{I}_\alpha$ is bounded from $BMO^\beta_{\mathcal{L}}(w)$ into $BMO^{\beta+\alpha}_{\mathcal{L}}(w)$.*

Proof. Since $\alpha > 0$, $BMO^{\beta+\alpha}_{\mathcal{L}}(w) = \Lambda^{\beta+\alpha}_{\mathcal{L}}(w)$ with equivalent norms, due to Proposition 4. Hence we can prove boundedness from $BMO^\beta_{\mathcal{L}}(w)$ into $\Lambda^{\beta+\alpha}_{\mathcal{L}}(w)$. Let $f \in BMO^\beta_{\mathcal{L}}(w)$. We will see that for x and y in $\mathbb{R}^d$, we have

$$\left|\mathcal{I}_\alpha f(x) - \mathcal{I}_\alpha f(y)\right| \leqslant C\|f\|_{BMO^\beta_{\mathcal{L}}(w)}\left[W_{\beta+\alpha}\left(x, |x-y|\right) + W_{\beta+\alpha}\left(y, |x-y|\right)\right] \tag{18}$$

provided $|x - y| < \rho(x)$, and

$$\int_{B(x,\rho(x))} \left|\mathcal{I}_\alpha f(u)\right| du \leqslant \|f\|_{BMO^\beta_{\mathcal{L}}(w)} \rho(x)^{\beta+\alpha} w\left(B\left(x, \rho(x)\right)\right). \tag{19}$$

The above inequalities (18) and (19) would imply that $\mathcal{I}_\alpha f$ belongs to $\Lambda^{\beta+\alpha}_{\mathcal{L}}(w)$ (see Remark 2).

Suppose $\|f\|_{BMO^{\beta}_{\mathcal{L}}(w)} = 1$ and let us start with (19). We split the inner integral, as usual, in local and global parts. If we call $B = B(x, \rho(x))$, then

$$\int_B \left|\mathcal{I}_\alpha f(u)\right| du \leqslant \int_B \left(\int_{2B} + \int_{(2B)^c} \right) K_\alpha(u,z)\left|f(z)\right| dz\, du.$$

By estimate (13), the first term is bounded by

$$\int_B \int_{2B} \frac{|f(z)|}{|u-z|^{d-\alpha}}\, dz\, du \leqslant \int_{2B} \left|f(z)\right| dz \int_B \frac{1}{|u-x|^{d-\alpha}}\, du \leqslant C\|f\|_{BMO^{\beta}_{\mathcal{L}}(w)} \rho(x)^{\alpha+\beta} w(B).$$

For the second term, using Lemma 2 and the change of variables $s = \frac{|u-z|^2}{Ct}$, we have

$$\int_B \int_0^\infty \int_{(2B)^c} k_t(u,z)\left|f(z)\right| dz\, t^{\alpha/2} \frac{dt}{t}\, du \leqslant C \int_B \int_0^\infty \int_{(2B)^c} t^{-(d-\alpha+N)/2} e^{-\frac{|u-z|^2}{Ct}} \rho(u)^N \left|f(z)\right| dz\, \frac{dt}{t}\, du$$
$$\leqslant C \int_0^\infty s^{(d-\alpha+N)/2} e^{-s} \frac{ds}{s} \int_B \int_{(2B)^c} \rho(u)^N \frac{|f(z)|}{|u-z|^{d-\alpha+N}}\, dz\, du.$$

If $u \in B(x, \rho(x))$ then $\rho(u) \leqslant C\rho(x)$ (Proposition 1), and also $|u-z| > |x-z|/2$ for all $z \in B(x, 2\rho(x))^c$. Hence, the last expression is bounded by

$$C\rho(x)^{N+d} \int_{(2B)^c} \frac{|f(z)|}{|x-z|^{d-\alpha+N}}\, dz. \tag{20}$$

If we call $B_j = 2^j B$, we may split the last integral into annuli, use that $f \in BMO^{\beta}_{\mathcal{L}}(w)$ and $w \in D_\eta$ to obtain

$$\int_{(2B)^c} \frac{|f(z)|}{|x-z|^{d-\alpha+N}}\, dz \leqslant \sum_{k=1}^\infty \int_{B_{k+1}\setminus B_k} \frac{|f(z)|}{|x-z|^{d-\alpha+N}}\, dz$$
$$\leqslant \rho(x)^{-d+\alpha-N} \sum_{k=1}^\infty 2^{-k(d-\alpha+N)} \int_{B_{k+1}} \left|f(z)\right| dz$$
$$\leqslant C\rho(x)^{-d+\alpha+\beta-N} \sum_{k=1}^\infty 2^{-k(d-\alpha-\beta+N)} w(B_{k+1})$$
$$\leqslant C\rho(x)^{-d+\alpha+\beta-N} w(B) \sum_{k=1}^\infty 2^{-k(d-\alpha-\beta+N-d\eta)}.$$

If we choose N large enough, the last sum is finite, thus (20) is bounded by a constant times

$$\rho(x)^{\alpha+\beta} w\bigl(B\bigl(x, \rho(x)\bigr)\bigr),$$

and we have shown that (19) is satisfied.

To see (18), let $|x-y| < \rho(x)$,

$$\left|\mathcal{I}_\alpha f(x) - \mathcal{I}_\alpha f(y)\right| \leqslant \left| \int_0^{\rho(x)^2} \int_{\mathbb{R}^d} \bigl[k_t(x,z) - k_t(y,z)\bigr] f(z)\, dz\, t^{\alpha/2} \frac{dt}{t} \right| + \left| \int_{\rho(x)^2}^\infty \int_{\mathbb{R}^d} \bigl[k_t(x,z) - k_t(y,z)\bigr] f(z)\, dz\, t^{\alpha/2} \frac{dt}{t} \right|. \tag{21}$$

For the first term, if $t > \rho(x)^2$, since $|x-y| < \rho(x)$, we have $|x-y| < \sqrt{t}$, hence Lemma 3 allows us to get

$$\int_{\rho(x)^2}^\infty \int_{\mathbb{R}^d} \left|k_t(x,z) - k_t(y,z)\right| \left|f(z)\right| dz\, t^{\frac{\alpha}{2}-1}\, dt \leqslant C_\delta |x-y|^\delta \int_{\rho(x)^2}^\infty \int_{\mathbb{R}^d} e^{-\frac{|x-z|^2}{Ct}} \left|f(z)\right| dz\, t^{(-d+\alpha-\delta)/2} \frac{dt}{t}, \tag{22}$$

for each $0 < \delta < \delta_0$. If $t > \rho(x)^2$, calling $B = B(x, \sqrt{t})$ we estimate the inner integral as

$$\int_{\mathbb{R}^d} e^{-\frac{|x-z|^2}{Ct}} \left|f(z)\right| dz \leqslant C \int_B |f| + t^{M/2} \sum_{k=0}^\infty \int_{2^{k+1}B\setminus 2^k B} \frac{|f(z)|}{|x-z|^M}\, dz,$$

for some $M > 1$ to be chosen. Since $f \in BMO_{\mathcal{L}}^{\beta}(w)$ and $t > \rho(x)^2$, the first integral is bounded by $w(B)t^{\beta/2}$. To deal with the sum in k, we use again $f \in BMO_{\mathcal{L}}^{\beta}(w)$, and then $w \in D_\eta$, to obtain

$$\begin{aligned} t^{M/2}\sum_{k=0}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\frac{|f(z)|}{|x-z|^M}\,dz &\leqslant 2\sum_{k=0}^{\infty}2^{-kM}\int_{2^{k+1}B}|f| \\ &\leqslant Ct^{\beta/2}\sum_{k=0}^{\infty}2^{-k(M-\beta)}w\big(2^{k+1}B\big) \\ &\leqslant Ct^{\beta/2}w(B)\sum_{k=0}^{\infty}2^{-k(M-\beta-d\eta)}, \end{aligned}$$

and the sum is finite for M large enough. Therefore, since $|x-y| < \rho(x) < \sqrt{t}$ and $-d+\alpha+\beta-\delta+d\eta < 0$ choosing δ close to δ_0, (22) is bounded by

$$\begin{aligned} |x-y|^{\delta}\int_{\rho(x)^2}^{\infty}w\big(B\big(x,\sqrt{t}\big)\big)t^{(-d+\alpha+\beta-\delta)/2}\frac{dt}{t} &\leqslant C|x-y|^{\delta-d\eta}w\big(B\big(x,|x-y|\big)\big)\int_{|x-y|^2}^{\infty}t^{(-d+\alpha+\beta-\delta+d\eta)/2}\frac{dt}{t} \\ &\leqslant Cw\big(B\big(x,|x-y|\big)\big)|x-y|^{-d+\alpha+\beta} \\ &\leqslant CW_{\alpha+\beta}\big(x,|x-y|\big). \end{aligned}$$

To deal with the second term of (21), we set

$$q_t(x,y) = k_t(x,y) - \tilde{k}_t(x,y),$$

for all $x, y \in \mathbb{R}^d$ and $t > 0$, where $\tilde{k}_t$ is the classical heat kernel as before. Then we have

$$\left|\int_0^{\rho(x)^2}\int_{\mathbb{R}^d}\big[k_t(x,z)-k_t(y,z)\big]f(z)\,dz\,t^{\alpha/2}\,\frac{dt}{t}\right| \leqslant I + II,$$

where

$$I = \left|\int_0^{\rho(x)^2}\int_{\mathbb{R}^d}\big[q_t(x,z)-q_t(y,z)\big]f(z)\,dz\,t^{\alpha/2}\frac{dt}{t}\right|$$

and

$$II = \left|\int_0^{\rho(x)^2}\int_{\mathbb{R}^d}\big[\tilde{k}_t(x,z)-\tilde{k}_t(y,z)\big]f(z)\,dz\,t^{\alpha/2}\,\frac{dt}{t}\right|.$$

To estimate I, calling $B = B(x, 4|x-y|)$, we split $\mathbb{R}^d$ into two regions and write

$$I \leqslant I_1 + I_2 + I_3,$$

with

$$I_1 = \int_0^{\rho(x)^2}\int_{B^c}\big|q_t(x,z)-q_t(y,z)\big|\big|f(z)\big|\,dz\,t^{\alpha/2}\,\frac{dt}{t},$$

$$I_2 = \int_0^{\rho(x)^2}\int_{B}\big|q_t(x,z)\big|\big|f(z)\big|\,dz\,t^{\alpha/2}\frac{dt}{t}$$

and

$$I_3 = \int_0^{\rho(x)^2}\int_{B}\big|q_t(y,z)\big|\big|f(z)\big|\,dz\,t^{\alpha/2}\frac{dt}{t}.$$

If $z \in B^c$, we are in the hypothesis of Lemma 5 and therefore, given $0 < \delta < \delta_0$, there exists a rapidly decaying function ψ such that

$$I_1 \leqslant C|x-y|^{\delta} \int_0^{\rho(x)^2} \int_{B^c} \frac{\psi_t(z-x)}{\rho(z)^{\delta}} \big|f(z)\big|\, dz\, t^{\alpha/2} \frac{dt}{t}$$

$$\leqslant C\left(\frac{|x-y|}{\rho(x)}\right)^{\delta} \int_0^{\rho(x)^2} \int_{B^c} \left(1+\frac{|x-z|}{\rho(x)}\right)^{\delta k_0} \psi_t(z-x)\big|f(z)\big|\, dz\, t^{\alpha/2} \frac{dt}{t},$$

where in the last inequality we have used Proposition 1.

The inner integral is

$$\int_{B^c} \left(1+\frac{|x-z|}{\rho(x)}\right)^{\delta k_0} \psi_t(z-x)\big|f(z)\big|\, dz = \sum_{j=0}^{\infty} \int_{B_j \setminus B_{j-1}} \left(1+\frac{|x-z|}{\rho(x)}\right)^{\delta k_0} \psi_t(z-x)\big|f(z)\big|\, dz,$$

where $B_j = B(x, 2^{j+3}|x-y|)$. Thus $I_1 \leqslant I_{11} + I_{12}$, where

$$I_{11} = C\left(\frac{|x-y|}{\rho(x)}\right)^{\delta} \int_0^{\rho(x)^2} \sum_{j=0}^{j_0} \int_{B_j \setminus B_{j-1}} \left(1+\frac{|x-z|}{\rho(x)}\right)^{\delta k_0} \psi_t(z-x)\big|f(z)\big|\, dz\, t^{\alpha/2} \frac{dt}{t},$$

with $j_0 = \lfloor \log_2(\frac{\rho(x)}{|x-y|}) \rfloor$, and I_{12} the same but summing up from j_0+1. If $j \leqslant j_0$ and $z \in B_j \setminus B_{j-1}$, then $(1+\frac{|x-z|}{\rho(x)})^{\delta k_0} \leqslant C$, and since $\psi_t(z-x) \leqslant Ct^{\epsilon/2}/|x-z|^{d+\epsilon}$, for some $\epsilon > 0$ fixed, we obtain

$$I_{11} \leqslant C\left(\frac{|x-y|}{\rho(x)}\right)^{\delta} \int_0^{\rho(x)^2} t^{(\alpha+\epsilon)/2} \frac{dt}{t} \sum_{j=0}^{j_0} \int_{B_j \setminus B_{j-1}} \frac{|f(z)|}{|x-z|^{d+\epsilon}}\, dz$$

$$\leqslant C \frac{|x-y|^{\delta-d-\epsilon}}{\rho(x)^{\delta-\alpha-\epsilon}} \sum_{j=0}^{j_0} 2^{-j(d+\epsilon)} \int_{B_j} \big|f(z)\big|\, dz.$$

From Lemma 6 and the fact that $w \in D_\eta$, in the case $\eta > 1$ or $\beta > 0$,

$$\sum_{j=0}^{j_0} 2^{-j(d+\epsilon)} \int_{B_j} \big|f(z)\big|\, dz \leqslant C \sum_{j=0}^{j_0} 2^{-j(d+\epsilon)} w(B_j)|B_j|^{\beta/d} \left(\frac{\rho(x)}{2^{j+3}|x-y|}\right)^{d\eta-d+\beta}$$

$$\leqslant C \frac{\rho(x)^{d\eta-d+\beta}}{|x-y|^{d\eta-d}} w(B) \sum_{j=0}^{j_0} 2^{-j\epsilon}$$

$$\leqslant C \frac{\rho(x)^{d\eta-d+\beta}}{|x-y|^{d\eta-d}} w(B).$$

Therefore, we have

$$I_{11} \leqslant C\left(\frac{|x-y|}{\rho(x)}\right)^{\delta-\alpha-\beta-d\eta+d-\epsilon} \frac{w(B)}{|x-y|^{d-\alpha-\beta}} \leqslant C \frac{w(B)}{|x-y|^{d-\alpha-\beta}}, \tag{23}$$

since by hypothesis $1 \leqslant \eta < \frac{\delta_0-\alpha-\beta}{d} + 1$ and $|x-y| < \rho(x)$, and thus $\delta - \alpha - \beta - d\eta + d - \epsilon > 0$, choosing ϵ small enough and δ close to δ_0.

As for the case $\beta = 0$ and $\eta = 1$, using Lemma 6 and the inequality

$$1 + \log(t) \leqslant Ct^{\epsilon/2}, \tag{24}$$

for $t > 1/8$, we arrive to the same estimate of I_{11} proceeding as before.

Next we estimate I_{12}. For $M > \delta k_0 + d\eta + \beta$, we have $\psi_t(z-x) \leqslant C\frac{t^{(M-d)/2}}{|z-x|^M}$. Also if $z \in B_j \setminus B_{j-1}$ for $j > j_0$, then $|x-z| > \rho(x)$. Therefore

$$I_{12} \leqslant C\left(\frac{|x-y|}{\rho(x)}\right)^{\delta+\delta k_0} \int_0^{\rho(x)^2} t^{(M-d+\alpha)/2}\frac{dt}{t} \sum_{j=j_0+1}^{\infty} 2^{j\delta k_0} \int_{B_j\setminus B_{j-1}} \frac{|f(z)|}{|z-x|^M}\,dz$$
$$\leqslant C\frac{|x-y|^{\delta+\delta k_0-M}}{\rho(x)^{\delta+\delta k_0-M+d-\alpha}} \sum_{j=j_0+1}^{\infty} 2^{-j(M-\delta k_0)} \int_{B_j} \left|f(z)\right|dz.$$

Since for $j > j_0$, the radius of B_j is $2^{j+3}|x-y| > \rho(x)$, then

$$\int_{B_j} \left|f(z)\right|dz \leqslant C w(B_j)|B_j|^{\beta/d} \leqslant C2^{j(d\eta+\beta)}|x-y|^{\beta} w(B),$$

and thus

$$I_{12} \leqslant C\left(\frac{|x-y|}{\rho(x)}\right)^{-M+\delta k_0+\delta-\alpha+d} \frac{w(B)}{|x-y|^{d-\alpha-\beta}} \sum_{j=j_0+1}^{\infty} 2^{-j(M-\delta k_0-d\eta-\beta)}$$
$$\leqslant C\left(\frac{|x-y|}{\rho(x)}\right)^{d-d\eta+\delta-\alpha-\beta} \frac{w(B)}{|x-y|^{d-\alpha-\beta}}$$
$$\leqslant C\frac{w(B)}{|x-y|^{d-\alpha-\beta}}, \tag{25}$$

with an appropriate choice of δ.

To deal with I_2, let $M > d$. From Lemma 4, being $t < \rho(x)^2$,

$$\left|q_t(x,z)\right| \leqslant C\left(\frac{\sqrt{t}}{\rho(x)}\right)_0^{\delta} \frac{1}{t^{d/2}}\left(1+\frac{|x-z|}{\sqrt{t}}\right)^{-M}. \tag{26}$$

Then we may write

$$I_2 = I_{21} + I_{22},$$

where

$$I_{21} = C\int_0^{|x-y|^2}\int_B \left|q_t(x,z)\right|\left|f(z)\right|dz\, t^{\alpha/2}\frac{dt}{t}$$

and

$$I_{22} = \int_{|x-y|^2}^{\rho(x)^2}\int_B \left|q_t(x,z)\right|\left|f(z)\right|dz\, t^{\alpha/2}\frac{dt}{t}.$$

To take care of I_{21} let $B_t = B(x,\sqrt{t})$ and $N = \lfloor \log_2(\frac{4|x-y|}{\sqrt{t}})\rfloor$. Using estimate (26), we have

$$\int_B \left|q_t(x,z)\right|\left|f(z)\right|dz \leqslant \frac{t^{\frac{\delta_0-d}{2}}}{\rho(x)_0^{\delta}}\left(\int_{B_t}|f| + t^{M/2}\int_{B\setminus B_t}\frac{|f(z)|}{|x-z|^M}\,dz\right)$$
$$\leqslant \frac{t^{\frac{\delta_0-d}{2}}}{\rho(x)_0^{\delta}}\left(\int_{B_t}|f| + t^{M/2}\sum_{j=0}^{N}\int_{2^{j+1}B_t\setminus 2^j B_t}\frac{|f(z)|}{|x-z|^M}\,dz\right)$$
$$\leqslant C\frac{t^{\frac{\delta_0-d}{2}}}{\rho(x)_0^{\delta}}\left(\sum_{j=0}^{N+1} 2^{-jM}\int_{2^j B_t}|f|\right),$$

and since every ball in the last sum has its radius less than $8\rho(x)$, we can apply Lemma 6 and that $w \in D_\eta$, to obtain

$$\int_B |q_t(x,z)||f(z)|\,dz \leqslant C\frac{t^{(\delta_0-d\eta)/2}}{\rho(x)^{\delta_0-d\eta+d-\beta}}\left(\sum_{j=0}^{N}2^{-j(M-d+d\eta)}w\big(2^jB_t\big)\right)$$
$$\leqslant C\frac{t^{(\delta_0-d\eta)/2}}{\rho(x)^{\delta_0-d\eta+d-\beta}}w(B_t)\left(\sum_{j=0}^{\infty}2^{-j(M-d)}\right)$$
$$\leqslant C\frac{t^{(\delta_0-d\eta)/2}}{\rho(x)^{\delta_0-d\eta+d-\beta}}w(B_t),$$

where the last sum is finite since $M > d$.

Hence,

$$I_{21} \leqslant \frac{C}{\rho(x)^{\delta_0-d\eta+d-\beta}}\int_0^{|x-y|^2} t^{(\delta_0+\alpha-d\eta)/2}w(B_t)\frac{dt}{t}$$
$$\leqslant \frac{C}{\rho(x)^{\delta_0-d\eta+d-\beta}}\int_0^{|x-y|^2} t^{(\delta_0-\beta-d\eta+d)/2}\,W_{\alpha+\beta}\big(x,\sqrt{t}\big)\frac{dt}{t}$$
$$\leqslant \frac{C}{\rho(x)^{\delta_0-d\eta+d-\beta}}\int_0^{|x-y|^2} t^{(\delta_0-\beta-d\eta+d)/2}\frac{dt}{t}W_{\alpha+\beta}\big(x,|x-y|\big)$$
$$\leqslant C\left(\frac{|x-y|}{\rho(x)}\right)^{\delta_0-\beta-d\eta+d}W_{\alpha+\beta}\big(x,|x-y|\big). \tag{27}$$

Since $\delta_0-\beta-d\eta+d > \alpha > 0$, and $|x-y| < \rho(x)$, we have $I_{21} \leqslant W_{\alpha+\beta}(x,|x-y|)$.

To deal with I_{22} we use again (26) and Lemma 6, to get

$$I_{22} = \int_{|x-y|^2}^{\infty}\int_B |q_t(x,z)||f(z)|\,dz\,t^{\alpha/2}\frac{dt}{t}$$
$$\leqslant C\rho(x)^{-\delta_0}\int_{|x-y|^2}^{\infty} t^{(\alpha+\delta_0-d)/2}\frac{dt}{t}\int_B |f|$$
$$\leqslant C\frac{w(B)}{|x-y|^{d-\alpha-\beta}}\left(\frac{|x-y|}{\rho(x)}\right)^{d-d\eta+\delta_0-\beta}$$
$$\leqslant C\frac{w(B)}{|x-y|^{d-\alpha-\beta}} \leqslant CW_{\alpha+\beta}\big(x,|x-y|\big), \tag{28}$$

since $d-d\eta+\delta_0-\beta > \alpha > 0$, and $|x-y| < \rho(x)$.

The case $\beta = 0$ and $\eta = 1$ is performed using Lemma 6 and inequality (24) with $\epsilon < \delta_0$, following the same steps as in (27) and (28) respectively.

We can also obtain that

$$I_3 \leqslant CW_{\alpha+\beta}\big(x,|x-y|\big) \tag{29}$$

following the same lines as in I_2 but exchanging x by y and integrating over $B(y, 8|x-y|)$.

From (23), (25), (27)–(29) we obtain

$$I \leqslant CW_{\alpha+\beta}\big(x,|x-y|\big).$$

To see $II \leqslant CW_{\alpha+\beta}(x,|x-y|)$ we refer to the reader to [5, p. 238]. In fact, since $\tilde{k}_t$ is a convolution kernel,

$$\int_{\mathbb{R}^d}\big[\tilde{k}_t(x,z)-\tilde{k}_t(y,z)\big]\,dz = 0.$$

So we have

$$II = \left| \int_0^{\rho(x)^2} \int_{\mathbb{R}^d} \left[\tilde{k}_t(x,z) - \tilde{k}_t(y,z)\right]\left[f(z) - f_B\right] dz\, t^{\alpha/2} \frac{dt}{t} \right| \leqslant II_1 + II_2,$$

where

$$II_1 = \int_0^{\rho(x)^2} \int_B \left|\tilde{k}_t(x,z) - \tilde{k}_t(y,z)\right| \left|f(z) - f_B\right| dz\, t^{\alpha/2} \frac{dt}{t}$$

and

$$II_2 = \int_0^{\rho(x)^2} \int_{B^c} \left|\tilde{k}_t(x,z) - \tilde{k}_t(y,z)\right| \left|f(z) - f_B\right| dz\, t^{\alpha/2} \frac{dt}{t}$$

with $B = B(x, |x-y|)$.

Applying

$$\left|\tilde{k}_t(x,z) - \tilde{k}_t(y,z)\right| \leqslant C \frac{e^{-\frac{|x-y|}{Ct}}}{t^{d/2-1}} |x-y||x-z|$$

and changing variables $s = \frac{t}{|x-y|}$, we have

$$II_2 \leqslant |x-y| \int_{B^c} \left|f(z) - f_B\right| |x-z| \int_0^{\rho(x)^2} \frac{e^{-\frac{|x-y|}{Ct}}}{t^{d/2-1}} t^{\alpha/2} \frac{dt}{t}\, dz \leqslant |x-y| \int_{B^c} \frac{|f(z) - f_B|}{|x-z|^{d-\alpha+1}}\, dz.$$

Since $w \in D_\eta$, from Lemma 4.7 in [5], the last expression is bounded by

$$|x-y| \int_{B^c} \frac{w(z)}{|x-z|^{d-\alpha-\beta+1}}\, dz \leqslant C w(B) |x-y|^{d-\alpha-\beta},$$

where the last inequality is due to Lemma 3.9 in [5].

To deal with II_1,

$$\int_0^{\rho(x)^2} \int_B \left|\tilde{k}_t(x,z)\right| \left|f(z) - f_B\right| dz\, t^{\alpha/2} \frac{dt}{t} \leqslant C \int_0^{\infty} \int_B \frac{e^{-\frac{|x-y|}{Ct}}}{t^{d/2}} \left|f(z) - f_B\right| dz\, t^{\alpha/2} \frac{dt}{t} = C \int_B \frac{|f(z) - f_B|}{|x-z|^{d-\alpha}}\, dz$$

and denoting $B_j = 2^{-j} B$, we obtain

$$\begin{aligned}
\int_B \frac{|f(z) - f_B|}{|x-z|^{d-\alpha}}\, dz &= \sum_{k=0}^{\infty} \int_{B_j \setminus B_{j+1}} \frac{|f(z) - f_B|}{|x-z|^{d-\alpha}} \\
&\leqslant C \sum_{k=0}^{\infty} \left(\frac{2^j}{|x-y|}\right)^{d-\alpha} \int_{B_j} \left|f(z) - f_B\right| \\
&\leqslant C \sum_{k=0}^{\infty} \left(\frac{2^j}{|x-y|}\right)^{d-\alpha-\beta} w(B_j) \\
&\leqslant C \sum_{k=0}^{\infty} \left(\frac{2^j}{|x-y|}\right)^{d-\alpha-\beta} w(B_j \setminus B_{j+1}) \\
&\leqslant C \sum_{k=0}^{\infty} \int_{B_j \setminus B_{j+1}} \frac{w(z)}{|x-z|^{d-\alpha-\beta}}\, dz \\
&= C \int_B \frac{w(z)}{|x-z|^{d-\alpha-\beta}}\, dz,
\end{aligned}$$

finishing the proof of the theorem. □

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