JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **49**, 575–611 (1975)

New Approach to the Asymptotic Theory of Nonlinear Oscillations and Wave-Propagation

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Submitted by J. L. Lions

1. INTRODUCTION

The asymptotic method of Krilov–Bogolioubov–Mitropolski is well-known and has frequently been described in the litterature (for example, Bogolioubov and Mitropolski [2]). To define the ideas let us briefly outline some of the main results.

Suppose $\mathbf{Y}(t, \epsilon)$ is an n-dimensional vector function of time-like variable t and a small parameter ϵ. $\mathbf{Y}(t, \epsilon)$ is defined as solution of the initial value problem

$$d\mathbf{Y}/dt = \epsilon\mathbf{F}(\mathbf{Y}, t, \epsilon), \qquad \mathbf{Y}(0, \epsilon) = \mathbf{Y}_0,$$

where $\mathbf{F}\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$.

According to the asymptotic method $\mathbf{Y}(t, \epsilon)$ can be approximated by a function $\eta(\epsilon t)$, defined as solution of

$$d\eta/dt = \epsilon\mathbf{F}_0(\eta); \qquad \eta(0) = \mathbf{Y}_0,$$

where

$$\mathbf{F}_0(\eta) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{F}(\eta, t, 0)\, dt.$$

The approximation is valid in the sense that $| \mathbf{Y}(t, \epsilon) - \eta(\epsilon t)|$ tends to zero as ϵ tends to zero. The validity is assured in an interval

$$0 \leqslant t \leqslant L/\epsilon$$

where L is an arbitrary constant (independent of ϵ). In the special case of periodic systems, that is $\mathbf{F}(\mathbf{Y}, t + T, \epsilon) = \mathbf{F}(\mathbf{Y}, t, \epsilon)$ for some constant T, a better estimate can be obtained. In that case one can show that

$$| \mathbf{Y}(t, \epsilon) - \eta(\epsilon t)| \leqslant \epsilon c(L),$$

where $c(L)$ is a constant which only depends on L. Furthermore, higher approximations of $\mathbf{Y}(t, \epsilon)$ can also be defined.

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There are several reasons which have led the present author to reconsider this well-known and well-developed theory: The proofs, given in any standard text, are difficult. The easiest case is that of periodic systems, when one supposes differentiability of $\mathbf{F}(\mathbf{Y}, t, \epsilon)$ with respect to $\mathbf{Y}$ and uniform boundness of the derivatives on $0 \leqslant t < \infty$. When one drops this differentiability hypothesis the proof becomes already much more complicated, and is still more complicated in the general nonperiodic case. (In this authors opinion, the best proofs given sofar are those of Besjes [1]).

Furthermore, in any standard presentation, the theory is not deductive. By this we mean that without any convincing motivation one asserts that $\boldsymbol{\eta}$ is an approximation of $\mathbf{Y}$, and one proceeds to proof the assertion by classical procedure of estimation of integrals. There is little in the development of the theory (except for the results) that justifies the name "asymptotic method."

Finally, one wonders whether it could be possible to extend the validity of the results to larger time-intervals. Such extension has sofar only been established for the case of periodic solutions or solutions starting sufficiently near stable periodic solutions.

In the study described in this paper a deductive asymptotic theory is developed, which uses from the outset concepts and methods of asymptotic analysis. The necessary preliminaries are given in Sections 2 and 3. Section 4 introduces the fundamental tool of our method of analysis: in a suitable (asymptotic) sense a local average value of the function $\mathbf{Y}$ is defined. With the aid of this concept a deductive procedure establishes the fundamental theorem of Krilov–Bogolioubov–Mitropolski under the most general conditions (Section 5). Next we deduce improved results for periodic systems and show how higher approximations can be obtained. These results are well-known, but they are reproduced here by a relatively simple deductive asymptotic analysis.

In the remaining part of the paper we show that our approach also permits to establish new results. Notably we show that if $\mathbf{F}_0(\eta)$ has an asymptotically stable singular point (and $\mathbf{Y}_0$ is any initial value within the domain of attraction of that singular point) then $\eta(\epsilon t)$ is an asymptotic approximation of $\mathbf{Y}(t, \epsilon)$ uniformly valid on $0 \leqslant t < \infty$ (Sections 8 and 9). Finally, we explore the possibility of applying our method of analysis to partial differential equations describing wavepropagation phenomena. In Section 10 we study a class of perturbed wave-equations, previously investigated by Chikwendu and Kevorkian [3]. These authors have proposed a formal method of construction of asymptotic approximation (without proof). Using the concept of local average values we rederive (by a deductive procedure) the fundamental result of Chikwendu and Kevorkian and thus give proof of the validity of the approximation.

A remark on the formulation used in this paper for problems in terms of ordinary differential equations should be made. Throughout our analysis we study vector equations

$$d\mathbf{Y}/dt = \epsilon\mathbf{F}(\mathbf{Y}, t, \epsilon),$$

which we have termed "standard systems for slowly modulated processes." A large class of problems can be transformed into this form. Well-known example is given by systems of perturbed linear oscillators:

$$\frac{d^2x_i}{dt^2} + w_i^2x_i = \epsilon g_i\left(x_1, ..., x_m, \frac{dx_1}{dt}, ..., \frac{dx_m}{dt}, t\right); \qquad i = 1, ..., m.$$

If now $\mathbf{X}$ is the vector with components $x_1, ..., x_m$, then the correspondence $\mathbf{X} \to \mathbf{Y}$ is achieved by the Van der Pol transformation. However, also non-linear perturbed systems, of the general form

$$\frac{d\mathbf{z}}{dt} = \mathbf{H}(\mathbf{z}, t) + \epsilon\mathbf{G}(\mathbf{z}, t)$$

can, under certain conditions, be transformed into a slowly modulated standard system. Roughly speaking this is possible when the "unperturbed system"

$$\frac{d\mathbf{z}^0}{dt} = \mathbf{H}(\mathbf{z}^0, t),$$

possesses a "general solution". The transformation is then in essence achieved by the method of variation of constants. The conditions that arise in the course of the calculations can be found in Volosov [11].

2. Orders of Magnitude and Asymptotic Approximations of Functions in Unbounded Domains

Let t be a real variable to be interpreted as time ($t_0 \leqslant t < \infty$), and let ϵ be a real small parameter ($0 \leqslant \epsilon \leqslant \epsilon_0$). We shall study real-valued vector functions $(t, \epsilon) \to \mathbf{Y}(t, \epsilon)$, of which the components are: $y_1(t, \epsilon), ..., y_n(t, \epsilon)$. $\mathbf{Y}(t, \epsilon)$ is defined for $t \in J$; the interval J may be unbounded, or may be such that the extent of J tends to infinity as ϵ tends to zero. We shall investigate asymptotic approximations of $\mathbf{Y}(t, \epsilon)$, valid for $\epsilon \downarrow 0$.

The definitions of asymptotic approximations are closely related to the definitions of orders of magnitude of functions. It is for this reason that we develop first the necessary concepts of asymptotic orders of magnitude.

Let $\mathbf{\Phi}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ be any vector function with components $\phi_1, ..., \phi_n$. We define

$$| \mathbf{\Phi}(t, \epsilon)| = \sum_{i=1}^{n} | \phi_i(t, \epsilon)| ,$$

$I \subset J$ will denote any bounded closed interval (which may depend on ϵ), and such that $\mathbf{\Phi}$ is continuous on I. We shall use as norm of $\mathbf{\Phi} \mid I$:

$$\| \mathbf{\Phi} \|_I = \operatorname*{Max}_{t \in I} | \mathbf{\Phi}(t, \epsilon)| ,$$

$\delta(\epsilon)$ will denote any real, positive, continuous function of ϵ with the property that $\lim_{\epsilon \to 0} \delta(\epsilon)$ exists. Such functions are called order functions. In comparing any two order functions the well-known order-of-magnitude definitions of Landau will be used. (For various properties of order functions see Eckhaus [5]). With these preliminaries, we have:

DEFINITION 1. $\mathbf{\Phi}(t, \epsilon) = O(\delta)$ in I if there exists a constant k such that $\| \mathbf{\Phi} \|_I \leqslant k\, \delta(\epsilon)$ for $0 \leqslant \epsilon \leqslant \epsilon_0$. $\mathbf{\Phi}(t, \epsilon) = o(\delta)$ in I if $\lim_{\epsilon \to 0} \| \mathbf{\Phi} \|_I / \delta = 0$.

For the study of large time-intervals (as $\epsilon \downarrow 0$) a transformation of time will be used. Such transformation is obtained by introducing a new time-scale, as follows:

DEFINITION 2. δ_s^{-1} is the time-scale of the transformation, $\tau = \delta_s(\epsilon) t$ with $\delta_s(\epsilon) = o(1)$. Furthermore: $\mathbf{\Phi}^*(\tau, \epsilon) = \mathbf{\Phi}(\tau/\delta_s, \epsilon)$.

If now I^* is the image of I under the transformation given above, then obviously

$$\| \mathbf{\Phi} \|_I = \| \mathbf{\Phi}^* \|_{I^*} = \operatorname*{Max}_{\tau \in I^*} | \mathbf{\Phi}^* | .$$

Furthermore,

$$\{\mathbf{\Phi}(t, \epsilon) = O(\delta) \text{ in } I\} \Leftrightarrow \{\mathbf{\Phi}^*(\tau, \epsilon) = O(\delta) \text{ in } I^*\}.$$

In the two definitions that follow now we introduce a notion of uniform behavior.

DEFINITION 3. If for a given time-scale δ_s^{-1} there exists an order function δ such that $\mathbf{\Phi}^*(\tau, \epsilon) = O(\delta)$ for *all bounded closed, ϵ-independent intervals* I^* then we shall say that $\mathbf{\Phi}^*(\tau, \epsilon) = O(\delta)$ *uniformly on the time scale* δ_s^{-1}. Similarly, if $\mathbf{\Phi}^*(\tau, \epsilon) = o(\delta)$ for all bounded closed ϵ-independent I^* then $\mathbf{\Phi}^*(\tau, \epsilon) = o(\delta)$ uniformly on the time scale δ_s^{-1}.

DEFINITION 4. If there exists an order function δ such that $\mathbf{\Phi}^*(\tau, \epsilon) = O(\delta)$ uniformly on *any* time scale δ_s^{-1} then we shall say that $\mathbf{\Phi}(t, \epsilon) = O(\delta)$

uniformly for $t_0 \leqslant t < \infty$. Similarly, if $\mathbf{\Phi}^*(\tau, \epsilon) = o(\delta)$ uniformly on any time scale, then $\mathbf{\Phi}(t, \epsilon) = o(\delta)$ uniformly in $t_0 \leqslant t < \infty$.

Remarks. It is not difficult to show that if, in the sense of Definition 4, a function $\mathbf{\Phi}(t, \epsilon) = O(\delta)$ uniformly in $t_0 \leqslant t < \infty$, then there exists a constant k such that $| \mathbf{\Phi} | \leqslant k\delta$ for all $t_0 \leqslant t < \infty$.

Thus, the uniform behavior in the sense of Definition 4 is in accordance with more usual definition and in fact one can show that the two definitions are equivalent; furthermore the same is true in the case of $\mathbf{\Phi}(t, \epsilon) = o(\delta)$ uniformly in $t_0 \leqslant t < \infty$. However, as we shall see in the sequal, only in exceptional cases can uniform results on $t_0 \leqslant t < \infty$ be attained. Usually the best results that can be obtained are uniform on some time scale in the sense of Definition 3 (and it is for this reason that Definition 3 has been introduced). Furthermore, in cases in which results uniform on $t_0 \leqslant t < \infty$ can be obtained, the analysis of uniform behavior on different time scales is a necessary preliminary. We are now ready to define asymptotic approximations.

DEFINITION 5. (A) $\eta(t, \epsilon)$ is a (uniform) asymptotic approximation of $\mathbf{Y}(t, \epsilon)$ in I if $\mathbf{Y}(t, \epsilon) - \eta(t, \epsilon) = o(1)$ in I.

(B) $\eta(t, \epsilon)$ is a uniform asymptotic approximation of $\mathbf{Y}(t, \epsilon)$ on a time scale δ_s^{-1} if $\mathbf{Y}^*(\tau, \epsilon) - \eta^*(\tau, \epsilon) = o(1)$ uniformly on that time scale.

(C) $\eta(t, \epsilon)$ is a uniform asymptotic approximation of $\mathbf{Y}(t, \epsilon)$ in $t_0 \leqslant t < \infty$ if $\mathbf{Y}(t, \epsilon) - \eta(t, \epsilon) = o(1)$ uniformly in $t_0 \leqslant t < \infty$.

3. Elementary Properties of Slowly Modulated Standard Systems

We study vector functions $\mathbf{Y}(t, \epsilon)$ which are defined as solutions of the initial value problem

$$dY/dt = \epsilon \mathbf{F}(\mathbf{Y}, t, \epsilon); \qquad \mathbf{Y}(0, \epsilon) = \mathbf{Y}_0 ,$$

where $\mathbf{F}: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is a vector function with components $f_i(Y_1, ..., Y_n, t, \epsilon)$, defined in some connected subset $G \subset \mathbb{R}^{n+2}$.

The above differential equation, which represents a system of n first order differential equations for the components $y_1, ..., y_n$ of $\mathbf{Y}$, will be called a *standard system for slowly modulated processes.*

We shall suppose throughout our analysis that the function $\mathbf{F}$ satisfies the following conditions:

(i) $\mathbf{F}$ is a continuous and uniformly bounded function in G, where $G = \{\mathbf{Y} \mid \mathbf{Y} \in \bar{D}\} \times \{t \mid 0 \leqslant t < \infty\} \times \{\epsilon \mid 0 \leqslant \epsilon \leqslant \epsilon_0\}$, D is some open bounded subset of $\mathbb{R}^n$. $\lim_{\epsilon \to 0} \{\mathbf{F}(\mathbf{Y}, t, \epsilon) - \mathbf{F}(\mathbf{Y}, t, 0)\} = 0$ uniformly in

$$\{\mathbf{Y} \mid \mathbf{Y} \in \bar{D}\} \times \{t \mid 0 \leqslant t < \infty\}$$

(ii) $\mathbf{F}$ is Lipschitz-continuous with respect to $\mathbf{Y}$ in $\mathbf{G}$, that is: there exists a constant λ such that for any pair $(\mathbf{Y}_1, \mathbf{Y}_2) \in D$ we have in G

$$|\mathbf{F}(\mathbf{Y}_1, t, \epsilon) - \mathbf{F}(\mathbf{Y}_2, t, \epsilon)| \leqslant \lambda | \mathbf{Y}_1 - \mathbf{Y}_2 |$$

Remarks. The above conditions are for the most part the classical conditions needed to assure existence and uniqueness of the solution $\mathbf{Y}(t, \epsilon)$ In addition, uniform behavior as $\epsilon \downarrow 0$ appears as a necessary condition for the existence of asymptotic approximations valid for $\epsilon \downarrow 0$. Finally, uniform boundness on the whole time-axis $0 \leqslant t < \infty$ has been imposed to assure existence of solutions on a sufficiently large time interval. This becomes apparent from the following result.

LEMMA 1. *If* $\mathbf{F}$ *satisfies conditions* (i) *and* (ii) *and* $\mathbf{Y}_0 \in D$, *then there exists a unique solution* $\mathbf{Y}(t, \epsilon)$ *of*

$$d\mathbf{Y}/dt = \epsilon\mathbf{F}(\mathbf{Y}, t, \epsilon); \qquad \mathbf{Y}(0, \epsilon) = \mathbf{Y}_0, \quad in \quad 0 \leqslant t \leqslant T$$

with $T = d/\epsilon M$, *where* d *is the distance of* $\mathbf{Y}_0$ *to the boundary of* D, *and* M *is defined by,*

$$M = \operatorname*{Sup}_G | \mathbf{F} | .$$

The proof of Lemma 1 is obtained by an almost trivial modification of the classical existence and uniqueness theorem, as given for example in Roseau [10].

By virtue of Lemma 1 it is meaningful to study $\mathbf{Y}(t, \epsilon)$ on the *natural time scale* ϵ^{-1}. Introduce therefore: $\tau = \epsilon t$ writing $\mathbf{Y}^*(\tau, \epsilon) = \mathbf{Y}(\tau/\epsilon, \epsilon)$ we have the initial value problem

$$d\mathbf{Y}^*/d\tau = \mathbf{F}(\mathbf{Y}^*, \tau/\epsilon, \epsilon); \qquad \mathbf{Y}^*(0, \epsilon) = \mathbf{Y}_0 .$$

Unique solution exists in some interval $0 \leqslant \tau \leqslant T^*$ where $T^* > 0$ is a number independent of ϵ.

Continuation of the solution can be obtained by the following classical corollary of the existence and uniqueness theorem:

LEMMA 2. *Let* I *be a closed interval such that for* $\tau \in I$ *unique solution* $\mathbf{Y}^*(\tau, \epsilon)$ *exists and* $\mathbf{Y}^*(\tau, \epsilon) \in K$ *where* K *is a compact subset of* D. *Then a unique continuation of* $\mathbf{Y}^*(\tau, \epsilon)$ *exists in some open interval containing* I. *Furthermore, the solution* $\mathbf{Y}^*(\tau, \epsilon)$ *may be continued to all values of* τ, *for which the continuation remains in a compact subset of* D.

In our analysis the continuation of the solution $\mathbf{Y}^*(\tau, \epsilon)$ will be obtained from consideration of approximate solutions. For the study of approximate solutions, valid as $\epsilon \downarrow 0$, a general approximation theorem can be formulated. As a preliminary we remind the reader that the initial value problem is equivalent with the integral equation

$$\mathbf{Y}(\tau, \epsilon) = \mathbf{Y}_0 + \int_0^\tau \mathbf{F}\left[\mathbf{Y}(\tau', \epsilon), \frac{\tau'}{\epsilon}, \epsilon\right] d\tau'.$$

As a second preliminary the definition of certain special subsets of D will be given, because such subset will frequently appear in the subsequent analysis.

DEFINITION 6. $D_0 \subset D$ is an interior subset if the distance between the boundary of D_0 and the boundary of D is bounded from below by a positive constant, independent of ϵ, for all $0 \leqslant \epsilon \leqslant \epsilon_0$.

We now have:

THEOREM I. *Consider two functions* $\mathbf{Y}^{(1)}$ *and* $\mathbf{Y}^{(2)}$,

$$\mathbf{Y}^{(1)}(\tau, \epsilon) = \mathbf{Y}_0^{(1)} + \int_0^\tau \mathbf{F}_1\left[\mathbf{Y}^{(1)}(\tau', \epsilon), \frac{\tau'}{\epsilon}, \epsilon\right] d\tau',$$

$$\mathbf{Y}^{(2)}(\tau, \epsilon) = \mathbf{Y}_0^{(2)} + \int_0^\tau \mathbf{F}_2\left[\mathbf{Y}^{(2)}(\tau', \epsilon), \frac{\tau'}{\epsilon}, \epsilon\right] d\tau'.$$

Suppose

(i) $\mathbf{Y}_0^{(1)} \in D_0$, $\mathbf{Y}_0^{(1)} \in D_0$, $|\mathbf{Y}_0^{(1)} - \mathbf{Y}_0^{(2)}| \leqslant \delta_0(\epsilon)$, $\delta_0(\epsilon) = o(1)$.

(ii) *For all* $\mathbf{Y} \in \bar{D}$ *and* $0 \leqslant \tau \leqslant A$,

$$\left| \mathbf{F}_1\left(\mathbf{Y}, \frac{\tau}{\epsilon}, \epsilon\right) - \mathbf{F}_2\left(\mathbf{Y}, \frac{\tau}{\epsilon}, \epsilon\right) \right| \leqslant \delta_f(\epsilon), \qquad \delta_f(\epsilon) = o(1).$$

(iii) *Solution* $\mathbf{Y}^{(2)}(\tau, \epsilon)$ *exists for* $0 \leqslant \tau \leqslant A$ *and* $\mathbf{Y}^{(2)} \in D_0$. *Then: Solution* $\mathbf{Y}_0^{(1)}(\tau, \epsilon)$ *exists for* $0 \leqslant \tau \leqslant A$ *and in that interval*

$$|\mathbf{Y}^{(1)}(\tau, \epsilon) - \mathbf{Y}^{(2)}(\tau, \epsilon)| \leqslant \delta_0(\epsilon)\, e^{\lambda\tau} + \delta_f(\epsilon) \frac{1}{\lambda} (e^{\lambda\tau} - 1).$$

For the proof of Theorem I we shall use Gronwall's lemma in a form given for example in Coddington and Levinson [4]:

LEMMA 3. *Let* $\lambda(\tau) \geqslant 0$ *be an integrable function, while* $u(\tau)$ *and* $\phi(\tau)$ *are absolutely continuous functions for* $\tau_0 \leqslant \tau \leqslant \tau_1$, *and let* $\phi'(\tau)$ *exist. If,*

$$u(\tau) \leqslant \phi(\tau) + \int_{\tau_0}^\tau \lambda(\tau')\, u(\tau')\, d\tau',$$

then

$$u(\tau) \leqslant \phi(\tau_0) \exp \int_{\tau_0}^{\tau} \lambda(\tau')\, d\tau' + \int_{\tau_0}^{\tau} \frac{d\phi(\tau'')}{d\tau''} \left[\exp \int_{\tau''}^{\tau} \lambda(\tau')\, d\tau'\right] d\tau''.$$

Proof of Theorem I.

$$\begin{aligned} &| \mathbf{Y}^{(1)}(\tau, \epsilon) - \mathbf{Y}^{(2)}(\tau, \epsilon)| \\ &\leqslant \delta_0(\epsilon) + \int_0^{\tau} \left| \mathbf{F}_1 \left[\mathbf{Y}^{(1)}(\tau', \epsilon), \frac{\tau'}{\epsilon}, \epsilon\right] - \mathbf{F}_2 \left[\mathbf{Y}^{(2)}(\tau', \epsilon), \frac{\tau'}{\epsilon}, \epsilon\right] \right| d\tau' \\ &\leqslant \delta_0(\epsilon) + \int_0^{\tau} \left| \mathbf{F}_1 \left[\mathbf{Y}^{(1)}(\tau', \epsilon), \frac{\tau'}{\epsilon}, \epsilon\right] - \mathbf{F}_1 \left[\mathbf{Y}^{(2)}(\tau', \epsilon), \frac{\tau'}{\epsilon}, \epsilon\right] \right| d\tau' \\ &\quad + \int_0^{\tau} \left| \mathbf{F}_1 \left[\mathbf{Y}^{(2)}(\tau', \epsilon), \frac{\tau'}{\epsilon}, \epsilon\right] - \mathbf{F}_2 \left[\mathbf{Y}^{(2)}(\tau', \epsilon), \frac{\tau'}{\epsilon}, \epsilon\right] \right| d\tau'. \end{aligned}$$

Since $\mathbf{Y}^{(1)}(\tau, \epsilon)$ is a continuous function, $\mathbf{Y}^{(1)}$ will certainly remain in D_0 for some $0 \leqslant \tau \leqslant \tau_1$. We can therefore use Lipschitz-continuity, and furthermore property (ii). We obtain in $0 \leqslant \tau \leqslant \tau_1$

$$\begin{aligned} &| \mathbf{Y}^{(1)}(\tau, \epsilon) - \mathbf{Y}^{(2)}(\tau, \epsilon)| \\ &\quad \leqslant \delta_0(\epsilon) + \delta_f(\epsilon)\, \tau + \lambda \int_0^{\tau} | \mathbf{Y}^{(1)}(\tau', \epsilon) - \mathbf{Y}^{(2)}(\tau', \epsilon)|\, d\tau'. \end{aligned}$$

Using Gronwall's lemma we find in $0 \leqslant \tau \leqslant \tau_1$

$$| \mathbf{Y}^{(1)}(\tau, \epsilon) - \mathbf{Y}^{(2)}(\tau, \epsilon)| \leqslant \delta_0(\epsilon)\, e^{\lambda\tau} + \delta_f(\epsilon) \frac{1}{\lambda} (e^{\lambda\tau} - 1).$$

From Lemma 2 we obtain a continuation of the solution $\mathbf{Y}^{(1)}(\tau, \epsilon)$, which remains in D_0 for sufficiently small ϵ. For every continuation the above estimate of $| \mathbf{Y}^{(1)} - \mathbf{Y}^{(2)} |$ remains valid, as long as $\mathbf{Y}^{(2)}$ remains in D_0. Hence the continuation and the estimate are valid in $0 \leqslant \tau \leqslant A$.

4. Local Average Values

When studying the initial value problem on the natural time scale

$$d\mathbf{Y}^*/d\tau = \mathbf{F}(\mathbf{Y}^*, \tau/\epsilon, \epsilon); \qquad \mathbf{Y}^*(0, \epsilon) = \mathbf{Y}_0,$$

one is often confronted with the case in which $\lim_{\epsilon\to 0} \mathbf{F}(\mathbf{Y}^*, \tau/\epsilon, \epsilon)$ does not exist. Such is for example the case when $\mathbf{F}(\mathbf{Y}, t, \epsilon)$ is a periodic function of t, with period T independent of ϵ. Nevertheless, as is well-known from the

Krilov–Bogolioubov theory, an asymptotic approximation $\eta(\tau)$ of $\mathbf{Y}^*(\tau, \epsilon)$ may exist. In our analysis the essential tool for the study of such problems is the concept of local average values that will be introduced now.

DEFINITION 7. Consider a function $(t, \epsilon) \to \mathbf{\Phi}(t, \epsilon)$ and a transformation $\tau = \delta_s(\epsilon)t$, $\mathbf{\Phi}(\tau/\delta_s, \epsilon) = \mathbf{\Phi}^*(\tau, \epsilon)$. A local average value $\bar{\mathbf{\Phi}}(\tau, \epsilon)$ of $\mathbf{\Phi}(t, \epsilon)$ on the time scale δ_s^{-1} is given by

$$\bar{\mathbf{\Phi}}(\tau, \epsilon) = \frac{1}{\delta(\epsilon)} \int_0^{\delta(\epsilon)} \mathbf{\Phi}^*(\tau + \tau', \epsilon)\, d\tau'$$

where δ is some order function with $\delta(\epsilon) = o(1)$.

Remarks. In the definition above the function $\mathbf{\Phi}(t, \epsilon)$ is in fact averaged (in the usual sense of the word) over a "small" distance on the δ_s^{-1} time scale. The "smallness" of the distance over which the averaging is performed is in asymptotic sense, and is measured by the order function $\delta(\epsilon)$. It is obvious that a "small" distance on the δ_s^{-1} time scale may be a "large" distance in the original t time variable. The average $\bar{\mathbf{\Phi}}(\tau, \epsilon)$ depends on the choice of $\delta(\epsilon)$, which leaves us with a degree of liberty to be exploited later on in the analysis. Naturally, $\bar{\mathbf{\Phi}}(\tau, \epsilon)$ also depends on the time scale δ_s^{-1}, on which $\mathbf{\Phi}(t, \epsilon)$ is being investigated. The asymetry in the definition of $\bar{\mathbf{\Phi}}(\tau, \epsilon)$ ("forward" integration, $\tau' \geqslant \tau$) is chosen, because otherwise $\bar{\mathbf{\Phi}}(0, \epsilon)$ could not be defined. Finally we remark that for the purpose of calculation it is often advantageous to introduce an obvious change of the integration variable $\tau' = \delta\bar{\tau}$, which yields

$$\bar{\mathbf{\Phi}}(\tau, \epsilon) = \int_0^1 \mathbf{\Phi}^*(\tau + \delta\bar{\tau}, \epsilon)\, d\bar{\tau}.$$

The usefulness of the local average values immediately appears from the following fundamental result on the natural time scale $\tau = \epsilon t$:

LEMMA 4. *Let* $\mathbf{Y}^*(\tau, \epsilon)$ *be the solution in* $0 \leqslant \tau \leqslant A$ *of*

$$d\mathbf{Y}^*/d\tau = \mathbf{F}(\mathbf{Y}^*, \tau/\epsilon, \epsilon); \qquad \mathbf{Y}^*(0, \epsilon) = \mathbf{Y}_0,$$

then

$$\mathbf{Y}^*(\tau, \epsilon) = \bar{\mathbf{Y}}(\tau, \epsilon) + O(\delta).$$

For the proof of Lemma 4 we need, as a preliminary, a result which although elementary will be stated in a separate Lemma because it will frequently be used in the sequal.

LEMMA 5. *Let* $\mathbf{Y}^*(\tau, \epsilon)$ *be defined as in Lemma* 4. *Then*

$$| \mathbf{Y}^*(\tau + \delta\bar{\tau}, \epsilon) - \mathbf{Y}^*(\tau, \epsilon)| \leqslant M\, \delta\bar{\tau},$$

where $M = \operatorname{Sup}_G | \mathbf{F} |$.

Proof of Lemma 5. From the differential equation it follows that

$$| \mathbf{Y}^*(\tau + \delta\bar{\tau}, \epsilon) - \mathbf{Y}^*(\tau, \epsilon)| \leqslant \int_\tau^{\tau+\delta\bar{\tau}} \left| \mathbf{F}\left(\mathbf{Y}^*(\tau', \epsilon), \frac{\tau'}{\epsilon}, \epsilon\right) \right| d\tau' \leqslant M\delta\bar{\tau}.$$

Proof of Lemma 4. From the definition of the average values we have

$$\bar{\mathbf{Y}}(\tau, \epsilon) = \int_0^1 \mathbf{Y}^*(\tau - \delta\bar{\tau}, \epsilon)\, d\bar{\tau} = \mathbf{Y}^*(\tau, \epsilon) + \int_0^1 \{\mathbf{Y}^*(\tau + \delta\bar{\tau}, \epsilon) - \mathbf{Y}^*(\tau, \epsilon)\}\, d\bar{\tau}$$

Using now Lemma 5 we obtain

$$| \bar{\mathbf{Y}}(\tau, \epsilon) - \mathbf{Y}^*(\tau, \epsilon)| \leqslant \tfrac{1}{2}M\delta,$$

which proofs Lemma 4.

Finally we investigate the relation between the local averages in the sense of Definition 7, and the classical averages of Krilov and Bogolioubov defined by:

DEFINITION 8. $\mathbf{F}(\mathbf{Y}, t, \epsilon)$ is a K.-B. function if

$$\mathbf{F}_0(\mathbf{Y}, \epsilon) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{F}(\mathbf{Y}, t, \epsilon)\, dt$$

exists.

Now, in the sense of Definition 7, a local average value of $\mathbf{F}(\mathbf{Y}^*, \tau/\delta_s, \epsilon)$ is given by

$$\bar{\mathbf{F}}\left(\mathbf{Y}^*, \frac{\tau}{\delta_s}, \epsilon\right) = \frac{1}{\delta(\epsilon)} \int_0^{\delta(\epsilon)} \mathbf{F}\left(\mathbf{Y}^*, \frac{\tau}{\delta_s} + \frac{\tau'}{\delta_s}, \epsilon\right) d\tau'.$$

We shall now prove the following correspondence:

LEMMA 6. *If* $\mathbf{F}$ *is a* K.-B. *function then there exists an order function* $\delta_1(\epsilon) = o(1)$ *such that for any choice of* $\delta(\epsilon)$ *in the averaging process, satisfying*

$$\delta_1/\delta = o(1), \qquad \delta_s/\delta = o(1),$$

we have, uniformly on any time scale δ_s^{-1}

$$\bar{\mathbf{F}}(\mathbf{Y}^*, \tau/\epsilon, \epsilon) = \mathbf{F}_0(\mathbf{Y}^*, \epsilon) + o(1).$$

Proof. In Definition 8 and in Lemma 6 the variable $\mathbf{Y}^*$ only appears as a parameter. We therefore write, to simplify the notation

$$\mathbf{F}(\mathbf{Y}, t, \epsilon) = \mathbf{\Phi}(t, \epsilon); \qquad \mathbf{F}_0(\mathbf{Y}, \epsilon) = \mathbf{\Phi}_0.$$

If $\mathbf{\Phi}(t, \epsilon)$ is a finite sum of functions periodic with respect to t (and this is the case which most often occurs in applications) then the proof of Lemma 6 can be obtained from straightforward computation of the average $\bar{\mathbf{\Phi}}(\tau, \epsilon)$, as given in Sections 6 and 8. One then finds

$$\bar{\mathbf{\Phi}}(\tau, \epsilon) - \mathbf{\Phi}_0 = O(\delta_s/\delta).$$

In the most general case, that is without supposing periodicity of $\mathbf{\Phi}(t, \epsilon)$, the proof is somewhat more involved. We are given that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \{\mathbf{\Phi}(t, \epsilon) - \mathbf{\Phi}_0\}\, dt = 0.$$

Hence there exists a positive, continuous, monotonic function $\phi(z)$ with the property

$$\lim_{z\to 0} \phi(z) = 0,$$

such that, for say $T > T_0$:

$$\left| \frac{1}{T} \int_0^T \{\mathbf{\Phi}(t, \epsilon) - \mathbf{\Phi}_0\}\, dt \right| \leqslant \phi\left(\frac{1}{T}\right).$$

On the other hand, by Definition 7,

$$\bar{\mathbf{\Phi}}(\tau, \epsilon) = \int_0^1 \mathbf{\Phi}\left(\frac{\tau}{\delta_s} + \frac{\delta}{\delta_s}\tau', \epsilon\right) d\tau' = \frac{\delta_s}{\delta} \int_{\tau/\delta_s}^{(\tau+\delta)/\delta_s} \mathbf{\Phi}(t, \epsilon)\, dt.$$

We now write,

$$\bar{\mathbf{\Phi}}(\tau, \epsilon) - \mathbf{\Phi}_0 = \frac{\delta_s}{\delta} \int_0^{(\tau+\delta)/\delta_s} \{\mathbf{\Phi}(t, \epsilon) - \mathbf{\Phi}_0\}\, dt - \frac{\delta_s}{\delta} \int_0^{\tau/\delta_s} \{\mathbf{\Phi}(\tau, \epsilon) - \mathbf{\Phi}_0\}\, dt.$$

Consequently,

$$| \bar{\mathbf{\Phi}}(\tau, \epsilon) - \mathbf{\Phi}_0 | \leqslant J_1 + J_2,$$

$$J_1 = \left| \frac{\delta_s}{\delta} \int_0^{(\tau+\delta)/\delta_s} \{\mathbf{\Phi}(t, \epsilon) - \mathbf{\Phi}_0\}\, dt \right|,$$

$$J_2 = \left| \frac{\delta_s}{\delta} \int_0^{\tau/\delta_s} \{\mathbf{\Phi}(t, \epsilon) - \mathbf{\Phi}_0\}\, dt \right|.$$

We first investigate J_1. Using the fundamental property of the K.-B. functions we find

$$| J_1 | \leqslant \frac{\tau + \delta}{\delta} \phi\left(\frac{\delta_s}{\tau + \delta}\right).$$

Let I^* be any bounded, closed, ϵ independent interval

$$I^* = \{\tau \mid 0 \leqslant \tau \leqslant A^*\},$$

$$\| J_1 \|_{I^*} \leqslant \operatorname*{Max}_{\tau \in I^*} \left\{ \frac{\tau + \delta}{\delta} \phi \left(\frac{\delta_s}{\tau + \delta} \right) \right\} \leqslant \frac{A^* + \delta}{\delta} \phi \left(\frac{\delta_s}{\delta} \right).$$

If now,

$$\delta_s / \delta = o(1),$$

then

$$\delta_1(\epsilon) = \phi(\delta_s(\epsilon)/\delta(\epsilon)) = o(1).$$

Hence

$$J_1 = O(\delta_1/\delta) + O(\delta_1).$$

We next investigate J_2. The analysis is somewhat more delicate then in the case of J_1, because the fundamental estimate of K.-B. functions cannot be applied for all $\tau \in I^*$. We therefore subdivide I^* as follows:

(1) $0 \leqslant \tau \leqslant \delta_s$: Elementary estimation shows

$$J_2 = O(\delta_s/\delta).$$

(2) $\delta_s \leqslant \tau \leqslant \delta'$ where δ' is an order function such that $\delta_s/\delta' = o(1)$ and $\delta'/\delta = o(1)$. Again using elementary estimation we have

$$| J_2 | \leqslant \frac{\delta_s}{\delta} \int_0^{\tau/\delta_s} | \mathbf{\Phi}(t, \epsilon) - \mathbf{\Phi}_0 | \, dt \Rightarrow J_2 = O\left(\frac{\delta'}{\delta} \right).$$

(3) $\delta' \leqslant \tau \leqslant \delta$, with δ' defined as before. We now can use the fundamental property of K.-B. functions and obtain

$$| J_2 | \leqslant (\tau/\delta)\, \phi(\delta_s/\tau) \leqslant \phi(\delta_s/\delta').$$

This again implies $J_2 = o(1)$.

(4) $\delta \leqslant \tau \leqslant A^*$. Here the estimation proceeds as in the case of J_1.

$$| J_2 | \leqslant \frac{\tau}{\delta} \phi \left(\frac{\delta_s}{\tau} \right) \leqslant \frac{A^*}{\delta} \phi \left(\frac{\delta_s}{\delta} \right) \Rightarrow J_2 = O\left(\frac{\delta_1}{\delta} \right).$$

Hence, under the conditions specified in Lemma 6 we have

$$J_1 = o(1) \qquad \text{and} \qquad J_2 = o(1),$$

which proves the Lemma.

5. The Fundamental Theorem

We now investigate local averages of the function $\mathbf{Y}^*(\tau, \epsilon)$ defined as solution of

$$d\mathbf{Y}^*/d\tau = \mathbf{F}(\mathbf{Y}^*, \tau/\epsilon, \epsilon); \qquad \mathbf{Y}^*(0, \epsilon) = \mathbf{Y}_0 .$$

We have

$$\mathbf{Y}^*(\tau, \epsilon) = \mathbf{Y}_0 + \int_0^\tau \mathbf{F}\left[\mathbf{Y}^*(\tau'', \epsilon), \frac{\tau''}{\epsilon}, \epsilon\right] d\tau''.$$

Using Definition 7 we find for the local average:

$$\overline{\mathbf{Y}}(\tau, \epsilon) = \mathbf{Y}_0 + \int_0^1 \left\{\int_0^{\tau+\delta\bar{\tau}} \mathbf{F}\left[\mathbf{Y}^*(\tau'', \epsilon), \frac{\tau''}{\epsilon}, \epsilon\right] d\tau''\right\} d\bar{\tau}.$$

We shall deduce from this expression a relation for $\overline{\mathbf{Y}}(\tau, \epsilon)$, not containing $\mathbf{Y}^*(\tau, \epsilon)$.

We rewrite the right-hand side as follows

$$\overline{\mathbf{Y}}(\tau, \epsilon) = \mathbf{Y}_0 + \int_0^1 \left\{\int_{\delta\bar{\tau}}^{\tau+\delta\bar{\tau}} \mathbf{F}\left[\mathbf{Y}^*(\tau'', \epsilon), \frac{\tau''}{\epsilon}, \epsilon\right] d\tau''\right\} d\bar{\tau} + I_1 ,$$

where

$$I_1 = \int_0^1 \int_0^{\delta\bar{\tau}} \mathbf{F}\left[\mathbf{Y}^*(\tau'', \epsilon), \frac{\tau''}{\epsilon}, \epsilon\right] d\tau''\, d\bar{\tau}.$$

It is immediately obvious that

$$| I_1 | \leqslant \tfrac{1}{2} M\delta.$$

In the reminding integral on the right-hand side of the expression for $\overline{\mathbf{Y}}(\tau, \epsilon)$ we introduce a change of the integration variables

$$\tau'' = \tau' + \delta\bar{\tau},$$

and we subsequently interchange the order of integration. It follows that

$$\overline{\mathbf{Y}}(\tau, \epsilon) = \mathbf{Y}_0 + \int_0^\tau \left\{\int_0^1 \mathbf{F}\left[\mathbf{Y}^*(\tau' + \delta\bar{\tau}, \epsilon), \frac{\tau'}{\epsilon} + \frac{\delta}{\epsilon}\bar{\tau}, \epsilon\right] d\bar{\tau}\right\} d\tau' + I_1 .$$

Finally, we write

$$\overline{\mathbf{Y}}(\tau, \epsilon) = \mathbf{Y}_0 + \int_0^\tau \left\{\int_0^1 \mathbf{F}\left[\overline{\mathbf{Y}}(\tau', \epsilon), \frac{\tau'}{\epsilon} + \frac{\delta}{\epsilon}\bar{\tau}, \epsilon\right] d\bar{\tau}\right\} d\tau' + I_1 + I_2 ,$$

where

$$I_2 = \int_0^\tau \int_0^1 \left\{ \mathbf{F}\left[\mathbf{Y}^*(\tau' + \delta\bar{\tau}, \epsilon), \frac{\tau'}{\epsilon} + \frac{\delta}{\epsilon}\bar{\tau}, \epsilon\right] - \mathbf{F}\left[\bar{\mathbf{Y}}(\tau', \epsilon), \frac{\tau'}{\epsilon} + \frac{\delta}{\epsilon}\bar{\tau}, \epsilon\right] \right\} d\bar{\tau}\, d\tau'.$$

In order to estimate I_2 we use the Lipschitz-continuity of $\mathbf{F}$ and obtain

$$| I_2 | \leqslant \lambda \int_0^\tau \int_0^1 | \mathbf{Y}^*(\tau' + \delta\bar{\tau}, \epsilon) - \bar{\mathbf{Y}}(\tau', \epsilon) |\, d\bar{\tau}\, d\tau'.$$

Using now Lemmas 4 and 5, we have

$$| I_2 | \leqslant \lambda M \,\delta\tau.$$

The above results are summarized in:

Lemma 7. *If,*

$$\mathbf{Y}^*(\tau, \epsilon) = \mathbf{Y}_0 + \int_0^\tau \mathbf{F}\left[\mathbf{Y}^*(\tau', \epsilon), \frac{\tau'}{\epsilon}, \epsilon\right] d\tau',$$

then

$$\bar{\mathbf{Y}}(\tau, \epsilon) = \mathbf{Y}_0 + \int_0^\tau \int_0^1 \mathbf{F}\left[\bar{\mathbf{Y}}(\tau', \epsilon), \frac{\tau'}{\epsilon} + \frac{\delta}{\epsilon}\bar{\tau}, \epsilon\right] d\bar{\tau}\, d\tau' + I_1 + I_2,$$

where $| I_1 | \leqslant \frac{1}{2} M\delta$, $| I_2 | \leqslant \lambda M \,\delta\tau$.

Using now Theorem I and Lemma 4, we obtain

Lemma 8. *If*

$$\mathbf{Y}^*(\tau, \epsilon) = \mathbf{Y}_0 + \int_0^\tau \mathbf{F}\left[\mathbf{Y}^*(\tau, \epsilon), \frac{\tau}{\epsilon}, \epsilon\right] d\tau',$$

then the local average $\bar{\mathbf{Y}}(\tau, \epsilon)$ *of* $\mathbf{Y}^*(\tau, \epsilon)$ *can be approximated by the functio* $\tilde{\mathbf{Y}}(\tau, \epsilon)$, *satisfying*

$$\tilde{\mathbf{Y}}(\tau, \epsilon) = \mathbf{Y}_0 + \int_0^\tau \int_0^1 \mathbf{F}\left[\tilde{\mathbf{Y}}(\tau', \epsilon), \frac{\tau'}{\epsilon} + \frac{\delta}{\epsilon}\bar{\tau}, \epsilon\right] d\bar{\tau}\, d\tau'.$$

We have

$$\bar{\mathbf{Y}}(\tau, \epsilon) = \tilde{\mathbf{Y}}(\tau, \epsilon) + O(\delta)$$

and

$$\mathbf{Y}^*(\tau, \epsilon) = \bar{\mathbf{Y}}(\tau, \epsilon) + O(\delta),$$

the estimates being valid on any closed interval on which $\tilde{\mathbf{Y}}(\tau, \epsilon)$ *exists an* $\tilde{\mathbf{Y}} \in D_0$.

Lemma 8 is the general and fundamental result, permitting to approximate $\mathbf{Y}^*(\tau, \epsilon)$ by a function which is an approximation of the local average of $\mathbf{Y}^*(\tau, \epsilon)$. In the case in which $\mathbf{F}(\mathbf{Y}, t, \epsilon)$ is a K.-B. function Lemma 8 reduces to the well-known fundamental theorem of the asymptotic theory of standard systems. We then have:

THEOREM II. *Let* $\mathbf{Y}^*(\tau, \epsilon)$ *be the solution of* $d\mathbf{Y}^*/d\tau = \mathbf{F}(\mathbf{Y}^*, \tau/\epsilon, \epsilon)$; $\mathbf{Y}^*(0, \epsilon) = \mathbf{Y}_0 \in D_0$ *and let* $\boldsymbol{\eta}(\tau)$ *be the solution of*

$$d\boldsymbol{\eta}/d\tau = \mathbf{F}_0(\boldsymbol{\eta}); \qquad \boldsymbol{\eta}(0) = \mathbf{Y}_0,$$

where

$$\mathbf{F}_0(\boldsymbol{\eta}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{F}(\boldsymbol{\eta}, t, 0)\, dt.$$

Suppose $\boldsymbol{\eta}(\tau)$ *exists for* $0 \leqslant \tau \leqslant A$, *and* $\boldsymbol{\eta} \in D_0$, *then* $\mathbf{Y}^*(\tau, \epsilon)$ *exists in the same interval and*

$$\mathbf{Y}^*(\tau, \epsilon) = \boldsymbol{\eta}(\tau) + o(1).$$

Proof. By condition (i) of Section 3 there exists an order function $\delta_2(\epsilon) = o(1)$ such that for all $\mathbf{Y} \in D$ and $0 \leqslant t < \infty$:

$$|\mathbf{F}(\mathbf{Y}, t, \epsilon) - \mathbf{F}(\mathbf{Y}, t, 0)| \leqslant \delta_2(\epsilon); \qquad \delta_2(\epsilon) = o(1).$$

Because $\mathbf{F}(\mathbf{Y}, t, \epsilon)$ is a K.-B. function, we may use the result of Lemma 6. It follows that

$$\tilde{\mathbf{Y}}(\tau, \epsilon) = \mathbf{Y}_0 + \int_0^\tau F_0(\tilde{\mathbf{Y}}(\tau'))\, d\tau' + I_3,$$

where

$$|I_3| \leqslant M'(\delta_1/\delta + \delta_2)\tau$$

and M' is a constant independent of ϵ. Using now Theorem I we find

$$|\tilde{\mathbf{Y}}(\tau, \epsilon) - \boldsymbol{\eta}(\tau)| \leqslant (1/\lambda)\, M'(\delta_1/\delta + \delta_2)(e^{\lambda\tau} - 1).$$

Hence, for any interval $0 \leqslant \tau \leqslant A$, for which $\eta(\tau)$ exists, $\tilde{\mathbf{Y}}(\tau, \epsilon)$ exists and

$$\tilde{\mathbf{Y}}(\tau, \epsilon) = \boldsymbol{\eta}(\tau) + O(\delta_1/\delta) + O(\delta_2).$$

From Lemma 8 it follows now that

$$\mathbf{Y}^*(\tau, \epsilon) = \boldsymbol{\eta}(\tau) + O(\delta_1/\delta) + O(\delta) + O(\delta_2).$$

Since $\delta(\epsilon) = o(1)$ is an order function such that $\delta_1/\delta = o(1)$, $\boldsymbol{\eta}(\tau)$ indeed is an asymptotic approximation of $\mathbf{Y}^*(\tau, \epsilon)$, that is $\mathbf{Y}^*(\tau, \epsilon) = \boldsymbol{\eta}(\tau) + o(1)$.

Remarks. In applications **F** often is independent of ϵ. In that case the order function $\delta_2(\epsilon)$ does not intervene and the explicit estimate of the result given in Theorem II reads

$$\mathbf{Y}^*(\tau, \epsilon) = \eta(\tau) + O(\delta_1/\delta) + O(\delta).$$

We recall that $\delta_1(\epsilon) = o(1)$ is an order function determined by the structure of the function **F** (see proof of Lemma 6) while $\delta(\epsilon)$ may yet be chosen arbitrarily, provided that

$$\delta_1/\delta = o(1) \qquad \text{and} \qquad \epsilon/\delta = o(1).$$

If $\delta_1/\delta = O(1)$ then the best choice for δ is

$$\delta = O(\sqrt{\delta_1})$$

If $\epsilon/\delta_1 = O(1)$ then the best choice is

$$\delta = O(\sqrt{\epsilon}).$$

Thus, the best general result, for the case in which **F** is independent of ϵ, appears to be either

$$\mathbf{Y}^*(\tau, \epsilon) = \eta(\tau) + O(\sqrt{\delta_1}),$$

or

$$\mathbf{Y}^*(\tau, \epsilon) = \eta(\tau) + O(\sqrt{\epsilon}).$$

As we shall see in the next sections, for the particular case of functions $\mathbf{F}(\mathbf{Y}, t, \epsilon)$ periodic in t the above result can yet considerably be improved by a further refinement of the theory.

6. Improved Results for Periodic Systems

For simplicity of calculations we suppose now that **F** is a function $(\mathbf{Y}, t) \to \mathbf{F}(\mathbf{Y}, t)$, independent of ϵ. The results of this section are easily extended to the case $F(\mathbf{Y}, t, \epsilon) = F(\mathbf{Y}, t, 0) + O(\epsilon)$. A periodic system will be defined by:

Definition 9. $d\mathbf{Y}/dt = \epsilon\mathbf{F}(\mathbf{Y}, t)$ is a periodic system if,

$$\mathbf{F}(\mathbf{Y}, t) = \sum_{p=1}^{m} \mathbf{F}_p(\mathbf{Y}, t),$$

$$\mathbf{F}_p(\mathbf{Y}, t) = \mathbf{F}_p(\mathbf{Y}, t + T_p).$$

We compute now averages in the sense of Definition 6, that is:

$$\bar{\mathbf{F}}_p(\mathbf{Y}, \tau) = \int_0^1 \mathbf{F}_p\left(\mathbf{Y}, \frac{\tau}{\epsilon} + \frac{\delta}{\epsilon}\bar{\tau}\right) d\bar{\tau}.$$

Expanding each $\mathbf{F}_p$ in Fourier series

$$\mathbf{F}_p(\mathbf{Y}, t) = f_0^{(p)}(\mathbf{Y}) + \sum_{n+1}^{\infty} \left\{ \mathbf{f}_{n,p}(\mathbf{Y}) \sin \frac{nt}{T_p} + \tilde{\mathbf{f}}_{n,p}(\mathbf{Y}) \cos \frac{nt}{T_p} \right\}.$$

We obtain by a straight forward computation

$$\bar{\mathbf{F}}^{(p)}(\mathbf{Y}, \tau) = \mathbf{f}_0^{(p)}(Y) + \frac{\epsilon}{\delta}\, \mathbf{f}_1^{(p)}\left(\mathbf{Y}, \frac{\tau}{\epsilon}, \epsilon\right)$$

where

$$\mathbf{f}_1^{(p)}\left(\mathbf{Y}, \frac{\tau}{\epsilon}\, \epsilon\right) = \sum_{n=1}^{\infty} \left\{ \mathbf{f}_{n,p}^{(1)}(\mathbf{Y}, \epsilon) \sin \frac{n}{T_p}\frac{\tau}{\epsilon} + \tilde{\mathbf{f}}_{n,p}^{(1)}(\mathbf{Y}, \epsilon) \cos \frac{n}{T_p}\frac{\tau}{\epsilon} \right\},$$

$$\mathbf{f}_{n,p}^{(1)} = \frac{T_p}{n} \left\{ \mathbf{f}_{n,p}(\mathbf{Y}) \sin n \frac{\delta}{\epsilon T_p} + \tilde{\mathbf{f}}_{n,p}(\mathbf{Y}) \left[\cos n \frac{\delta}{\epsilon T_p} - 1\right] \right\},$$

$$\tilde{\mathbf{f}}_{n,p}^{(1)} = \frac{T_p}{n} \left\{ \tilde{\mathbf{f}}_{n,p}(\mathbf{Y}) \sin n \frac{\delta}{\epsilon T_p} - \mathbf{f}_{n,p}(\mathbf{Y}) \left[\cos n \frac{\delta}{\epsilon T_p} - 1\right] \right\}.$$

It is obvious that $f_1^{(p)}$ again is a uniformly convergent Fourier-series, and that

$$\mathbf{f}_1^{(p)} = O(1)$$

uniformly for $0 \leqslant t < \infty$, $\mathbf{Y} \in D$.

We next introduce the concept of *higher* averages, defined by the formula

$$\bar{\mathbf{F}}_p^{(n)}(\mathbf{Y}, \tau) = \int_0^1 \bar{\mathbf{F}}_p^{(n-1)}\left(\mathbf{Y}, \frac{\tau}{\epsilon} + \frac{\delta}{\epsilon}\bar{\tau}\right) d\bar{\tau}, \qquad n \geqslant 2,$$

$$\bar{\mathbf{F}}_p^{(1)}(\mathbf{Y}, \tau) = \bar{\mathbf{F}}_p(\mathbf{Y}, \tau).$$

Repeating the calculation of the averages we find

$$\bar{\mathbf{F}}^{(n)}(\mathbf{Y}, \tau) = \sum_{p=1}^{m} \bar{\mathbf{F}}_p^{(n)}(\mathbf{Y}, \tau) = \mathbf{F}_0(\mathbf{Y}) + \left(\frac{\epsilon}{\delta}\right)^n \mathbf{f}^{(n)}\left(Y, \frac{\tau}{\epsilon}, \epsilon\right),$$

where

$$\bar{\mathbf{F}}_0(\mathbf{Y}) = \sum_{p=1}^{m} \mathbf{f}_0^{(n)}(\mathbf{Y}) \qquad \text{and} \qquad \mathbf{f}^{(n)} = O(1).$$

In order to exploit these results we introduce

$$\bar{\mathbf{Y}}^{(n)}(\tau, \epsilon) = \int_0^1 \bar{\mathbf{Y}}^{(n-1)}(\tau + \delta\bar{\tau})\, d\bar{\tau},$$

$$\bar{\mathbf{Y}}^{(1)}(\tau, \epsilon) = \bar{\mathbf{Y}}(\tau, \epsilon).$$

Reconsidering now the proofs of Lemma 4 and Lemma 7 one easily deduces the following:

LEMMA 9. *If,*

$$\mathbf{Y}^*(\tau, \epsilon) = \mathbf{Y}_0 + \int_0^\tau \mathbf{F}\left(\mathbf{Y}^*(\tau', \epsilon), \frac{\tau'}{\epsilon}\right) d\tau'$$

then,

$$\mathbf{Y}^*(\tau, \epsilon) = \bar{\mathbf{Y}}^{(1)}(\tau, \epsilon) + O(\delta),$$

$$\bar{\mathbf{Y}}^{(n)}(\tau, \epsilon) = \bar{\mathbf{Y}}^{(n-1)}(\tau, \epsilon) + O(\delta), \qquad n \geqslant 2$$

and

$$\bar{\mathbf{Y}}^{(n)}(\tau, \epsilon) = \mathbf{Y}_0 + \int_0^\tau \bar{f}^{(n)}[\bar{\mathbf{Y}}^{(n)}(\tau', \epsilon), \tau']\, d\tau' + O(\delta).$$

We are now able to demonstrate Theorem III.

THEOREM III. *Let* $\mathbf{Y}^*(\tau, \epsilon)$ *be the solution of the periodic system*

$$d\mathbf{Y}^*/d\tau = \mathbf{F}(\mathbf{Y}^*, \tau/\epsilon); \qquad \mathbf{Y}^*(0, \epsilon) = \mathbf{Y}_0 \in D_0$$

and let $\eta(\tau)$ *be the solution of*

$$d\eta/d\tau = \mathbf{F}_0(\eta); \qquad \eta(0) = \mathbf{Y}_0,$$

where

$$\mathbf{F}_0(\eta) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{F}(\eta, t)\, dt.$$

Suppose $\eta(\tau)$ *exists for* $0 \leqslant \tau \leqslant A$, *and* $\eta \in D_0$, *then* $\mathbf{Y}^*(\tau, \epsilon)$ *exists for* $0 \leqslant \tau \leqslant A$ *and*

$$\mathbf{Y}^*(\tau, \epsilon) = \eta(\tau) + O(\epsilon^{1-\gamma}),$$

where γ *is an arbitrarily small positive number.*

Proof. According to the explicit computation of the average $\bar{\mathbf{F}}^{(n)}$ we may write (from Lemma 9)

$$\bar{\mathbf{Y}}^{(n)}(\tau, \epsilon) = \mathbf{Y}_0 + \int_0^\tau \mathbf{F}_0(\bar{\mathbf{Y}}^{(n)}(\tau', \epsilon))\, d\tau' + O\left(\left(\frac{\epsilon}{\delta}\right)^n\right) + O(\delta).$$

Using Theorem I, it follows that

$$\bar{\mathbf{Y}}^{(n)}(\tau, \epsilon) - \boldsymbol{\eta}(\tau) = O((\epsilon/\delta)^n) + O(\delta).$$

Using again Lemma 9 we have

$$\mathbf{Y}^*(\tau, \epsilon) = \boldsymbol{\eta}(\tau) + O((\epsilon/\delta)^n) + O(\delta).$$

Take now

$$\delta = \epsilon^{1-\gamma},$$

where γ is an arbitrarily positive number. For every γ there exists an integer n such that

$$\gamma n \geqslant 1 - \gamma,$$

which completes the proof of the theorem.

The preceeding results can easily be generalized to the so-called *slowly varying periodic systems* studied by Mitropolski [9]. For such systems one has

$$d\mathbf{Y}/dt = \epsilon\mathbf{F}(\mathbf{Y}, t, \tau); \qquad \tau = \epsilon t,$$

where $\mathbf{F}\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is a vector function which can be written as a finite sum of functions periodic in the t-variable. To obtain a generalisation of Theorem III only minor modifications of the preceding analysis are needed. The average of $\mathbf{F}$ in the sense of Definition 6 is

$$\bar{\mathbf{F}}(\mathbf{Y}, \tau) = \int_0^1 \mathbf{F}\left(\mathbf{Y}, \frac{\tau}{\epsilon} + \frac{\delta}{\epsilon}\bar{\tau}, \tau + \delta\bar{\tau}\right) d\bar{\tau}.$$

However, using continuity of $\mathbf{F}$ with respect to τ one gets

$$\bar{\mathbf{F}}(\mathbf{Y}, \tau) = \int_0^1 \mathbf{F}\left(\mathbf{Y}, \frac{\tau}{\epsilon} + \frac{\delta}{\epsilon}\bar{\tau}, \tau\right) d\bar{\tau} + O(\delta).$$

Furthermore, using Fourier series expansion in the t variable one obtains

$$\bar{\mathbf{F}}^{(n)}(\mathbf{Y}, \tau) = \mathbf{F}_0(\mathbf{Y}, \tau) + O((\epsilon/\delta)^n) + O(\delta),$$

where

$$\mathbf{F}_0(\mathbf{Y}, \tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{F}(\mathbf{Y}, t, \tau)\, dt.$$

With this result, and following the reasoning used in proof of Theorem III, one easily demonstrates:

THEOREM III bis. *Let* $\mathbf{Y}^*(\tau, \epsilon)$ *be the solution of the slowly varying periodic system*

$$d\mathbf{Y}^*/d\tau = \mathbf{F}(\mathbf{Y}^*, \tau/\epsilon, \tau); \qquad \mathbf{Y}^*(0, \epsilon) = \mathbf{Y}_0 \in D_0$$

and let $\eta(\tau)$ *be the solution of*

$$d\eta/d\tau = \mathbf{F}_0(\eta, \tau); \qquad \eta(0) = \mathbf{Y}_0,$$

where

$$\mathbf{F}_0(\eta, \tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{F}(\eta, t, \tau)\, dt.$$

Suppose $\eta(\tau)$ *exists for* $0 \leqslant \tau \leqslant A$, *and* $\eta \in D_0$. *Then* $\mathbf{Y}^*(\tau, \epsilon)$ *exists for* $0 \leqslant \tau \leqslant A$ *and*

$$\mathbf{Y}^*(\tau, \epsilon) = \eta(\tau) + O(\epsilon^{1-\gamma}),$$

where γ *is an arbitrarily small positive number.*

Remark. The results given in Theorem III and III bis are not yet the best possible results for periodic systems. The error in the asymptotic approximation is estimated to be $\epsilon^{1-\gamma}$, where γ is arbitrarily small, but nonzero. In the next section, we shall show, by computing higher approximations, that γ can in fact be taken equal zero. However, for the computation of higher approximations, some additional conditions on the function $\mathbf{F}$ must be imposed.

7. Higher Approximations for Periodic Systems

We shall show now that the method of analysis developed in the preceeding sections permits also to deduce higher approximations. We shall content ourselves with the next approximation, and not attempt to develop a systematic procedure for the construction of approximations of an arbitrarily high order. Such procedures has been given in the past, for example by Hoogstraten and Kaper [7]. However, the complexity of the results is such that the practical applicability diminishes rapidly as the number of terms in the expansion increases.

In order to construct higher approximations additional hypothesis on the structure of $\mathbf{F}(\mathbf{Y}, t)$ must be made. Usually one assumes that, in the vicinity of $\mathbf{Y} = \eta$, $\mathbf{F}$ can be expanded in convergent power series of $\mathbf{Y} - \eta$. In order to construct the next approximation only, it is sufficient to assume that

$$\mathbf{F}(\eta + \zeta, t) - \mathbf{F}(\eta, t) = A(t, \eta) \cdot \zeta + \mathbf{F}^*(\eta + \zeta, t),$$

where $\mathbf{F}^*$ is such that

$$\lim_{|\zeta|\to 0} \frac{|\mathbf{F}^*(\eta + \zeta, t)|}{|\zeta|} = 0,$$

uniformly for $0 \leqslant t < \infty$, $\eta \in D_0$.

In what follows it is assumed that $\mathbf{F}$ has the structure given above.

Taking now as source of inspiration the result of Theorem III we write

$$\mathbf{Y}(t, \epsilon) = \boldsymbol{\eta}(\tau) + \epsilon\mathbf{Z}(t, \epsilon); \qquad \tau = \epsilon t,$$

$\mathbf{Z}(t, \epsilon)$ must then be solution of

$$\frac{d\mathbf{Z}}{dt} = \mathbf{F}(\boldsymbol{\eta} + \epsilon\mathbf{Z}, t) - \mathbf{F}_0(\boldsymbol{\eta}); \qquad \mathbf{Z}(0, \epsilon) = 0.$$

In accordance with the structure of $\mathbf{F}$ we write

$$\frac{d\mathbf{Z}}{dt} = \epsilon A(t, \boldsymbol{\eta}) \cdot \mathbf{Z} + \{\mathbf{F}(\boldsymbol{\eta}, t) - \mathbf{F}_0(\boldsymbol{\eta})\} + \mathbf{F}^*(\boldsymbol{\eta} + \epsilon\mathbf{Z}, t).$$

From the property of $\mathbf{F}^*$ given above it follows that

$$\mathbf{F}^*(\boldsymbol{\eta} + \epsilon\mathbf{Z}, t) = o(\epsilon),$$

for all finite $\mathbf{Z}$ and uniformly in $0 \leqslant t \leqslant A/\epsilon$ (that is in the interval in which $\boldsymbol{\eta}(\tau)$ exists and $\boldsymbol{\eta} \in D_0$). The equation for $\mathbf{Z}$ does not yet have the appearance of a standard system for slowly modulated processes, because on the right-hand side

$$\mathbf{F}(\boldsymbol{\eta}, t) - \mathbf{F}_0(\boldsymbol{\eta}) = O(1).$$

We therefore introduce

$$\mathbf{Z} = \mathbf{u} + \boldsymbol{\phi},$$

$$\mathbf{u}(t, \epsilon) = \int_0^t \{\mathbf{F}(\boldsymbol{\eta}(\epsilon t'), t') - \mathbf{F}_0(\boldsymbol{\eta}(\epsilon t'))\}\, dt'$$

and obtain

$$\frac{d\boldsymbol{\phi}}{dt} = \epsilon \left\{ A(t, \boldsymbol{\eta}) \cdot \boldsymbol{\phi} + A(t, \boldsymbol{\eta}) \cdot \mathbf{u} + \frac{1}{\epsilon}\mathbf{F}(\boldsymbol{\eta} + \epsilon\mathbf{u} + \epsilon\boldsymbol{\phi}, t) \right\},$$

with the initial condition $\boldsymbol{\phi}(0, \epsilon) = 0$.

Obviously $\mathbf{u}(t, \epsilon)$ is a function that can be computed explicitly. For the periodic systems under consideration $\mathbf{u}(t, \epsilon)$ is bounded as $\epsilon \downarrow 0$ for all t in the interval $0 \leqslant t \leqslant A/\epsilon$ (that is in the interval in which $\boldsymbol{\eta}(\tau)$ exists). In fact, from the Fourier series expansion one easily finds

$$\mathbf{u}(t, \epsilon) = \sum_{p=1}^{m} \sum_{n=1}^{\infty} \frac{T_n}{n} \left\{ \mathbf{f}_{n,p}(\boldsymbol{\eta}(\tau)) \left(1 - \cos\frac{nt}{T_p}\right) + \mathbf{f}_{n,p}(\boldsymbol{\eta}(\tau)) \sin\frac{nt}{T_p} \right\} + O(\epsilon).$$

Hence we may study $\boldsymbol{\phi}$ on the natural time scale given by the transformation $\tau = \epsilon t$:

$$\frac{d\boldsymbol{\phi}^*}{d\tau} = A\left(\frac{\tau}{\epsilon}, \boldsymbol{\eta}\right) \cdot \boldsymbol{\phi}^* + A\left(\frac{\tau}{\epsilon}, \boldsymbol{\eta}\right) \cdot \mathbf{u}\left(\frac{\tau}{\epsilon}, \epsilon\right) + \frac{1}{\epsilon}\mathbf{F}^*\left(\boldsymbol{\eta} + \epsilon\mathbf{u} + \epsilon\boldsymbol{\phi}^*, \frac{\tau}{\epsilon}\right);$$

$$\boldsymbol{\phi}^*(0, \epsilon) = 0,$$

where

$$\frac{1}{\epsilon}\mathbf{F}^* = o(1) \qquad \text{for } 0 \leqslant \tau \leqslant A, \qquad \boldsymbol{\phi}^* \text{ finite.}$$

In accordance with Theorem I we can approximate $\boldsymbol{\phi}^*$ as follows:

$$\boldsymbol{\phi}^* = \boldsymbol{\phi}_0 + o(1).$$

Where $\boldsymbol{\phi}_0(\tau, \epsilon)$ is the solution of

$$\frac{d\boldsymbol{\phi}_0}{d\tau} = A\left(\frac{\tau}{\epsilon}, \boldsymbol{\eta}\right) \cdot \boldsymbol{\phi}_0 + A\left(\frac{\tau}{\epsilon}, \boldsymbol{\eta}\right) \cdot \mathbf{u}\left(\frac{\tau}{\epsilon}, \epsilon\right); \qquad \boldsymbol{\phi}_0(0, \epsilon) = 0.$$

Thus the problem is reduced to a *linear* nonhomogeneous system. Further simplification is achieved by observing that the equation for $\boldsymbol{\phi}_0$ represents a slowly varying periodic system. This follows from the fact that $A(t, \boldsymbol{\eta})$ is periodic in the t-variable, and $\boldsymbol{\eta} = \boldsymbol{\eta}(\tau)$, while the function $\mathbf{u}(t, \epsilon)$ (see its Fourier series expansion) can be written as follows:

$$\mathbf{u}(t, \epsilon) = \mathbf{u}^*(t, \tau),$$

where $\mathbf{u}^*(t, \tau)$ is periodic in the t-variable. Hence Theorem III bis applies and we finally have

$$\boldsymbol{\phi}^* = \boldsymbol{\phi}_0{}^* + o(1),$$

$$d\boldsymbol{\phi}_0{}^*/d\tau = A_0 \cdot \boldsymbol{\phi}_0{}^* + \mathbf{g}(\tau); \qquad \boldsymbol{\phi}_0{}^*(0, \epsilon) = 0.$$

Here A_0 is a matrix obtained by

$$A_0 = \lim_{T\to\infty} \frac{1}{T} \int_0^T A(t, \boldsymbol{\eta})\, dt,$$

furthermore

$$\mathbf{g}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T A(t, \boldsymbol{\eta}) \cdot \mathbf{u}(t, \tau)\, dt.$$

Summarising our results, we have demonstrated that for $0 \leqslant \tau \leqslant A$, that is in the interval in which $\eta(\tau)$ exists, we have

$$\mathbf{Y}^*(\tau, \epsilon) = \eta(\tau) + \epsilon\{\mathbf{u}(t, \tau) + \boldsymbol{\phi}_0^*(\tau) + o(1)\}.$$

This result implicitly shows that in Theorem III we can take $\gamma = 0$.

8. Average Values on Arbitrary Time-Scales

All the preceding results are restricted in validity to the natural time-scale ϵ^{-1}, that is to time-intervals $0 \leqslant t \leqslant A/\epsilon$, where A is some constant. In the next section we shall show that for certain classes of problems the interval of validity can be extended to $0 \leqslant t < \infty$. As a preliminary we study in this section the formal properties of average values on arbitrary time-scales. The formal aspect of the analysis of this section consists of the assumption that $\mathbf{Y}(t, \epsilon)$ as the solution of

$$d\mathbf{Y}/dt = \epsilon\mathbf{F}(\mathbf{Y}, t, \epsilon); \qquad \mathbf{Y}(0, \epsilon) = \mathbf{Y}_0,$$

exists in $0 \leqslant t \leqslant A^*/\delta_s(\epsilon)$, $\delta_s(\epsilon) = o(\epsilon)$. Under this hypothesis average values on time-scale $\delta_s^{-1}(\epsilon)$ can be computed and their properties can be studied.

Reviewing the proofs of Lemma 4 and 5 one easily deduces:

Lemma 4 bis. *Suppose $\mathbf{Y}(t, \epsilon)$ exists as the solution of*

$$d\mathbf{Y}/dt = \epsilon\mathbf{F}(\mathbf{Y}, t, \epsilon); \qquad \mathbf{Y}(0, \epsilon) = \mathbf{Y}_0,$$

in the interval $0 \leqslant t \leqslant A^*/\delta_s(\epsilon)$, and $\mathbf{Y} \in D_0$.
Let $\tau = \delta_s(\epsilon)t$ and $\mathbf{Y}(\tau/\delta_s, \epsilon) = \mathbf{Y}^*(\tau, \epsilon)$. Then

$$|\mathbf{Y}^*(\tau, \epsilon) - \overline{\mathbf{Y}}(\tau, \epsilon)| \leqslant (\epsilon/\delta_s)\delta.$$

Lemma 5 bis. *Let $\mathbf{Y}^*(\tau, \epsilon)$ be defined as in Lemma 4 bis. Then*

$$|\mathbf{Y}^*(\tau + \delta\bar{\tau}, \epsilon) - \mathbf{Y}^*(\tau, \epsilon)| \leqslant M(\epsilon/\delta_s)\,\delta\bar{\tau},$$

where $M = \sup_G |\mathbf{F}|$.

Next we deduce an equivalent of Lemma 7 in a somewhat different formulation.

Let $\mathbf{Y}(t, \epsilon)$ be the solution of

$$d\mathbf{Y}/dt = \epsilon\mathbf{F}(\mathbf{Y}, t, \epsilon); \qquad \mathbf{Y}(t_1, \epsilon) = \mathbf{Y}_1.$$

Introduce the transformation $\tau = \delta_s t$. We have the equivalent integral equation

$$\mathbf{Y}^*(\tau, \epsilon) = \mathbf{Y}_1 + \frac{\epsilon}{\delta_s} \int_{\tau_1}^{\tau} \mathbf{F}\left[\mathbf{Y}^*(\tau'', \epsilon), \frac{\tau''}{\delta_s}, \epsilon\right] d\tau''$$

where $\tau_1 = \delta_s t_1$. Proceeding as in Section 5 one finds:

$$\bar{\mathbf{Y}}(\tau, \epsilon) = \mathbf{Y}_1 + \frac{\epsilon}{\delta_s}\left\{\int_{\tau_1}^{\tau}\int_0^1 \mathbf{F}\left[\bar{\mathbf{Y}}(\tau', \epsilon), \frac{\tau'}{\delta_s} + \frac{\delta}{\delta_s}\bar{\tau}, \epsilon\right] d\bar{\tau}\, d\tau' + I_1 + I_2\right\},$$

where

$$I_1 = \int_0^1 \int_{\tau_1}^{\tau_1 + \delta\bar{\tau}} \mathbf{F}\left[\mathbf{Y}^*(\tau', \epsilon), \frac{\tau'}{\delta_s}\right] d\tau'\, d\bar{\tau},$$

$$I_2 = \int_{\tau_1}^{\tau} \tilde{I}_2(\tau')\, d\tau',$$

$$\tilde{I}_2 = \int_0^1 \left\{\mathbf{F}\left[\mathbf{Y}^*(\tau' + \delta\bar{\tau}, \epsilon), \frac{\tau'}{\delta_s} + \frac{\delta}{\delta_s}\bar{\tau}, \epsilon\right] - \mathbf{F}\left[\bar{\mathbf{Y}}(\tau', \epsilon), \frac{\tau'}{\delta_s} + \frac{\delta}{\delta_s}\bar{\tau}, \epsilon\right]\right\} d\bar{\tau}.$$

Obviously, I_1 is a constant and for all $\mathbf{Y}^* \in D_0$, $\tau_1 \leqslant \tau < \infty$ we have the estimate

$$|I_1| \leqslant \tfrac{1}{2}M\delta.$$

Furthermore, using Lipschitz-continuity and the Lemmas 4 bis and 5 bis, we have for all $\mathbf{Y}^* \in D_0$, $\bar{\mathbf{Y}} \in D_0$, $\tau_1 \leqslant \tau < \infty$ the estimate

$$|\tilde{I}_2| \leqslant \lambda M(\epsilon/\delta_s)\delta.$$

We thus obtain:

Lemma 7 bis. *Let* $\mathbf{Y}(t, \epsilon)$ *be the solution of*

$$d\mathbf{Y}/dt = \epsilon\mathbf{F}(\mathbf{Y}, t, \epsilon); \qquad \mathbf{Y}(t_1, \epsilon) = \mathbf{Y}_1.$$

Let $\tau = \delta_s(\epsilon)t$; $\mathbf{Y}(\tau/\delta_s, \epsilon) = \mathbf{Y}^*(\tau, \epsilon)$. *Then*

$$\frac{d\bar{\mathbf{Y}}}{d\tau} = \frac{\epsilon}{\delta_s}\left\{\int_0^1 \mathbf{F}\left[\bar{\mathbf{Y}}(\tau, \epsilon), \frac{\tau}{\delta_s} + \frac{\delta}{\delta_s}\bar{\tau}, \epsilon\right] d\bar{\tau} + \tilde{I}_2\right\},$$

$$\bar{\mathbf{Y}}(\tau_1, \epsilon) = \mathbf{Y}_1 + I_1,$$

$$|I_1| \leqslant \frac{1}{2}M\delta; \qquad |\tilde{I}_2| \leqslant \lambda M \frac{\epsilon}{\delta_s}\delta.$$

The estimates of I_1 *and* $\tilde{I}_2$ *are valid for all* $\mathbf{Y}^* \in D_0$, $\bar{\mathbf{Y}} \in D_0$, $\tau_1 \leqslant \tau < \infty$.

Finally we compute the average of $\mathbf{F}$ for the special and important case of periodic systems. Again for simplicity of calculations we take $\mathbf{F}$ to be independent of ϵ.

Straightforward computation as in Section 6 shows that

$$\int_0^1 \mathbf{F}\left[\overline{\mathbf{Y}}(\tau, \epsilon), \frac{\tau}{\delta_s} + \frac{\delta}{\delta_s}\bar{\tau}\right] d\bar{\tau} = \mathbf{F}_0(\overline{\mathbf{Y}}) + \frac{\delta_s}{\delta}\mathbf{F}^*(\overline{\mathbf{Y}}, \tau),$$

where $\mathbf{F}^*(\overline{\mathbf{Y}}, \tau)$ is uniformly bounded for all $\overline{\mathbf{Y}} \in D_0$, $0 \leqslant \tau < \infty$. Collecting our results we have, for periodic systems:

LEMMA 10. *Let* $\mathbf{Y}(t, \epsilon)$ *be the solution of a periodic system*

$$d\mathbf{Y}/dt = \epsilon\mathbf{F}(\mathbf{Y}, t); \qquad \mathbf{Y}(t_1, \epsilon) = \mathbf{Y}_1.$$

Let furthermore $\tau = \delta_s t$. *Then*:

$$\frac{d\overline{\mathbf{Y}}}{d\tau} = \frac{\epsilon}{\delta_s}\left\{\mathbf{F}_0(\overline{\mathbf{Y}}) + \frac{\delta_s}{\delta}\mathbf{F}^*(\overline{\mathbf{Y}}, \tau) + \frac{\epsilon}{\delta_s}\delta\mathbf{F}^{**}(\mathbf{Y}^*, \overline{\mathbf{Y}}, \tau)\right\},$$

$$\overline{\mathbf{Y}}(\tau_1, \epsilon) = \mathbf{Y}_1 + \delta\boldsymbol{\rho}.$$

$\mathbf{F}^*$, $\mathbf{F}^{**}$ *and* $\boldsymbol{\rho}$ *are uniformly bounded for* $\mathbf{Y}^* \in D_0$, $\overline{\mathbf{Y}} \in D_0$, $\tau_1 \leqslant \tau < \infty$.

9. UNIFORM VALIDITY ON $0 \leqslant t < \infty$

In the asymptotic theory of periodic systems the following result is well-known (see, for example, Roseau [10]):

Suppose the associated system

$$d\boldsymbol{\eta}/d\tau = \mathbf{F}_0(\boldsymbol{\eta}); \qquad \tau = \epsilon t,$$

has a singular point ξ which is asymptotically stable in linear approximations. Then there exists a periodic solution $\tilde{\mathbf{Y}}(t, \epsilon)$ such that

$$\lim_{\epsilon \to 0} | \tilde{\mathbf{Y}}(t, \epsilon) - \boldsymbol{\xi} | = 0,$$

uniformly in t.

We shall now prove, under similar conditions:

THEOREM IV. *Let* $\mathbf{Y}(t, \epsilon)$ *be the solution of the periodic system*

$$d\mathbf{Y}/dt = \epsilon\mathbf{F}(\mathbf{Y}, t); \qquad \mathbf{Y}(0, \epsilon) = \mathbf{Y}_0 \in D_0.$$

Let $\eta(\tau)$, *with* $\tau = \epsilon t$, *be the solution of the associated system*

$$d\eta/d\tau = \mathbf{F}_0(\eta); \qquad \eta(0) = \mathbf{Y}_0 ,$$

where

$$\mathbf{F}_0(\eta) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{F}(\eta, t)\, dt.$$

Suppose that:

(i) $\eta = \xi$ *is a singular point of the associated system*; $\eta = \xi$ *is asymptotically stable in the linear approximation.*

(ii) $\mathbf{Y}_0$ *belongs to the domain of attraction of* ξ.

(iii) $\eta(\tau) \in D_0$ *for* $0 \leqslant \tau < \infty$.

Then

$$\mathbf{Y}(t, \epsilon) = \eta(\epsilon t) + o(1),$$

uniformly on $0 \leqslant t < \infty$.

Proof. The reasoning leading to the result given in Theorem IV is rather lengthy and it is therefore usefull to subdivide the proof in a number of steps:

1. Since $\eta(\tau)$, $\tau = \epsilon t$, exists for all $0 \leqslant \tau < \infty$ and $\eta \in D_0$, then from Theorem III (and using results of Section 7), we have

$$\mathbf{Y}^*(\tau, \epsilon) = \eta(\tau) + O(\epsilon),$$

uniformly on the time scale ϵ^{-1} (in the sense of Definition 5B and Definition 4).

2. Next we use an analytic result which in the asymptotic theory is called *The Extension Theorem.* Adapted to the problem under consideration here, the theorem states:

Suppose $f^*(\tau, \epsilon)$ is a continuous function and

$$\lim_{\epsilon\to 0}\{f^*(\tau, \epsilon) - \eta(\tau)\} = 0,$$

uniformly for $0 \leqslant \tau \leqslant A < \infty$, where A is *any* positive number. Then there exists an order function $\delta'(\epsilon) = o(1)$ such that

$$\lim_{\epsilon\to 0} \left\{ f^* \left(\frac{\tau'}{\delta'}, \epsilon\right) - \eta\left(\frac{\tau'}{\delta'}\right) \right\} = 0,$$

uniformly for $0 \leqslant \tau' \leqslant A' < \infty$, where A' is any positive number.

Proof of the extension theorem (in a more general setting) can be found for example in Eckhaus [5]. For the sake of completeness of the present analysis an elementary proof of the result stated above is given in the appendix.

Adapting the above result, and using elementary continuation argument for the existence of $\mathbf{Y}^*$ on the larger time interval, we find: There exists a time scale $(\delta_s')^{-1}$ given by

$$\delta_s' = \epsilon\delta'; \qquad \delta' = o(1)$$

and an order function $\tilde{\delta}' = o(1)$ such that

$$\mathbf{Y}^*(\tau', \epsilon) = \boldsymbol{\eta}\left(\frac{\epsilon}{\delta_s'}\tau'\right) + O(\tilde{\delta}'); \qquad \tau' = \delta_s' t,$$

uniformly on the time scale $(\delta_s')^{-1}$.

3. We now formulate a new initial value problem for the function $\mathbf{Y}(t, \epsilon)$ in $t_1 \leqslant t$:

$$d\mathbf{Y}/dt = \epsilon\mathbf{F}(\mathbf{Y}, t); \qquad \mathbf{Y}(t_1, \epsilon) = \mathbf{Y}_1,$$

where $t_1 = (1/\delta_s')\tau_1'$, τ_1' is some fixed number.

Since $\boldsymbol{\eta} = \boldsymbol{\xi}$ is an asymptotically stable singular point of the differential equation for $\boldsymbol{\eta}$ and the initial value of $\boldsymbol{\eta}$ lies in the domain of attraction, we have

$$\lim_{\tau\to\infty}\{\boldsymbol{\eta}(\tau) - \boldsymbol{\xi}\} = 0.$$

Now $\delta_s'/\epsilon = o(1)$, we can therefore conclude that there exists an order function $\tilde{\delta}'' = o(1)$ such that

$$\boldsymbol{\eta}((\epsilon'/\delta_s)\tau) = \boldsymbol{\xi} + O(\tilde{\delta}'').$$

Using the result of Step 2 of the proof above, we obtain:

$$\mathbf{Y}_1 = \boldsymbol{\xi} + O(\delta_1)$$

where $\delta_1(\epsilon)$ is some order function such that

$$\delta_1 = o(1).$$

4. We next proceed to the formal analysis of the average $\bar{\mathbf{Y}}$ of $\mathbf{Y}$ for $t_1 \leqslant t$, on arbitrary time scales. The formal aspect of the analysis at this stage

consists of the assumption that $\mathbf{Y}^*$ exists on any time scale under consideration and that $\mathbf{Y}^* \in D_0$. From Lemma 10 we have for $\tau > \tau_1$

$$\frac{d\bar{\mathbf{Y}}}{d\tau} = \frac{\epsilon}{\delta_s}\left\{\mathbf{F}_0(\bar{\mathbf{Y}}) + \frac{\delta_s}{\delta}\mathbf{F}^*(\bar{\mathbf{Y}}, \tau) + \frac{\epsilon}{\delta_s}\delta\mathbf{F}^{**}(\mathbf{Y}^*, \bar{\mathbf{Y}}, \tau)\right\},$$

$$\bar{\mathbf{Y}}(\tau_1, \epsilon) = \mathbf{Y}_1 + \delta\rho; \qquad \mathbf{Y}_1 = \xi + O(\delta_1),$$

$$\tau_1 = \delta_s t_1 = \delta_s \frac{1}{\delta_s'} \tau_1'.$$

Now Theorem IV states that $\boldsymbol{\eta} = \boldsymbol{\xi}$ is asymptotically stable in the lineai approximation. This means that $\mathbf{F}_0(\boldsymbol{\xi} + \boldsymbol{\zeta})$ can be written as follows:

$$\mathbf{F}_0(\boldsymbol{\xi} + \boldsymbol{\zeta}) = A \cdot \boldsymbol{\zeta} + \tilde{\mathbf{F}}_0(\boldsymbol{\zeta}),$$

where

$$\lim_{|\boldsymbol{\zeta}| \to 0} \frac{|\tilde{\mathbf{F}}_0(\boldsymbol{\zeta})|}{|\boldsymbol{\zeta}|} = 0.$$

A is a constant matrix of which *all* eigenvalues have negative real parts.

We introduce the fundamental matrix $\Phi((\epsilon/\delta_s)\tau)$, defined by

$$d\Phi/d\tau = (\epsilon/\delta_s)\, A \cdot \Phi; \qquad \Phi(0) = I,$$

where I is the identity matrix.

Because of the properties of eigenvalues of A, we have the estimate

$$\left|\Phi\left(\frac{\epsilon}{\delta_s}\tau\right)\right| \leqslant C \exp\left\{-\mu\frac{\epsilon}{\delta_s}\tau\right\}; \qquad \mu > 0.$$

With these preliminaries, the problem for the average $\bar{\mathbf{Y}}$ can be reformulated as an equivalent integral equation

$$\begin{aligned}\bar{\mathbf{Y}}(\tau, \epsilon) - \boldsymbol{\xi} &= \Phi\left[\frac{\epsilon}{\delta_s}(\tau - \tau_1)\right] \cdot [\mathbf{Y}(\tau_1, \epsilon) - \boldsymbol{\xi}] \\ &\quad + \int_{\tau_1}^{\tau} \Phi\left[\frac{\epsilon}{\delta_s}(\tau - \tau')\right] \cdot \tilde{\mathbf{F}}_0(\bar{\mathbf{Y}} - \boldsymbol{\xi})\, d\tau' \\ &\quad + \frac{\delta_s}{\delta}\int_{\tau_1}^{\tau} \Phi\left[\frac{\epsilon}{\delta_s}(\tau - \tau')\right] \cdot \mathbf{F}^*(\bar{\mathbf{Y}}, \tau')\, d\tau' \\ &\quad + \frac{\epsilon}{\delta_s}\delta\int_{\tau_1}^{\tau} \Phi\left[\frac{\epsilon}{\delta_s}(\tau - \tau')\right] \mathbf{F}^{**}(\mathbf{Y}^*, \bar{\mathbf{Y}}, \tau')\, d\tau'.\end{aligned}$$

Using the estimate for $|\Phi|$, and the fact that $\mathbf{F}^*$ and $\mathbf{F}^{**}$ are uniformly bounded, we find

$$\begin{aligned}|\bar{\mathbf{Y}}(\tau, \epsilon) - \boldsymbol{\xi}| \leqslant & \, C e^{-\mu(\epsilon/\delta_s)(\tau-\tau_1)} |\mathbf{Y}(\tau_1, \epsilon) - \boldsymbol{\xi}| \\ & + \frac{1}{\mu} C \left\{\frac{\delta_s^2}{\epsilon\delta} N_1 + \delta N_2\right\} \{1 - e^{-\mu(\epsilon/\delta_s)(\tau-\tau_1)}\} \\ & + C \int_{\tau_1}^{\tau} e^{-\mu(\epsilon/\delta_s)(\tau-\tau')} |\tilde{\mathbf{F}}_0(\bar{\mathbf{Y}} - \boldsymbol{\xi})| \, d\tau',\end{aligned}$$

where N_1 and N_2 are constants independent of ϵ.

Next we use the property of $\tilde{\mathbf{F}}_0$ given above, which can be translated as follows:

For every $p > 0$ there exists $q(p)$ such that

$$|\tilde{\mathbf{F}}_0(\bar{\mathbf{Y}} - \boldsymbol{\xi})| \leqslant p \, |\bar{\mathbf{Y}} - \boldsymbol{\xi}| \qquad \text{if} \quad |\bar{\mathbf{Y}} - \boldsymbol{\xi}| \leqslant q(p).$$

We thus obtain

$$e^{\mu(\epsilon/\delta_s)(\tau-\tau_1)} |\bar{\mathbf{Y}}(\tau, \epsilon) - \boldsymbol{\xi}| \leqslant \psi(\tau) + Cp \int_{\tau_1}^{\tau} e^{\mu(\epsilon/\delta_s)(\tau'-\tau_1)} |\bar{\mathbf{Y}}(\tau', \epsilon) - \boldsymbol{\xi}| \, d\tau',$$

where

$$\psi(\tau) = C \, |\bar{\mathbf{Y}}(\tau_1, \epsilon) - \boldsymbol{\xi}| + \frac{1}{\mu} C \left\{\frac{\delta_s^2}{\epsilon\delta} N_1 + \delta N_2\right\} \{e^{\mu(\epsilon/\delta_s)(\tau-\tau_1)} - 1\}.$$

Using Gronwall's lemma now it follows that

$$\begin{aligned}& |\bar{\mathbf{Y}}(\tau, \epsilon) - \boldsymbol{\xi}| \\ & \leqslant C \left\{ |\bar{\mathbf{Y}}(\tau_1, \epsilon) - \boldsymbol{\xi}| + \frac{1}{\mu} \left(\frac{\delta_s^2}{\epsilon\delta} N_1 + N_2\delta\right)\right\} \exp\left[-\left(\mu \frac{\epsilon}{\delta_s} - Cp\right)(\tau - \tau_1)\right] \\ & \quad + C \frac{1}{\mu} \frac{(\delta_s^2/\epsilon\delta) N_1 + N_2\delta}{1 - (1/\mu)(\delta_s/\epsilon) Cp} \left\{1 - \exp\left[-\left(\mu \frac{\epsilon}{\delta_s} - Cp\right)(\tau - \tau_1)\right]\right\}.\end{aligned}$$

In order to interpret the above estimate we recall that:

$$\delta = o(1); \qquad \bar{\mathbf{Y}}(\tau_1, \epsilon) - \boldsymbol{\xi} = o(1).$$

Furthermore we are interested in time scales for which

$$\delta_s/\epsilon = o(1).$$

We now assume δ chosen such that

$$\delta_s^2/\epsilon\delta = o(1).$$

It is obvious that p can be chosen such that

$$\mu(\epsilon/\delta_s) - Cp > 0.$$

We hence find that for $t_1 \leqslant t$; $t_1 = (1/\epsilon\delta')\tau_1$:

$$\bar{\mathbf{Y}}(\tau, \epsilon) - \boldsymbol{\xi} = O(\delta_1) + O(\delta) + O(\delta_s{}^2/\epsilon\delta),$$

uniformly on any time scale δ_s^{-1}.

5. We recall from Lemma 4 bis, that on any time scale

$$\mathbf{Y}^*(\tau, \epsilon) = \bar{\mathbf{Y}}(\tau, \epsilon) + O((\epsilon/\delta_s)\delta).$$

Hence, the average value is an approximation of $\mathbf{Y}^*$ if we impose on δ the requirement

$$(\epsilon/\delta_s)\delta = o(1).$$

From the final result of Step 4 of our proof we have that on any time scale and in fact for $t_1 \leqslant t < \infty$,

$$\bar{\mathbf{Y}} \in D_0 .$$

Therefore, by the elementary continuation argument the existence of $\mathbf{Y}(t, \epsilon)$ in $0 \leqslant t < \infty$ is assured.

Next we confront the various requirements imposed on the order function δ. We have assumed that

$$\delta = o(1); \qquad \frac{\delta_s{}^2}{\epsilon\delta} = o(1); \qquad \frac{\epsilon}{\delta_s}\,\delta = o(1); \qquad \frac{\delta_s}{\epsilon} = o(1).$$

The requirements can all be satisfied when choosing for example

$$\delta = \delta_s/\epsilon^{\gamma},$$

where γ is some fixed number with $0 < \gamma < 1$. With this choice we have:

$$\begin{aligned} \delta &= (\delta_s/\epsilon)\, \epsilon^{1-\gamma} = O(\epsilon^{1-\gamma}), \\ \delta_s{}^2/\epsilon\delta &= (\delta_s/\epsilon)\, \epsilon^{\gamma} = O(\epsilon^{\gamma}), \\ (\epsilon/\delta_s)\delta &= \epsilon^{1-\gamma}. \end{aligned}$$

We now summarize our results: We have found that

$$\mathbf{Y}(t, \epsilon) = \eta(\epsilon t) + O(\tilde{\delta}'); \qquad \tilde{\delta}' = o(1),$$

uniformly in $0 \leqslant t \leqslant t_1$, where

$$t_1 = \tau_1/\epsilon\delta'; \qquad \delta' = o(1).$$

Furthermore, for $t > t_1$, uniformly on any time scale δ_s^{-1}:

$$\mathbf{Y}^*(\tau, \epsilon) = \bar{\mathbf{Y}}(\tau, \epsilon) + O(\epsilon^{1-\gamma}),$$

$$\bar{\mathbf{Y}}(\tau, \epsilon) = \boldsymbol{\xi} + O(\delta_1) + O(\epsilon^{1-\gamma}) + O(\epsilon^{\gamma}); \qquad \delta_1 = o(1),$$

$$\boldsymbol{\eta}(\epsilon t) = \boldsymbol{\xi} + O(\tilde{\delta}''); \qquad \tilde{\delta}'' = o(1).$$

Hence we conclude that there exists an order function $\delta^* = o(1)$ such that for $t > t_1$, uniformly, on any time scale δ_s^{-1},

$$\mathbf{Y}(t, \epsilon) = \boldsymbol{\eta}(\epsilon t) + O(\delta^*).$$

Finally then

$$\mathbf{Y}(t, \epsilon) = \boldsymbol{\eta}(\epsilon t) + o(1),$$

uniformly for $0 \leqslant t < \infty$, which proves the assertion of our theorem.

10. Application of the Concept of Local Averages to Partial Differential Equations

In the preceding sections we have seen that using the concept of local average values (as introduced in Section 4) a deductive asymptotic theory of nonlinear oscillations can be developed, and that new results can be obtained (Section 9). It is tempting now to investigate whether this concept could also be useful in studying problems gouverned by partial differential equations of the type describing wave-propagation phenomena. For such problems various formal methods have been proposed, however the methods generally do not contain proof of the asymptotic validity of the results.

It is beyond the scope of this paper to attempt a new approach to the general problem of wave-propagation phenomena, along the lines of the preceding sections. However, we shall show by an example that such approach is indeed possible, at least for certain classes of problems.

We shall study a class of problems investigated by Chikwendu and Kevorkian [3]. These authors have proposed a formal method of construction of asymptotic approximations (without proof of the validity of the result). Using now local average values we shall rederive the fundamental result of Chikwendu and Kevorkian and deduce conditions for validity of the asymptotic approximation. In the analysis that follows various parts are closely analogous to some corresponding parts of the analysis of the preceding sections. For simplicity of exposition the reasoning needed in these parts shall only be outlined, with a reference to the corresponding reasoning of the preceding sections.

Consider a function $u(x, t)$ defined as solution of the perturbed wave equation

$$\partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 = \epsilon H(\partial u/\partial t, \partial u/\partial x).$$

We suppose that some initial conditions (and perhaps some boundary conditions) have been specified, but the nature of these conditions does not interfere in the course of our analysis. The function H will be assumed to be Lipschitz-continuous in some domain D.

The first step of the analysis is to transform the problem to a form analoguous to the standard slowly modulated systems of Section 3. For this purpose we introduce characteristic coordinates

$$\sigma = x - t; \qquad \xi = x + t.$$

Furthermore, inspired by the structure of the solutions of the wave-equation for $\epsilon = 0$, we introduce the transformation

$$\partial u/\partial x = \phi(\sigma, t) + \psi(\xi, t),$$
$$\partial u/\partial t = -\phi(\sigma, t) + \psi(\xi, t).$$

It is a matter of straight forward computation to deduce that

$$\partial\phi/\partial t = \tfrac{1}{2}\epsilon G(\phi, \psi); \qquad \partial\psi/\partial t = -\tfrac{1}{2}\epsilon G(\phi, \psi),$$

where $G(\phi, \psi) = H(\psi - \phi, \psi + \phi)$.

Writing: $\tau = \epsilon t$; $\phi(\sigma, \tau/\epsilon) = \phi^*(\sigma, \tau)$; $\psi(\xi, \tau/\epsilon) = \psi^*(\xi, \tau)$ and taking correct account of the fact that in the differential equations above we have partial derivatives with respect to t, we find as equivalent integral relations:

$$\phi^*(\sigma, \tau) = \phi^*(\sigma, 0) + \frac{1}{2}\int_0^\tau G\left\{\phi^*(\sigma, \tau'), \psi^*\left(\sigma + 2\frac{\tau'}{\epsilon}, \tau'\right)\right\} d\tau',$$

$$\psi^*(\xi, \tau) = \psi^*(\xi, 0) - \frac{1}{2}\int_0^\tau G\left\{\phi^*\left(\xi - 2\frac{\tau'}{\epsilon}, \tau'\right), \psi^*(\xi, \tau')\right\} d\tau'.$$

Next we introduce local average values as follows:

$$\bar{\phi}(\sigma, \tau) = \int_0^1 \phi^*(\sigma, \tau + \delta\bar{\tau})\, d\bar{\tau},$$

$$\bar{\psi}(\xi, \tau) = \int_0^1 \psi^*(\xi, \tau + \delta\bar{\tau})\, d\bar{\tau}.$$

In analogy to Lemmas 4 and 5 one easily shows that

$$|\phi^*(\sigma, \tau + \delta\bar{\tau}) - \phi^*(\sigma, \tau)| \leqslant M\delta\bar{\tau},$$
$$|\psi^*(\xi, \tau + \delta\bar{\tau}) - \psi^*(\xi, \tau)| \leqslant M\delta\bar{\tau},$$

where M is a constant independent of ϵ. Furthermore,

$$|\bar{\phi}(\sigma, \tau) - \phi^*(\sigma, \tau)| \leqslant \tfrac{1}{2}M\delta,$$
$$|\bar{\psi}(\xi, \tau) - \psi^*(\xi, \tau)| \leqslant \tfrac{1}{2}M\delta.$$

We proceed to compute the average values from the equivalent integral relations, in a way analogous to Section 5. We indicate here the analysis of $\bar{\phi}(\sigma, \tau)$.

$$\bar{\phi}(\sigma, \tau) = \phi^*(\sigma, 0) + \frac{1}{2}\int_0^1\int_0^{\tau+\delta\bar{\tau}} G\left\{\phi^*(\sigma, \tau'), \psi^*\left(\sigma + 2\frac{\tau'}{\epsilon}, \tau'\right)\right\} d\tau'\, d\bar{\tau}$$
$$= \phi^*(\sigma, 0)$$
$$+ \frac{1}{2}\int_0^\tau\int_0^1 G\left\{\phi^*(\sigma, \tau' + \delta\bar{\tau}'), \psi^*\left(\sigma + 2\frac{\tau'}{\epsilon} + 2\frac{\delta}{\epsilon}\bar{\tau}, \tau' + \delta\bar{\tau}\right)\right\} d\bar{\tau}\, d\tau' + I_1,$$

where

$$I_1 = \int_0^1\int_0^{\delta\bar{\tau}} G\left\{\phi^*(\sigma, \tau'), \psi^*\left(\sigma + 2\frac{\tau'}{\epsilon}, \tau'\right)\right\} d\tau'\, d\bar{\tau}.$$

It is immediately obvious that

$$|I_1| \leqslant \tfrac{1}{2}M\delta.$$

Furthermore, using Lipschitz-continuity, and the estimates analogous to Lemmas 4 and 5 (given above), we find

$$\bar{\phi}(\sigma, \tau) = \phi(\sigma, 0) + \frac{1}{2}\int_0^\tau\int_0^1 G\left\{\bar{\phi}(\sigma, \tau'), \bar{\psi}\left(\sigma + 2\frac{\tau'}{\epsilon} + 2\frac{\delta}{\epsilon}\bar{\tau}, \tau'\right)\right\} d\bar{\tau}\, d\tau'$$
$$+ I_1 + \int_0^\tau \tilde{I}_2(\tau')\, d\tau',$$

where

$$|\tilde{I}_2(\tau')| \leqslant 4M\lambda\delta.$$

This result can be rewritten as follows:

$$\frac{\partial\bar{\phi}(\sigma, \tau)}{\partial\tau} = \frac{1}{2}\int_0^1 G\left\{\bar{\phi}(\sigma, \tau), \bar{\psi}\left(\sigma + 2\frac{\tau}{\epsilon} + 2\frac{\delta}{\epsilon}\bar{\tau}, \tau\right)\right\} d\bar{\tau} + O(\delta),$$
$$\bar{\phi}(\sigma, 0) = \phi^*(\sigma, 0) + O(\delta).$$

In an entirely analogous way one finds that

$$\frac{\partial\bar{\psi}(\xi, \tau)}{\partial\tau} = -\frac{1}{2}\int_0^1 G\left\{\bar{\phi}\left(\xi - 2\frac{\tau}{\epsilon} - 2\frac{\delta}{\epsilon}\bar{\tau}, \tau\right), \bar{\psi}(\xi, \tau)\right\} d\bar{\tau} + O(\delta),$$
$$\bar{\psi}(\xi, \tau) = \psi^*(\xi, 0) + O(\delta).$$

Suppose now that $\psi(\xi, t)$ is a periodic function of ξ, with a period ξ_0 independent of ϵ, and similarly that $\phi(\sigma, t)$ is a periodic function of σ with a period σ_0 independent of ϵ. Then by computation closely analogous to Section 6 one finds that

$$\frac{\partial\bar{\phi}(\sigma,\tau)}{\partial\tau} = \frac{1}{2}\lim_{T\to\infty}\frac{1}{T}\int_0^T G\left\{\bar{\phi}(\sigma,\tau), \bar{\psi}(\sigma+2t,\tau)\right\} dt + O\left(\frac{\epsilon}{\delta}\right) + O(\delta),$$

$$\frac{\partial\bar{\psi}(\xi,\tau)}{\partial\tau} = -\frac{1}{2}\lim_{T\to\infty}\frac{1}{T}\int_0^T G\left\{\bar{\phi}(\xi-2t,\tau), \bar{\psi}(\xi,\tau)\right\} dt + O\left(\frac{\epsilon}{\delta}\right) + O(\delta)$$

If one leaves out in the differential equations for $\bar{\phi}$ and $\bar{\psi}$ above the terms indicated by $O(\delta)$ and $O(\epsilon/\delta)$, then one obtains the differential equation which according to the formal analysis of Chikwendu and Kevorkian [3] should produce the asymptotic approximation of ϕ^* and ψ^*.

Now, we already know that $\bar{\phi}$ is an asymptotic approximation ϕ^*, and $\bar{\psi}$ is an asymptotic approximation of ψ^*, if $\delta = o(1)$. Therefore, in order to prove the assertion of Chikwendu and Kevorkian the following final step is needed:

Let us abbreviate the equations for $\bar{\phi}$ and $\bar{\psi}$, by:

$$\partial\bar{\phi}/\partial\tau = F_1(\bar{\phi}, \bar{\psi}) + r_1,$$
$$\partial\bar{\psi}/\partial\tau = F_2(\bar{\phi}, \bar{\psi}) + r_2,$$
$$r_1 = O(\epsilon/\delta) + O(\delta); \qquad r_2 = O(\epsilon/\delta) + O(\delta).$$

It is necessary to prove that the differential equations for $\bar{\phi}$ and $\bar{\psi}$ have a structure such that the solutions continuously depend on r_1 and r_2 in the vicinity $r_1 = 0$, $r_2 = 0$.

However, the structure of F_1 and F_2 can only be obtained from actual computation in any given problem. Therefore, this final step of the proof must be verified separately in any individual problem. In the paper of Chikwendu and Kevorkian [3] various applications are studied, in which F_1 and F_2 are computed. One can show that in these applications the differential equations for $\bar{\phi}$ and $\bar{\psi}$ indeed possess the structure needed for the asymptotic validity of the results.

APPENDIX

Proof of the Extension Theorem

The proof given here follows (with slight modifications) the reasoning outlined in Kaplun [8].

We are given that the function

$$\mathbf{g}(\tau, \epsilon) = \mathbf{f}^*(\tau, \epsilon) - \boldsymbol{\eta}(\tau)$$

has the property

$$\lim_{\epsilon \to 0} \mathbf{g}(\tau, \epsilon) = 0,$$

uniformly for $0 \leqslant \tau \leqslant A$, where A is *any* positive number. This means that for any number $p > 0$ there exists a number $q > 0$ such that

$$|\mathbf{g}(\tau, \epsilon)| < p \qquad \text{for} \quad 0 < \epsilon \leqslant q, \quad 0 \leqslant \tau \leqslant A.$$

Obviously q depends on p and on A. We therefore write $q = q(p, A)$. We can choose $q(p, A)$ to be a continuous monotonic function, decreasing with decreasing p and with increasing A, and furthermore such that

$$\lim_{p \to 0} q(p, A) = 0; \qquad \lim_{A \to \infty} q(p, A) = 0.$$

Consider any monotonic decreasing sequence p_n, such that $\lim_{n \to \infty} p_n = 0$, and simultaneously a monotonic increasing sequence A_n, such that $\lim_{n \to \infty} A_n = \infty$. We have

$$|\mathbf{g}(\tau, \epsilon)| < p_n \qquad \text{for} \quad 0 < \epsilon \leqslant q(p_n, A_n), \quad 0 \leqslant \tau \leqslant A_n.$$

We define now (by any convenient construction) a continuous monotonic decreasing function $\tau \to \xi(\tau)$ such that for $\tau > A_1$

$$\xi(A_n) = q(p_{n+1}, A_{n+1}) = q_{n+1}.$$

The construction is indicated in Fig. 1.

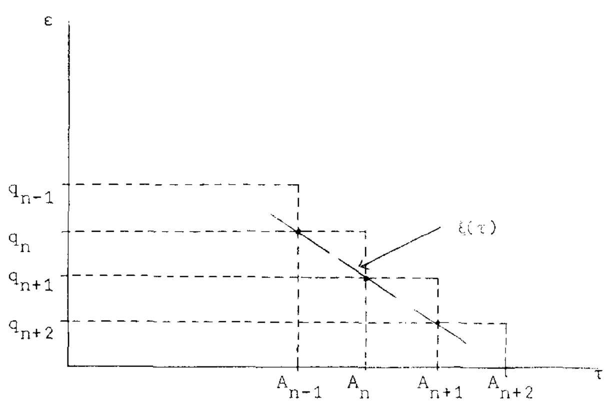

FIGURE 1

Finally we define the inverse function of the function $\xi(\tau)$. Let this inverse function be $\epsilon \to \tilde{\tau}(\epsilon)$, for $0 < \epsilon \leqslant q_1$. We obviously have the property

$$\lim_{\epsilon \to 0} \tilde{\tau}(\epsilon) = \infty.$$

From our construction it follows that for every p_n there exists a number q_n such that

$$| \mathbf{g}(\tau, \epsilon)| < p_n \qquad \text{for} \quad 0 < \epsilon \leqslant q_n, \quad 0 \leqslant \tau \leqslant \tilde{\tau}(\epsilon).$$

Furthermore, because of the monotonic decreasing behavior of the sequences p_n and q_n, we can affirm that for every $p > 0$ there exists a number $q(p)$ such that

$$| \mathbf{g}(\tau, \epsilon)| < p \qquad \text{for} \quad 0 < \epsilon \leqslant q(p), \quad 0 \leqslant \tau \leqslant \tilde{\tau}(\epsilon).$$

The above result is in essence the extension theorem. A more convenient interpretation of the theorem is obtained as follows: We can write

$$\tilde{\tau}(\epsilon) = 1/\delta_0(\epsilon),$$

where $\delta_0(\epsilon)$ is an order function such that $\delta_0 = o(1)$. Let now $\delta'(\epsilon)$ be an order function such that

$$\delta_0 = o(\delta'), \qquad \delta' = o(1).$$

Consider the values of τ given by

$$\tau = \tau'/\delta',$$

where $0 \leqslant \tau' \leqslant A'$, and A' is any positive number. For sufficiently small ϵ we certainly have

$$\tau = \tau'/\delta' < \tilde{\tau} = 1/\delta_0.$$

Hence, for these particular values of τ we also have

$$| \mathbf{g}(\tau'/\delta', \epsilon)| < p \qquad \text{for} \quad 0 < \epsilon \leqslant q(p).$$

We have thus demonstrated that there exists an order function $\delta'(\epsilon) = o(1)$ such that

$$\lim_{\epsilon \to 0} \mathbf{g}\left(\frac{\tau'}{\delta'}, \epsilon\right) = 0,$$

uniformly for $0 \leqslant \tau' \leqslant A'$, where A' is any positive number.

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