Oscillations of Delay Differential Equations with Variable Coefficients

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Consider the delay differential equation

\[ x'(t) + p(t)x(t - \tau) = 0, \quad t \geq t_0 \]

where \( p(t) \in C([t_0, \infty), R^+) \) and \( \tau \) is a positive constant. We obtain a sharp sufficient condition for the oscillation of this equation, which improves previously known results.

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1. INTRODUCTION

Consider the delay differential equation

\[ x'(t) + p(t)x(t - \tau) = 0, \quad t \geq t_0 \]  \hspace{1cm} (1)

where \( p(t) \in C([t_0, \infty), R^+) \) and \( \tau \) is a positive constant. It is well known (see [1, 2]) that every solution of (1) oscillates, provided that

\[ \liminf_{t \to \infty} \int_{t-\tau}^{t} p(s) \, ds > \frac{1}{e}. \]  \hspace{1cm} (2)

It is also known (see [2]) that the differential inequality

\[ x'(t) + p(t)x(t - \tau) \leq 0, \quad t \geq t_0 \]  \hspace{1cm} (1')

is satisfied for all solutions of (1).
has no eventually positive solution if (2) holds. This observation has been extensively exploited in the study of the oscillatory properties of solutions of various functional differential equations. See for example [3–5].

By [6, Corollary 3.2.2], inequality (1') has no eventually positive solution if and only if Eq. (1) has no eventually positive solution. Therefore, by obtaining sharper sufficient conditions for oscillation of (1), we expect many of the above-mentioned results can be improved.

Recently, Li [7] obtained a sharper sufficient condition by improving condition (2).

**THEOREM A.** Let \( p(t) \in C([t_0, \infty), R^+) \) and let \( \tau \) be a positive constant. Suppose that there exists a \( \tilde{t} > t_0 + \tau \) such that

\[
\int_{t-\tau}^{\tilde{t}} p(s) \, ds \geq \frac{1}{e}, \quad t \geq \tilde{t}, \tag{3}
\]

and

\[
\int_{t_0+\tau}^{\infty} p(t) \left[ \exp\left(\int_{t-\tau}^{t} p(s) \, ds - \frac{1}{e}\right) - 1 \right] dt = \infty. \tag{4}
\]

Then every solution of (1) oscillates.

In this paper, we use a different method to obtain new sufficient conditions for oscillation of (1) which improve conditions (3) and (4). As is customary, a solution of (1) is said to oscillate if it has arbitrarily large zeros.

2. MAIN RESULTS

Let \( p(t) \in C([t_0, \infty), R^+) \) and define the following sequences of function:

\[
p_1(t) = \int_{t-\tau}^{t} p(s) \, ds, \quad t \geq t_0 + \tau,
\]

\[
p_{k+1}(t) = \int_{t-\tau}^{t} p(s) p_k(s) \, ds, \quad t \geq t_0 + (k + 1) \tau,
\]

\[
\bar{p}_1(t) = \int_{t}^{t+\tau} p(s) \, ds, \quad t \geq t_0,
\]

\[
\bar{p}_{k+1}(t) = \int_{t}^{t+\tau} p(s) \bar{p}_k(s) \, ds, \quad t \geq t_0,
\]

\[
k = 1, 2, 3, \ldots.
\]
**Theorem 1.** Let \( p(t) \in C([t_0, \infty), R^+) \) and let \( \tau \) be a positive constant. Suppose that there exist a \( t_1 > t_0 + \tau \) and a positive integer \( n \) such that

\[
\begin{align*}
p_n(t) &\geq \frac{1}{e^n}, \quad p_n(t) \geq \frac{1}{e^n}, \quad t \geq t_1, \\
\int_{t_0+n\tau}^{\infty} p(t) \left[ \exp \left( e^{n-1} p_n(t) - \frac{1}{e} \right) - 1 \right] dt = \infty.
\end{align*}
\]

and

where \( p_n(t) \) and \( p_n(t) \) are defined by (5). Then every solution of (1) oscillates.

**Proof.** Assume, for the sake of contradiction, that Eq. (1) has an eventually positive solution \( x(t) \). Then there exists a \( t_2 \geq t_1 \) such that

\[
x(t - \tau) \geq x(t) > 0, \quad x'(t) \leq 0, \quad t \geq t_2.
\]

Set

\[
\omega(t) = \frac{x(t - \tau)}{x(t)}, \quad t \geq t_2.
\]

Then

\[
\omega(t) \geq 1, \quad t \geq t_2.
\]

Dividing the both sides of (1) by \( x(t) \), for \( t \geq t_2 \), we obtain

\[
\frac{x'(t)}{x(t)} + p(t) \omega(t) = 0, \quad t \geq t_2.
\]

Integrating the both sides of (10) from \( t - \tau \) to \( t \), yields

\[
\ln x(t) - \ln x(t - \tau) + \int_{t-\tau}^{t} p(s) \omega(s) ds = 0, \quad t \geq t_2 + \tau,
\]

or

\[
\omega(t) = \exp \left( \int_{t-\tau}^{t} p(s) \omega(s) ds \right), \quad t \geq t_2 + \tau.
\]

It is easy to show that \( e^c \geq ec \) for all \( c \geq 0 \), and so

\[
\omega(t) \geq e \int_{t-\tau}^{t} p(s) \omega(s) ds, \quad t \geq t_2 + \tau.
\]
Set
\[
\omega_1(t) = \int_{t-\tau}^{t} p(s) \omega(s) \, ds, \quad t \geq t_2 + \tau,
\]
\[
\omega_2(t) = \int_{t-\tau}^{t} p(s) \omega_2(s) \, ds, \quad t \geq t_2 + 2\tau,
\]
\[
\vdots
\]
\[
\omega_n(t) = \int_{t-\tau}^{t} p(s) \omega_{n-1}(s) \, ds, \quad t \geq t_2 + n\tau,
\]
(13)
and
\[
v(t) = \omega(t) - 1, \quad t \geq t_2,
\]
\[
v_1(t) = \int_{t-\tau}^{t} p(s) v(s) \, ds, \quad t \geq t_2 + \tau,
\]
\[
v_2(t) = \int_{t-\tau}^{t} p(s) v_1(s) \, ds, \quad t \geq t_2 + 2\tau,
\]
\[
\vdots
\]
\[
v_n(t) = \int_{t-\tau}^{t} p(s) v_{n-1}(s) \, ds, \quad t \geq t_2 + n\tau.
\]
(14)
By (9),
\[
v(t) \geq 0, \quad t \geq t_2, \quad v_i(t) \geq 0, \quad t \geq t_2 + i\tau, \quad i = 1, 2, \ldots, n.
\]
(15)
From (11) and (12), we easily obtain
\[
\omega(t) \geq e^{n-1} \omega_{n-1}(t), \quad t \geq t_2 + (n - 1)\tau,
\]
(16)
and
\[
\omega(t) \geq \exp \left( e^{n-1} \int_{t-\tau}^{t} p(s) \omega_{n-1}(s) \, ds \right), \quad t \geq t_2 + n\tau.
\]
(17)
In view of (5), (14), and (15), (17) can be written as
\[
\omega(t) \geq \exp \left( e^{n-1} \int_{t-\tau}^{t} p(s) v_{n-1}(s) \, ds + e^{n-1} p_n(t) \right)
\]
\[
= \exp \left( e^{n-1} \int_{t-\tau}^{t} p(s) v_{n-1}(s) \, ds + \frac{1}{e} \right) \cdot \exp \left( e^{n-1} p_n(t) - \frac{1}{e} \right),
\]
\[
t \geq t_2 + n\tau,
\]
and so
\[
\omega(t) \geq \left( e^n \int_{t-\tau}^{t} p(s) v_{n-1}(s) \, ds + 1 \right) \cdot \exp \left( e^{n-1} p_n(t) - \frac{1}{e} \right),
\]
\[t \geq t_2 + n\tau. \tag{18}\]

By (6) and (15),
\[
p(t) \left[ \omega(t) - \left( e^n \int_{t-\tau}^{t} p(s) v_{n-1}(s) \, ds + 1 \right) \right] \\
\geq p(t) \left[ e^n \int_{t-\tau}^{t} p(s) v_{n-1}(s) \, ds + 1 \right] \left[ \exp \left( e^{n-1} p_n(t) - \frac{1}{e} \right) - 1 \right] \\
\geq p(t) \left[ \exp \left( e^{n-1} p_n(t) - \frac{1}{e} \right) - 1 \right], \quad t \geq t_2 + n\tau,
\]
or
\[
p(t) [v(t) - e^n v_n(t)] \geq p(t) \left[ \exp \left( e^{n-1} p_n(t) - \frac{1}{e} \right) - 1 \right], \quad t \geq t_2 + n\tau.
\]

By integrating the both sides from \( t_3 = t_2 + n\tau \) to \( T > t_3 + n\tau \) we obtain
\[
\int_{t_3}^{T} p(t) [v(t) - e^n v_n(t)] \, dt \geq \int_{t_3}^{T} p(t) \left[ \exp \left( e^{n-1} p_n(t) - \frac{1}{e} \right) - 1 \right] \, dt. \tag{19}
\]

From this and (7), we have
\[
\lim_{T \to \infty} \int_{t_3}^{T} p(t) [v(t) - e^n v_n(t)] \, dt = \infty. \tag{20}
\]

Since
\[
e^n \int_{t_3}^{T} p(t) v_n(t) \, dt = e^n \int_{t_3}^{T} p(t) \, dt \int_{t-\tau}^{t} p(s) v_{n-1}(s) \, ds \\
\geq e^n \int_{t_3}^{T-\tau} p(s) v_{n-1}(s) \, ds \int_{t-\tau}^{t} p(t) \, dt \\
= e^n \int_{t_3}^{T-\tau} p(t) \delta(t) \, dt \int_{t-\tau}^{t} p(s) v_{n-2}(s) \, ds
\]
\[ \geq e^n \int_{t_3}^{T-2\tau} p(s) v_n(s) \, ds \int_s^{s+\tau} p(t) \bar{p}_n(t) \, dt \]
\[ = e^n \int_{t_3}^{T-2\tau} p(t) \bar{p}_n(t) v_n(t) \, dt \]

we have
\[ e^n \int_{t_3}^{T} p(t) v_n(t) \, dt \geq e^n \int_{t_3}^{T-n\tau} p(t) v(t) \bar{p}_n(t) \, dt \geq \int_{t_3}^{T-n\tau} p(t) v(t) \, dt. \]  

(21)

Thus,
\[ \int_{t_3}^{T} p(t) [v(t) - e^n v_n(t)] \, dt \leq \int_{t_3}^{T} p(t) v(t) \, dt - \int_{t_3}^{T-n\tau} p(t) v(t) \, dt \]
\[ = \int_{T-n\tau}^{T} p(t) v(t) \, dt. \]

In view of (20), we have
\[ \lim_{T \to \infty} \int_{T-n\tau}^{T} p(t) v(t) \, dt = \infty. \]  

(22)

This shows that either
\[ \lim_{T \to \infty} \int_{T-n\tau}^{T} p(t) \, dt = \infty \]  

(23)

or
\[ \limsup_{t \to \infty} v(t) = \infty. \]  

(24)

If (23) holds, then
\[ \lim_{t \to \infty} \int_{t-\tau}^{t} p(s) \, ds = \infty. \]

By a known result in [8], every solution of (1) oscillates.

If (24) holds, then
\[ \limsup_{t \to \infty} \omega(t) = \infty. \]  

(25)
On the other hand, integrating the both sides of (1) from \( t - \tau \) to \( t \) we have

\[
x(t) - x(t - \tau) + \int_{t-\tau}^{t} p(s)x(s - \tau) \, ds = 0, \quad t \geq t_2,
\]

and so

\[
x(t - \tau) > \int_{t-\tau}^{t} p(s)x(s - \tau) \, ds, \quad t \geq t_2. \tag{26}
\]

From this, by successively substituting \((n-2)\) times and using the decreasing nature of \(x(t)\), it follows that

\[
x(t - \tau) > \int_{t-\tau}^{t} p(s)p_{n-2}(s)x(s - \tau) \, ds
\]

\[
> x(t - \tau)\int_{t-\tau}^{t} p(s)p_{n-2}(s) \, ds,
\]

and so

\[
x(t - \tau) > x(t - \tau)p_{n-1}(t), \quad t \geq t_2 + (n - 2)\tau. \tag{27}
\]

By (6), for any \( t \geq t_1 + \tau \) there exists a \( \xi \in (t - \tau, t) \) such that

\[
\int_{\xi}^{t} p(s)p_{n-1}(s) \, ds \geq \frac{1}{2e^{\tau}}, \quad \int_{t}^{\xi + \tau} p(s)p_{n-1}(s) \, ds \geq \frac{1}{2e^{\tau}}. \tag{28}
\]

By integrating the both sides of (1) over \([\xi, t]\) and \([t, \xi + \tau]\), we have

\[
x(t) - x(\xi) + \int_{\xi}^{t} p(s)x(s - \tau) \, ds = 0, \quad t \geq t_2 + (n - 1)\tau, \tag{29}
\]

and

\[
x(\xi + \tau) - x(t) + \int_{t}^{\xi + \tau} p(s)x(s - \tau) \, ds = 0, \quad t \geq t_2 + (n - 1)\tau. \tag{30}
\]
Substituting (27) into (29) and (30), omitting the first terms in (29) and (30) and using the decreasing nature of \( x(t) \) and (28), we see that

\[
-x(\xi) + \frac{1}{2e^n}x(t - \tau) < 0, \quad -x(t) + \frac{1}{4e^n}x(\xi) < 0,
\]

or

\[
x(t) > \frac{1}{2e^n}x(\xi) > \frac{1}{4e^n}x(t - \tau),
\]

or

\[
\omega(t) < 4e^{2n}, \quad t \geq t_2 + (n - 1)\tau. \tag{31}
\]

This contradicts (25) and completes the proof of the theorem.

**Theorem 2.** Let \( p(t) \in C([t_0, \infty), R^+) \) and let \( \tau \) be a positive constant. Suppose that there exists a \( t > t_0 + \tau \) such that (3) and (7) hold. Then every solution of (1) oscillates.

Because (3) implies (6), Theorem 1 implies Theorem 2.

**Remark 1.** Theorems 1 and 2 improve Theorem A.

**Corollary 1.** Let \( p(t) \in C([t_0, \infty), R^+) \) and let \( \tau \) be a positive constant. Suppose that, for some positive integer \( n \),

\[
\liminf_{t \to \infty} p_n(t) > \frac{1}{e^n} \quad \text{and} \quad \liminf_{t \to \infty} \bar{p}_n(t) > \frac{1}{e^n}, \tag{32}
\]

where \( p_n(t), \bar{p}_n(t) \) are defined by (5). Then every solution of (1) oscillates.

**Remark 2.** Condition (32) improves (2).

**Corollary 2.** Let \( p(t) \in C([t_0, \infty), R^+) \) and let \( \tau \) be a positive constant. If (3) holds, and for some positive integer \( n \),

\[
\int_{t_0 + n\tau}^{\infty} p(t) \left( e^{-n}p_n(t) - \frac{1}{e^n} \right) dt = \infty, \tag{33}
\]

where \( p_n(t) \) is defined by (5). Then every solution of (1) oscillates.

**Corollary 3.** Let \( p(t) \in C([t_0, \infty), R^+) \) and let \( \tau \) be a positive constant. If (6) and (33) hold, then every solution of (1) oscillates.
3. EXAMPLE

Consider the delay differential equation

$$x'(t) + \frac{1}{2e} (1 + \cos t)x(t - \pi) = 0, \quad t \geq 0. \quad (34)$$

Clearly, for $t \geq \pi$,

$$p_1(t) = \int_{t - \pi}^{t} \frac{1}{2e} (1 + \cos s) \, ds = \frac{1}{2e} (\pi + 2 \sin t)$$

$$\lim_{t \to \infty} \int_{t - \pi}^{t} \frac{1}{2e} (1 + \cos s) \, ds = \frac{1}{2e} (\pi - 2) < \frac{1}{e}.$$

This shows that (2) and (3) do not hold. But

$$p_2(t) = \int_{t - \pi}^{t} p(s) p_1(s) \, ds$$

$$= \frac{1}{4e^2} \int_{t - \pi}^{t} (1 + \cos s)(\pi + 2 \sin s) \, ds$$

$$= \frac{1}{4e^2} (\pi^2 + 2 \pi \sin t - 4 \cos t),$$

$$p_3(t) = \int_{t - \pi}^{t} p(s) p_2(s) \, ds$$

$$= \frac{1}{8e^3} \int_{t - \pi}^{t} (1 + \cos s)(\pi^2 + 2 \pi \sin s - 4 \cos s) \, ds$$

$$= \frac{1}{8e^3} \left[ \pi^3 - 2 \pi + (2 \pi^2 - 8) \sin t - 4 \pi \cos t \right],$$

$$p_4(t) = \int_{t - \pi}^{t} p(s) p_3(s) \, ds$$

$$= \frac{1}{16e^4} \int_{t - \pi}^{t} (1 + \cos s) \left[ \pi^3 - 2 \pi + (2 \pi^2 - 8) \sin s - 4 \pi \cos s \right] \, ds$$

$$= \frac{1}{16e^4} \left[ \pi^4 - 4 \pi^2 + 2(\pi^3 - 6 \pi) \sin t - 4(\pi^2 - 4) \cos t \right],$$

$$\lim_{t \to \infty} p_4(t) = \frac{1}{16e^4} \left[ \pi^4 - 4 \pi^2 - 2 \sqrt{(\pi^3 - 6 \pi)^2 + 4(\pi^2 - 4)^2} \right] > \frac{22}{16e^4},$$
and

\[
\bar{p}_1(t) = \int_t^{t+\pi} \frac{1}{2e}(1 + \cos s) \, ds = \frac{1}{2e}(\pi - 2\sin t),
\]

\[
\bar{p}_2(t) = \int_t^{t+\pi} p(s) \bar{p}_1(s) \, ds
= \frac{1}{4e^2} \int_t^{t+\pi} (1 + \cos s)(\pi - 2\sin s) \, ds
= \frac{1}{4e^2}(\pi^2 - 2\pi \sin t - 4\cos t),
\]

\[
\bar{p}_3(t) = \int_t^{t+\pi} p(s) \bar{p}_2(s) \, ds
= \frac{1}{8e^3} \int_t^{t+\pi} (1 + \cos s)(\pi^2 - 2\pi \sin s - 4\cos s) \, ds
= \frac{1}{8e^3}[\pi^3 - 2\pi - (2\pi^2 - 8)\sin t - 4\pi \cos t],
\]

\[
\bar{p}_4(t) = \int_t^{t+\pi} p(s) \bar{p}_3(s) \, ds
= \frac{1}{16e^4} \int_t^{t+\pi} (1 + \cos s)[\pi^3 - 2\pi - (2\pi^2 - 8)\sin s - 4\pi \cos s] \, ds
= \frac{1}{16e^4}[\pi^4 - 4\pi^2 - 2(\pi^3 - 6\pi)\sin t - 4(\pi^2 - 4)\cos t],
\]

\[
\liminf_{t \to \infty} \bar{p}_4(t) = \frac{1}{16e^4}
\left[\pi^4 - 4\pi^2 - 2\sqrt{(\pi^3 - 6\pi)^2 + 4(\pi^2 - 4)^2}\right] > \frac{22}{16e^4}.
\]

Then, by Corollary 1, every solution of (34) oscillates.

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