Topology and its Applications 158 (2011) 2140–2147

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Generic properties of compact metric spaces

Joël Rouyer

16, rue Philippe, 68 220 Hegenheim, France

ARTICLE INFO

Article history:
Received 27 October 2010
Received in revised form 1 July 2011
Accepted 5 July 2011

Keywords:
Gromov–Hausdorff space
Generic properties
Compact metric spaces
Dimension

ABSTRACT

We prove that there is a residual subset of the Gromov–Hausdorff space (*i.e.* the space of all compact metric spaces up to isometry endowed with the Gromov–Hausdorff distance) whose elements enjoy several unexpected properties. In particular, they have zero lower box dimension and infinite upper box dimension.

1. Introduction

It is rather known that a 'generic' continuous real function (or a 'typical' one, or 'most' of them) admits a derivative at no point. This is often stated as a kind of curiosity, for such a function is not so easy to exhibit, or even to fancy. The aim of this paper is to give some properties of a generic compact metric space.

When we say generic, we refer to the notion of Baire categories. We recall that a subset of a topological space B is said to be *rare* if the interior of its closure is empty. It is said to be *meager*, or of *first category*, if it is a countable union of rare subsets of B. The space B is called a *Baire space* if each meager subset of B has empty interior. Baire's theorem states that any complete metric space is a Baire space. The complement of a meager subset of a Baire space is said to be *residual*. At last, given a Baire space B, we say that a *generic* element of B enjoys a property if the set of elements which satisfy this property is residual.

In order to state the results of this article we need a few definitions. We say that a metric space X is *totally anisometric* if two distinct pairs of points have distinct distances. We say that three points $x, y, z \in X$ are *collinear* if one of the three distances between them equals the sum of the two others. Of course, this definition matches the classical one in a Euclidean space. A *perfect set* is a closed set without isolated points. The definitions of the upper and lower box dimensions are recalled in Section 5.

We will prove that a generic compact metric space X:

1. is totally discontinuous (Theorem 1);
2. is totally anisometric (Theorem 2);
3. has no collinear triples of (pairwise distinct) points (Theorem 3);
4. is perfect (Theorem 4);
5. is homeomorphic to the Cantor set (Corollary 5);
6. admits a set of distance values $\{d(x, y) \mid x, y \in X\}$ which is homeomorphic to the Cantor set (Theorem 5);
7. cannot be embedded in any Hilbert space (Theorem 6);

E-mail address: Joel.Rouyer@ymail.com.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.07.003

8. has zero Hausdorff and lower box dimensions (Theorem 7);
9. has infinite upper box dimension (Theorem 7).

Earlier studies were performed on a generic compact subset of some fixed complete metric space. It is known that the points 1, 4 and 5 hold for a generic compact subset of $\mathbb{R}^n$. The first proof in black and white of this fact is due to J.A. Wieacker, who, however, didn't claim its discovery [14]. In the same paper he proved that any subset with $n+1$ elements of a generic subset of $\mathbb{R}^n$ is affinely independent. This implies in particular that the point 3 holds for subsets of $\mathbb{R}^n$.

Almost simultaneously, P.M. Gruber investigated the dimension of a generic subset of a fixed complete metric space X. In [8], he proved that the point 8 holds in this framework. Clearly the point 9 cannot hold for any complete metric space X. However P.M. Gruber proved that, if X has some suitable property (*e.g.* $X = \mathbb{R}^n$), then a generic compact subset of X has an upper box dimension greater than or equal to n.

A decade later, A.V. Kuz'minykh proved that the points 2 and 6 hold in the case of subsets of the Euclidean spaces. As far we know, the point 7 is new.

Many other properties of a generic compact subset K of some fixed ambient space X have been investigated by these authors and some others. One can mention the porousness of K [15], the properties of the nearest point mapping from X to K [3,12,4], the properties of the convex hull of K [14], or the properties of the image of K by the spherical projection with respect to a given point [16,5]. However, all those properties involve the embedding of X in the whole space, and so, admit no counterpart in our framework.

2. The Gromov–Hausdorff space

The section is devoted to recall the definition and the properties of the so-called Gromov–Hausdorff space, *i.e.* the space of all isometry classes of compact metric spaces. We will use the same letter to designate both a metric space (*i.e.* a set endowed with a distance) and its underlying set. If X is a metric space, we denote by d^X its metric. If A is a part of X and ρ is a positive number, we denote by $A+\rho$ the ρ-neighborhood of A, namely

$$A+\rho = \{x \in X \mid \exists y \in A \text{ s.t. } d^X(x,y) < \rho\}.$$

If $A = \{a\}$ is a singleton, we denote by $B(a,\rho) = A+\rho$ the open ball. We recall that the Hausdorff distance $d_H^X(A,B)$ between two nonempty closed bounded subsets A and B of X is the infimum of those numbers ρ such that $A \subset B+\rho$ and $B \subset A+\rho$. It is easy to see that d_H^X is a distance on the set $\mathfrak{M}(X)$ of all nonempty compact subsets of X. Moreover, if X is complete (resp. compact), then $\mathfrak{M}(X)$ is complete (resp. compact) too [2, p. 253].

Let X and Y be two compact metric spaces. The Gromov–Hausdorff distance between them is defined by

$$d_{GH}(X,Y) = \inf d_H^Z\big(f(X), g(Y)\big),$$

where the infimum is taken over all metric spaces Z and all isometric injections $f: X \to Z$ and $g: Y \to Z$. It is known that d_{GH} is a distance on the Gromov–Hausdorff space $\mathfrak{M}$ of all compact metric spaces up to isometry. Moreover, $\mathfrak{M}$ is complete [13, p. 296].

A *correspondence* R between two metric spaces X and Y is a relation (*i.e.* a subset of $X \times Y$) such that each element of X is in relation with at least one element of Y, and conversely, each element of Y is in relation with at least one element of X. For $x \in X$ and $y \in Y$, we write xRy instead of $(x,y) \in R$. The *distortion* of a correspondence R between X and Y is the number

$$\operatorname{dis}(R) = \sup\Big\{\big|d^X(x,x') - d^Y(y,y')\big| \;\Big|\; x,x' \in X,\ y,y' \in Y,\ xRy,\ x'Ry'\Big\}.$$

The above notion is useful for it provides another way to compute the Gromov–Hausdorff distance.

Lemma 1. *([2, p. 257])*

$$d_{GH}(X,Y) = \frac{1}{2}\inf_R \operatorname{dis}(R)$$

where the infimum is taken over all correspondences R between X and Y.

Another useful result is the following

Lemma 2. *Let $(X_n)_{n\in\mathbb{N}}$ be a converging sequence of elements of $\mathfrak{M}$, and denote by Y its limit. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers. Then, there exist a compact metric space Z, an isometric embedding $g: Y \to Z$, and for each positive integer n, an isometric embedding $f_n: X_n \to Z$, such that $d_H^Z(f_n(X_n), g(Y)) < d_{GH}(X_n,Y) + \varepsilon_n$.*

Proof. First, we can assume without loss of generality that ε_n converges to 0. For each integer n there exists a compact metric space Z_n and two isometric injections $f'_n: X_n \to Z_n$ and $g'_n: Y \to Z_n$ such that

$$d_H^{Z_n}\big(f'_n(X_n), g'_n(Y)\big) < d_{GH}(X_n,Y) + \varepsilon_n.$$

We also assume that Z_n is minimum for inclusion, that is $Z_n = f'_n(X_n) \cup g'_n(Y)$. Let $Z' = \coprod_{n\in\mathbb{N}} Z_n$, endowed with the pseudo-distance

$$d^{Z'}(a,b) = d^{Z_n}(a,b), \quad \text{if } a,b \in Z_n,$$
$$d^{Z'}(a,b) = \min_{y\in Y}\bigl(d^{Z_n}\bigl(a, g'_n(y)\bigr) + d^{Z_m}\bigl(g'_m(y), b\bigr)\bigr), \quad \text{if } a \in Z_n \text{ and } b \in Z_m \text{ with } m \neq n.$$

We have called $d^{Z'}$ a pseudo-distance, but of course, this should be checked. Since the verification is straightforward, it is left to the reader. Let Z be the quotient of Z' by the equivalence relation $\sim$ defined by

$$a \sim b \iff d^{Z'}(a,b) = 0,$$

and let $\pi : Z' \to Z$ be the canonical surjection. We define $f_n = \pi \circ f'_n$ and $g = \pi \circ g'_n$ (g does not depend on n, since $d^{Z'}(g'_n(y), g'_m(y)) = 0$). It is clear that f_n and g are isometric embeddings, and that

$$d_H^Z\bigl(f_n(X_n), g(Y)\bigr) < d_{GH}(X_n, Y) + \varepsilon_n.$$

It remains to prove that Z is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in Z. Either there exists an integer n such that all but a finite number of terms of $(z_k)_{k\in\mathbb{N}}$ belong to $\pi(Z_n)$, or one can extract from $(z_k)_{k\in\mathbb{N}}$ a subsequence such that $z_k \in \pi(Z_{\nu(k)})$, where $\nu : \mathbb{N} \to \mathbb{N}$ is increasing. In the former case, since Z_n is compact, we can extract from (z_k) a converging subsequence, and the proof is over. In the latter case, there exists a sequence $(y_k)_{k\in\mathbb{N}}$ of points of Y such that $d^{Z_{\nu(k)}}(z'_k, g'_{\nu(k)}(y_k)) \leqslant d_{GH}(X_{\nu(k)}, Y) + \varepsilon_{\nu(k)}$, where $z'_k \in Z_{\nu(k)}$ is such that $\pi(z'_k) = z_k$. By extracting a suitable subsequence we may assume that y_k is converging to some point $y \in Y$. It follows that

$$\begin{aligned} d^Z\bigl(z_k, g(y)\bigr) &\leqslant d^{Z_{\nu(k)}}\bigl(z'_k, g'_{\nu(k)}(y_k)\bigr) + d^Z\bigl(g(y_k), g(y)\bigr) \\ &\leqslant d_{GH}(X_{\nu(k)}, Y) + \varepsilon_{\nu(k)} + d^Y(y_k, y) \to 0, \end{aligned}$$

whence $(z_k)_{k\in\mathbb{N}}$ is converging. Hence Z is compact. □

3. Finite spaces

The subset $\mathfrak{M}_F \subset \mathfrak{M}$ of finite metric spaces is playing a key role in the study of $\mathfrak{M}$ because it is dense in $\mathfrak{M}$ and simple enough to be described by mean of matrices.

We define the *codiameter* of a finite metric space X as the minimum of all non-zero distances in X: $\operatorname{cdm}(X) = \min_{x\neq y\in X} d^X(x,y)$.

A *distance matrix* is a symmetric matrix $D = (d_{ij})_{1\leqslant i\leqslant n,\, 1\leqslant j\leqslant n}$ with 0's on the (main) diagonal, and positive numbers elsewhere, such that for all indices i, j, k we have $d_{ij} \leqslant d_{ik} + d_{kj}$. We say that two distance matrices are equivalent, if we can pass from one to the other by applying the same permutation simultaneously to its rows and its columns.

We clearly can associate to a distance matrix D of order n the metric space X_D defined by

$$X_D = \{1, \ldots, n\},$$
$$d^{X_D}(i,j) = d_{ij}.$$

The spaces X_D and $X_{D'}$ are isometric if and only if D and D' are equivalent. Conversely, given a finite metric space $X = \{x_1, \ldots, x_n\}$, we can associate to it the distance matrix, $D = (d^X(x_i, x_j))_{1\leqslant i\leqslant n,\, 1\leqslant j\leqslant n}$. Of course D depends on the order in which the points of X are labeled, but not its class of equivalence. Moreover, two isometric spaces X and X' give the same class of equivalence of distance matrices. Hence the set $\mathfrak{M}_n \subset \mathfrak{M}$ of those metric spaces with cardinality n is bijectively mapped on the set of equivalence classes of distance matrices of order n. Furthermore the inequality

$$d_{GH}(X_D, X_{D'}) \leqslant \frac{1}{2}\max_{i,j}\bigl|d_{ij} - d'_{ij}\bigr| = \frac{1}{2}\bigl\|D - D'\bigr\|_\infty \tag{1}$$

follows from Lemma 1.

Conversely, let X and Y be two finite metric spaces with n elements, such that $d_{GH}(X,Y) < \frac{1}{2}\operatorname{cdm}(Y)$. Let θ be a real number such that $d_{GH}(X,Y) < \theta < \frac{1}{2}\operatorname{cdm}(Y)$. By definition of d_{GH}, there exist a metric space Z, and two subsets $X' = \{x_1, \ldots, x_n\}$ and $Y' = \{y_1, \ldots, y_n\}$ of Z, such that X is isometric to X', Y is isometric to Y' and $d_H^Z(X', Y') < \theta$. It follows that for each $i \in \{1, \ldots, n\}$, there exists $j \in \{1, \ldots, n\}$ such that $d^Z(x_i, y_j) < \theta$. Moreover j is unique: assume on the contrary that two such indices j_1 and j_2 exist, then

$$2\theta < \operatorname{cdm}(Y) \leqslant d^Y(y_{j_1}, y_{j_2}) \leqslant d^Z(y_{j_1}, x_i) + d^Z(x_i, y_{j_2}) < 2\theta.$$

Hence, by changing the labeling of elements of Y, we can assume that $d^Z(x_i, y_i)$ is less than θ for all indices i. Let D_X and D_Y be the distance matrices of X and Y corresponding to this order, then

$$\frac{1}{2}\|D_X - D_Y\|_\infty \leqslant \max_{i,j}\frac{1}{2}\bigl(d^Z(x_i, y_i) + d^Z(x_j, y_j)\bigr) < \theta.$$

Since this holds for all θ greater than $d_{GH}(X_D, X_{D'})$, it follows that

$$\frac{1}{2}\|D_X - D_Y\|_\infty \leqslant d_{GH}(X, Y).$$

This and (1) prove that, if $d_{GH}(X_D, X_{D'}) \leqslant \frac{1}{2}\operatorname{cdm}(X_D)$, then

$$d_{GH}(X_D, X_{D'}) = \frac{1}{2}\min_{D''\sim D'}\left\|D - D''\right\|_\infty. \tag{2}$$

In other words, the bijection between $\mathfrak{M}_n$ and the set of equivalence classes of distance matrix of order n is a local similitude. We will use this fact to prove the

Lemma 3. *Let $X \in \mathfrak{M}_n$ be a finite metric space with cardinality n and ε be a positive number. There exits a ball $B_0 \subset B(X,\varepsilon) \subset \mathfrak{M}_n$, each point of which is totally anisometric and without triples of collinear points.*

Proof. Put $m = \frac{n(n-1)}{2}$. Let D be a distance matrix associated to X. Since the space $\mathcal{D}_n$ of distance matrices of order n is defined by a finite number of linear inequalities ($d_{ij} > 0$, $d_{ij} + d_{jk} \geqslant d_{ik}$) it is a convex polytope of the set $\mathcal{S}_n$ of all symmetric matrices with zero on the diagonal, which in turn is isomorphic to $\mathbb{R}^m$ as a vector space. Moreover the distance matrix with 0 on the diagonal and 1 elsewhere clearly belongs to the interior of $\mathcal{D}_n$, that is therefore nonempty. Let R_1 be the union of the $\frac{(m-1)m}{2}$ hyperplanes of $\mathcal{S}_n$ defined by the equations $d_{ij} = d_{kl}$ ($1 \leqslant i < j \leqslant n$, $1 \leqslant k < l \leqslant n$, $(i,j) < (k,l)$) and R_2 the union of the $(n-2)m$ hyperplanes defined by the equations $d_{ij} + d_{jk} = d_{ik}$ ($1 \leqslant i < k \leqslant n$, $1 \leqslant j \leqslant n$, $j \neq k$, $j \neq i$). The matrix of a metric space of cardinality n which is not totally anisometric (resp. admits a triple of collinear points) should belong to R_1 (resp. R_2). Since $R_1 \cup R_2$ is clearly rare, there exists a ball $B_1 = B(\Delta, 2\eta)$ included in $B(D,\varepsilon) \cap (\mathcal{D}_n \backslash (R_1 \cup R_2))$. Assume moreover that $\eta < \frac{1}{2}\operatorname{cdm}(X_\Delta)$. Let $Y \in B_0 \stackrel{def}{=} B(X_\Delta, \eta) \subset \mathfrak{M}_n$. By (2), there exists a distance matrix D_Y associated to Y such that $\|\Delta - D_Y\| = 2d_{GH}(X_\Delta, Y) < 2\eta$. Whence $D_Y \in B_1$ and Y is totally anisometric and without collinear points. □

Corollary 4. *Totally anisometric spaces without collinear points are dense in $\mathfrak{M}$.*

4. Basic properties of generic compact metric spaces

Theorem 1. *A generic compact metric space is totally discontinuous.*

Proof. Let $P_n \subset \mathfrak{M}$ be the set of compact metric spaces admitting a connected component of diameter at least $\frac{1}{n}$. Since $\mathfrak{M}_F$ is dense in $\mathfrak{M}$, P_n has empty interior. The union $\bigcup_{n\in\mathbb{N}} P_n$ is the complement of the set of totally discontinuous compact metric spaces, thus we only need to prove that P_n is closed. Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of elements of P_n, converging to $X \in \mathfrak{M}$. Let $C_k \subset X_k$ be a closed connected subset whose diameter is at least $\frac{1}{n}$. By Lemma 2, we can assume without loss of generality that all X_k and X are subsets of a compact metric space Z and that $d_H^Z(X_k, X) \leqslant d_{GH}(X_k, X) + \frac{1}{k}$. Since $\mathfrak{M}(Z)$ is compact, we can extract from (C_k) a converging subsequence. Let C be its limit, it is easy to see that $C \subset X$. Since the diameter function is continuous, it is clear that $\operatorname{diam}(C) = \lim \operatorname{diam}(C_k) \geqslant \frac{1}{n}$. Moreover, it is a well-known fact that the set of connected compact subsets of Z is closed in $\mathfrak{M}(Z)$ [10]. Hence C is connected and X belongs to P_n. Thus P_n is closed. □

Theorem 2. *A generic compact metric space is totally anisometric.*

Proof. Let $P \subset \mathfrak{M}$ be the set of compact metric spaces which are not totally anisometric.

$$P = \left\{ X \;\middle|\; \begin{array}{r} \exists x, y, x', y' \in X \; d(x,y) = d(x',y') > 0 \\ \text{and } d(x,x') + d(y,y') > 0 \\ \text{and } d(x,y') + d(x',y) > 0 \end{array} \right\} = \bigcup_{n\in\mathbb{N}} P_{\frac{1}{n}},$$

where

$$P_\varepsilon = \left\{ X \;\middle|\; \begin{array}{r} \exists x, y, x', y' \in X \; d(x,y) = d(x',y') \geqslant \varepsilon \\ \text{and } d(x,x') + d(y,y') \geqslant \varepsilon \\ \text{and } d(x,y') + d(x',y) \geqslant \varepsilon \end{array} \right\}.$$

By virtue of Corollary 4, it is sufficient to prove that P_ε is closed. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of elements of P_ε tending to $X \in \mathfrak{M}$. There exists a sequence of correspondences R_n from X to X_n such that $\operatorname{dis}(R_n)$ tends to zero. Let $x_n, y_n, x'_n, y'_n \in X_n$ be such that

$$d(x_n, y_n) = d(x'_n, y'_n) \geqslant \varepsilon,$$
$$d(x_n, x'_n) + d(y_n, y'_n) \geqslant \varepsilon,$$
$$d(y_n, x'_n) + d(x_n, y'_n) \geqslant \varepsilon. \tag{3}$$

There exists $\tilde{x}_n, \tilde{x}'_n, \tilde{y}_n, \tilde{y}'_n \in X$ such that $\tilde{x}_n R_n x_n$, $\tilde{y}_n R_n y_n$, $\tilde{x}'_n R_n x'_n$, and $\tilde{y}'_n R_n y'_n$. Let $(\tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}')$ be the limit of a converging subsequence of $(\tilde{x}_n, \tilde{x}'_n, \tilde{y}_n, \tilde{y}'_n)$. Since $\operatorname{dis}(R_n)$ tends to zero, we can pass to the limit in (3):

$$d(\tilde{x}, \tilde{y}) = d(\tilde{x}', \tilde{y}') \geqslant \varepsilon,$$
$$d(\tilde{x}, \tilde{x}') + d(\tilde{y}, \tilde{y}') \geqslant \varepsilon,$$
$$d(\tilde{y}, \tilde{x}') + d(\tilde{x}, \tilde{y}') \geqslant \varepsilon.$$

Hence X belongs to P_ε. This completes the proof. □

Theorem 3. *In a generic compact metric space, three distinct points are never collinear.*

Proof. By Corollary 4, it is sufficient to prove that

$$P_\varepsilon \stackrel{def}{=} \left\{ X \in \mathfrak{M} \;\middle|\; \begin{array}{r} \exists x, y, z \in X \; d(x, y) = d(x, z) + d(z, y) \\ \text{and } d(x, z) \geqslant \varepsilon \text{ and } d(x, z) \geqslant \varepsilon \end{array} \right\}$$

is closed. The proof is totally similar to the one of Theorem 2. □

Theorem 4. *A generic compact metric space is perfect.*

Proof. Let $P_n = \{X \in \mathfrak{M} \mid \exists x \in X \text{ s.t. } \forall x' \in X \; d(x, x') \in A_n\}$, where $A_n \stackrel{def}{=} \{0\} \cup]\frac{1}{n}, +\infty[$. The set of non-perfect compact metric spaces is the union $\bigcup_{n \in \mathbb{N}} P_n$. It is therefore sufficient to prove that P_n is rare.

We claim that the set of perfect compact metric spaces is dense in $\mathfrak{M}$, and so the interior of P_n is empty. Indeed, it is sufficient to prove that any finite metric space F can be approximated by perfect spaces. Let ε be less than the codiameter of F and endow the product $F_\varepsilon \stackrel{def}{=} F \times [0, \varepsilon]$ with the distance

$$d^{F_\varepsilon}\big((a, s), (b, t)\big) = d^F(a, b) + s + t, \quad \text{if } a \neq b,$$
$$d^{F_\varepsilon}\big((a, t), (a, s)\big) = |t - s|.$$

It is easy to see that d^{F_ε} is a distance on F_ε, that F_ε is perfect and that $d_{GH}(F_\varepsilon, F) \leqslant \varepsilon$. This proves the claim.

It remains to prove that P_n is closed. Let $(X_k)_{k \in \mathbb{N}}$ be a sequence of elements of P_n converging to $Y \in \mathfrak{M}$. Let R_k be a correspondence between X_k and Y such that $\varepsilon_k \stackrel{def}{=} \operatorname{dis}(R_k) \leqslant 3 d_{GH}(X_k, Y)$. Let $x_k \in X_k$ be such that for all $x' \in X_k$, $d^{X_k}(x_k, x') \in A_n$. Let $y_k \in Y$ correspond to x_k by R_k. By extracting a suitable subsequence, we can assume that the sequence $(y_k)_{k \in \mathbb{N}}$ converges to some point $y \in Y$. Let y' be a point of Y and let $x'_k \in X_k$ correspond to y' by R_k. We have

$$\begin{aligned} d^Y(y, y') &\in \{d^Y(y_k, y')\} + 2 d^Y(y, y_k) \\ &\subset \{d^{X_k}(x_k, x'_k)\} + 2\big(d^Y(y, y_k) + \varepsilon_k\big) \\ &\subset A_n + 2\big(d^Y(y, y_k) + \varepsilon_k\big). \end{aligned}$$

Since the relation holds for arbitrary large k, we have $d^Y(y, y') \in A_n$, whence $Y \in P_n$. □

Corollary 5. *A generic compact metric space is a Cantor space.*

Proof. Brouwer's theorem [9, (7.4)], states that a topological space is a Cantor space if and only if it is nonempty, perfect, compact, totally disconnected, and metrizable. Thus the result follows from Theorems 1 and 4. □

Theorem 5. *Let X be a generic compact metric space. The set $d(X) \stackrel{def}{=} \{d^X(x, y) \mid x, y \in X\}$ is homeomorphic to the Cantor set.*

Proof. By virtue of Brouwer's theorem we quote in the proof of Corollary 5, it is sufficient to prove that $d(X)$ is totally discontinuous and perfect.

Let $P_{a,\varepsilon} = \{X \in \mathfrak{M} \mid [a, a + \varepsilon] \subset d(X)\}$. For $\varepsilon > 0$, $\mathfrak{M}_F \cap P_{a,\varepsilon} = \emptyset$, whence $P_{a,\varepsilon}$ has empty interior. The set $d(X)$ is not totally discontinuous if and only if X belongs to the countable union $\bigcup_{a \geqslant 0, a \in \mathbb{Q}} \bigcup_{n \in \mathbb{N}} P_{a, \frac{1}{n}}$. Hence we only have to prove that $P_{a,\varepsilon}$ is closed. Let $(X_k)_{k \in \mathbb{N}}$ be a sequence of elements of $P_{a,\varepsilon}$ converging to $X \in \mathfrak{M}$. Let R_k be a correspondence between X_k and X such that $\operatorname{dis}(R_k)$ tends to zero. Let b be a number of $[a, a+\varepsilon]$. By hypothesis, there exist $x_k, y_k \in X_k$ such that $d^{X_k}(x_k, y_k) = b$. Let $x'_k, y'_k \in X$ correspond by R_k to x_k and y_k respectively, and extract from (x'_k) and (y'_k) some subsequences converging to x and y respectively. Since

$$\begin{aligned}\left|b - d^X\left(x'_k, y'_k\right)\right| &= \left|d^{X_k}(x_k, y_k) - d^X\left(x'_k, y'_k\right)\right| \\ &\leqslant \operatorname{dis}(R_k) \to 0,\end{aligned}$$

$d^X(x, y) = b$, and $b \in d(X)$. This holds for all b in $[a, a+\varepsilon]$, whence $X \in P_{a,\varepsilon}$. Hence $P_{a,\varepsilon}$ is closed. It follows that for a generic $X \in \mathfrak{M}$, $d(X)$ is totally discontinuous.

We will now prove that $d(X)$ is perfect. Note that $d(X) = \bigcup_{x_0 \in X} f_{x_0}(X)$, where $f_{x_0} : X \to \mathbb{R}$ is the distance function from $x_0 \in X$. So, it is sufficient to prove that $f_{x_0}(X)$ is perfect. Since by Theorem 2 a generic $X \in \mathfrak{M}$ is totally anisometric, the functions f_{x_0} are injective, and so are homomorphisms between X and $f_{x_0}(X)$. Theorem 4 completes the proof. □

Theorem 6. *The set of compact metric spaces which contain a 4 points subspace that cannot be isometrically imbedded in $\mathbb{R}^3$ is open and dense in $\mathfrak{M}$. Therefore, a generic compact metric space cannot be embedded in any Hilbert space.*

Proof. It is well known (see for instance [1, (10.6.5)]) that the square of the volume of a (possibly degenerated) tetrahedron of $\mathbb{R}^3$ is given by the following formula (the so-called Cayley–Menger determinant)

$$\phi\left(r, s, t, r', s', t'\right) = \frac{1}{288}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & t^2 & s^2 & r'^2 \\ 1 & t^2 & 0 & r^2 & s'^2 \\ 1 & s^2 & r^2 & 0 & t'^2 \\ 1 & r'^2 & s'^2 & t'^2 & 0 \end{vmatrix},$$

where r, s and t are the lengths of the sides of a one of the faces of the tetrahedron, and r', s', t' are the lengths of the edges respectively opposite to the ones of length r, s, t. Whereas ϕ was initially defined only for the sextuples (r, s, t, r', s', t') which actually correspond to a tetrahedron, as a polynomial function, it can be extended to $\mathbb{R}^6$. Given a metric space $A = \{a_0, a_1, a_2, a_3\} \in \mathfrak{M}_4$, we put

$$\phi(A) \overset{def}{=} \phi\left(d^A(a_1, a_2), d^A(a_2, a_3), d^A(a_3, a_1), d^A(a_0, a_3), d^A(a_0, a_1), d^A(a_0, a_2)\right).$$

If $\phi(A) < 0$, then surely A cannot be isometrically embedded in $\mathbb{R}^3$, nor in any Hilbert space. We will prove that the set

$$P \overset{def}{=} \left\{X \in \mathfrak{M} \mid \exists A \subset X \ \operatorname{card}(A) = 4 \text{ and } \phi(A) < 0\right\}$$

is open and dense in $\mathfrak{M}$.

Let X be in P and $A = \{a_0, a_1, a_2, a_3\} \subset X$, such that $\operatorname{card}(A) = 4$ and $\phi(A) < 0$. Since ϕ is continuous, there exists $\eta > 0$ such that, for any $A' \in \mathfrak{M}_4$, if there exists a correspondence between A and A' whose distortion is less than η, then $\phi(A') < 0$. Let $Y \in U \overset{def}{=} B(X, \frac{1}{3}\min(\eta, \operatorname{cdm}(A)))$; there exists a correspondence R of distortion less than $\min(\eta, \operatorname{cdm}(A))$ between X and Y. Let $A' = \{a'_0, a'_1, a'_2, a'_3\} \subset Y$ be such that $a_i R a'_i$, $i = 0, \ldots, 3$. Since $\operatorname{dis}(R) < \operatorname{cdm}(A)$, A' is a 4 points set, and since $\operatorname{dis}(R) < \eta$, $\phi(A') < 0$. Hence $U \subset P$. It follows that P is open.

Denote by A_ε that the 4 points space whose distance matrix is

$$\begin{pmatrix} 0 & 2\varepsilon & 2\varepsilon & \varepsilon \\ 2\varepsilon & 0 & 2\varepsilon & \varepsilon \\ 2\varepsilon & 2\varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix}.$$

A direct computation shows that $\phi(A_\varepsilon) = -\frac{1}{9}\varepsilon^6$, and so, any space in which A_ε is isometrically embedded belongs to P. In order to show that P is dense, it is sufficient to prove that any finite metric space F can be approached by elements of P. Let $F = \{x_0, \ldots, x_n\} \in \mathfrak{M}_F$. For each $\varepsilon > 0$ we endow $F_\varepsilon \overset{def}{=} \{y_1, y_2, y_3, x_0, x_1, \ldots, x_n\}$ with the distance

$$\begin{aligned} d^{F_\varepsilon}(x_i, x_j) &= d^F(x_i, x_j) \quad (0 \leqslant i, j \leqslant n), \\ d^{F_\varepsilon}(x_i, y_j) &= d^F(x_i, x_0) + \varepsilon \quad (0 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant 3), \\ d^{F_\varepsilon}(y_i, y_j) &= 2\varepsilon \quad (1 \leqslant i, j \leqslant 3,\ i \neq j). \end{aligned}$$

It is easy to check that d^{F_ε} is a distance on F_ε. Moreover A_ε is embedded (as $\{y_1, y_2, y_3, x_0\}$) in F_ε, whence $F_\varepsilon \in P$. At last $d_{GH}(F, F_\varepsilon) \leqslant \varepsilon$, whence P is dense in $\mathfrak{M}$. □

5. Dimensions

We can associate to a compact metric space several (possibly coinciding) numbers which all deserve to be called *dimension*. Among the most used, one distinguishes the *topological dimension* ($\dim_T$), the *Hausdorff dimension* ($\dim_H$), the *lower* ($\dim_B$) or *upper* ($\dim^B$) *box dimension*. It is a well-known fact that for any compact metric space X

$$\dim_T(X) \leqslant \dim_H(X) \leqslant \dim_B(X) \leqslant \dim^B(X).$$

We refer to [11] or [6] for more details on this subject. For our purpose, we only need to recall that the upper and lower box (also called *box-counting*, *fractal* [11], *entropy* [8], *capacity*, *Kolmogorov*, *Minkowski*, or *Minkowski–Bouligand*) dimensions are defined as

$$\dim^B(X) = -\limsup_{\varepsilon\to 0} \frac{\log N(X,\varepsilon)}{\log \varepsilon},$$
$$\dim_B(X) = -\liminf_{\varepsilon\to 0} \frac{\log N(X,\varepsilon)}{\log \varepsilon},$$

where

$$N(X,\varepsilon) = \min\bigl\{\operatorname{card}(F) \bigm| F \subset X\ \forall x \in X\ d(x,F) \leqslant \varepsilon\bigr\}$$

stands for the minimum number of closed balls of radius ε which are required to cover X. It is easy to see that, for a given space X, the function $N(X,\bullet)$ is non-increasing and left-continuous.

If we put

$$M(X,\varepsilon) = \max\bigl\{\operatorname{card}(F) \bigm| F \subset X \text{ and } \operatorname{cdm}(F) \geqslant \varepsilon\bigr\},$$

we may replace N by M in the definitions of $\dim^B$ and $\dim_B$. This fact follows from the inequalities

$$N(X,\varepsilon) \leqslant M(X,\varepsilon) \leqslant N(X,\varepsilon/3),$$

which in turn, follow with a little effort from the definitions of M and N [8, p. 152].

The generic dimension of some compact subset of some fixed complete metric space has been studied in [8] by P.M. Gruber. He proved that, given a complete metric space X such that $\{A \in \mathfrak{M}(X) \mid \dim_B A \geqslant \alpha\}$ is dense in $\mathfrak{M}(X)$, a generic element of $\mathfrak{M}(X)$ has zero lower box dimension, and an upper box dimension greater than or equal to α. In this section, we transpose his result in the frame of the Gromov–Hausdorff space.

The result of Gruber is based on the following three lemmas

Lemma 6. *([8, p. 153]) Given a complete metric space X and a positive number ε, the functions $N(\bullet,\varepsilon): \mathfrak{M}(X) \to \mathbb{N}$ and $M(\bullet,\varepsilon): \mathfrak{M}(X) \to \mathbb{N}$ are respectively lower and upper semi-continuous.*

Lemma 7. *([7, p. 20]) Let B be a Baire space. Let $\alpha_1, \alpha_2, \ldots$ be positive real constants and $\phi_1, \phi_2, \ldots$ be non-negative upper-continuous real functions on B such that $\{x \in B \mid \phi_n(x) = o(\alpha_n)\}$ is dense in B. Then, for a generic point of B, the inequality $\phi_n(x) < \alpha_n$ holds for infinitely many n.*

Lemma 8. *([7, p. 20]) Let B be a Baire space. Let $\beta_1, \beta_2, \ldots$ be non-negative real constants and $\psi_1, \psi_2, \ldots$ be positive lower-continuous real functions on B such that $\{x \in B \mid \beta_n = o(\psi_n(x))\}$ is dense in B. Then, for a generic point of B, the inequality $\beta_n < \psi_n(x)$ holds for infinitely many n.*

We first transfer Lemma 6 to the Gromov–Hausdorff framework.

Lemma 9. *Given a positive number ε, the functions $N(\bullet,\varepsilon): \mathfrak{M} \to \mathbb{N}$ and $M(\bullet,\varepsilon): \mathfrak{M} \to \mathbb{N}$ are respectively lower and upper semi-continuous.*

Proof. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of compact metric spaces converging to X with respect to d_{GH}. By Lemma 2, we can assume without loss of generality that all X_n and X are subsets of some compact metric space Z, such that $d_H^Z(X_n, X)$ tends to zero. Hence, we can apply Lemma 6 in Z; it follows that $M(X,\varepsilon) \geqslant \limsup M(X_n,\varepsilon)$ and $N(X,\varepsilon) \leqslant \liminf N(X_n,\varepsilon)$. □

Theorem 7. *The lower box dimension of a generic compact metric space is zero, while its upper box dimension is infinite.*

Proof. The proof follows rather closely Gruber's one. Let $\tau > 0$. Since $\mathfrak{M}_F$ is included in

$$\left\{X \in \mathfrak{M} \mid M\left(X, \frac{1}{n}\right) = o\left(n^\tau\right)\right\},$$

this set is dense in $\mathfrak{M}$. Applying Lemma 7, we obtain that for a generic $X \in \mathfrak{M}$, the inequality $M(X, \frac{1}{n}) < n^\tau$ holds for infinitely many n, whence

$$\dim_B X \leqslant \liminf_{n\to\infty} -\frac{\log M(X, \frac{1}{n})}{\log \frac{1}{n}} \leqslant \tau.$$

In other words, the set $P_\tau \stackrel{def}{=} \{X \in \mathfrak{M} \mid \dim_B X > \tau\}$ is meager, and thus the set

$$\{X \in M \mid \dim_B X > 0\} = \bigcup_{k\in\mathbb{N}} P_{1/k}$$

is meager too.

Let D be a positive integer. We claim that set of D-dimensional (for any of the aforementioned notion of dimension) compact set is dense in $\mathfrak{M}$. Indeed, it is sufficient to prove that any finite metric space can be approximated by a D-dimensional one. Let F be a finite metric space, let $B \subset \mathbb{R}^D$ be a D-dimensional ball centered at 0 whose radius ε is less than $\operatorname{cdm}(F)$, and endow the product $F \times B$ with the distance

$$d^{F\times B}\big((a,u),(a',u')\big) = d^F(a,a') + \|u\| + \|u'\|, \quad \text{if } a \neq a',$$
$$d^{F\times B}\big((a,u),(a,u')\big) = \|u - u'\|.$$

Its easy to see that $d^{F\times B}$ is actually a distance, and that $d_{GH}(F, F\times B) \leqslant d_H^{F\times B}(F\times\{0\}, F\times B) = \varepsilon$. Moreover, as a disjoint union of D-dimensional balls, $F \times B$ is D-dimensional. This proves the claim.

If $\dim_B X = D$, then for n large enough $n^{\frac{2D}{3}} < N(X, \frac{1}{n})$, whence $n^{\frac{D}{3}} = o(n^{\frac{2D}{3}}) = o(N(X, \frac{1}{n}))$. It follows that the set of D-dimensional compact metric spaces is included in

$$\left\{X \in \mathfrak{M} \;\middle|\; n^{\frac{D}{3}} = o\left(N\left(X, \frac{1}{n}\right)\right)\right\},$$

which is thereby dense in $\mathfrak{M}$. Applying Lemma 8 we obtain that, for a generic metric space X, the inequality $N(X, \frac{1}{n}) > n^{D/3}$ holds for infinitely many n. Hence $\dim^B X \geqslant \limsup_{n\to\infty} -\frac{\log N(X, \frac{1}{n})}{\log \frac{1}{n}} \geqslant \frac{D}{3}$. In other words the set $Q_D \stackrel{def}{=} \{X \in M \mid \dim^B X < \frac{D}{3}\}$ is meager, and thus the set

$$\{X \in M \mid \dim^B X < \infty\} = \bigcup_{D\in\mathbb{N}} Q_D$$

is meager too. □

References

[1] Marcel Berger, Geometry. I, Universitext, Springer-Verlag, Berlin, 1994, translated from the 1977 French original by M. Cole and S. Levy, corrected reprint of the 1987 translation.
[2] Dmitri Burago, Yuri Burago, Sergei Ivanov, A Course in Metric Geometry, Grad. Stud. Math., vol. 33, American Mathematical Society, Providence, RI, 2001.
[3] F.S. De Blasi, T.I. Zamfirescu, Cardinality of the metric projection on typical compact sets in Hilbert spaces, Math. Proc. Cambridge Philos. Soc. 126 (1) (1999) 37–44.
[4] F.S. de Blasi, N.V. Zhivkov, Typical compacta with dense ambiguous loci, Rev. Roumaine Math. Pures Appl. 50 (5–6) (2005) 555–564, MR 2204135 (2006j:41047).
[5] F.S. de Blasi, N.V. Zhivkov, On typical sets in Banach spaces and a parametric Kuratowski–Ulam theorem, Monatsh. Math. 158 (4) (2009) 349–365, MR 2563186 (2011b:46025).
[6] Kenneth Falconer, Fractal Geometry. Mathematical Foundations and Applications, second ed., John Wiley & Sons Inc., Hoboken, NJ, 2003.
[7] Peter M. Gruber, In most cases approximation is irregular, Rend. Semin. Mat. Univ. Politec. Torino 41 (1) (1983) 19–33, (1984).
[8] Peter M. Gruber, Dimension and structure of typical compact sets, continua and curves, Monatsh. Math. 108 (2–3) (1989) 149–164.
[9] Alexander S. Kechris, Classical Descriptive Set Theory, Grad. Texts in Math., vol. 156, Springer-Verlag, New York, 1995.
[10] Ernest Michael, Topologies on spaces of subsets, Trans. Amer. Math. Soc. 71 (1951) 152–182.
[11] Józef Myjak, Some new concepts of dimension, in: Differential Equations, Chaos and Variational Problems, in: Progr. Nonlinear Differential Equations Appl., vol. 75, Birkhäuser, Basel, 2008, pp. 317–344.
[12] Józef Myjak, Ryszard Rudnicki, On the typical structure of compact sets, Arch. Math. (Basel) 76 (2) (2001) 119–126.
[13] Peter Petersen, Riemannian Geometry, second ed., Grad. Texts in Math., vol. 171, Springer, New York, 2006.
[14] J.A. Wieacker, The convex hull of a typical compact set, Math. Ann. 282 (4) (1988) 637–644.
[15] Tudor Zamfirescu, How many sets are porous?, Proc. Amer. Math. Soc. 100 (2) (1987) 383–387.
[16] Tudor Zamfirescu, The strange aspect of most compacta, J. Math. Soc. Japan 57 (3) (2005) 701–708.