A new application of δ-quasi-monotone and almost increasing sequences

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ABSTRACT
In Bor (2011) [5], we proved a main theorem dealing with absolute Nörlund summability factors of infinite series. In the present paper, this result has been further generalized under weaker conditions by using δ-quasi-monotone sequences.

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1. Introduction

A positive sequence \((b_n)\) is said to be almost increasing if there exist a positive increasing sequence \((c_n)\) and two positive constants \(A\) and \(B\) such that \(Ac_n \leq b_n \leq Bc_n\) (see [1]). Obviously, every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking an example, say \(b_n = ne^{(-1)n}\). A sequence \((d_n)\) of positive numbers is said to be δ-quasi-monotone if \(d_n > 0\) ultimately and \(\Delta d_n \geq -\delta_n\), where \((\delta_n)\) is a sequence of positive numbers (see [2]). Let \(\sum a_n\) be a given infinite series with the sequence of partial sums \((s_n)\) and \(w_n = na_n\). By \(u^\alpha_n\) and \(t^\alpha_n\) we denote the \(n\)th Cesàro means of order \(\alpha\), with \(\alpha > -1\), of the sequences \((s_n)\) and \((na_n)\), respectively, i.e.,

\[
\begin{align*}
\quad u^\alpha_n &= \frac{1}{A^\alpha_n} \sum_{v=0}^{n} A^{\alpha-1}_{n-v} s_v, \\
\quad t^\alpha_n &= \frac{1}{A^\alpha_n} \sum_{v=1}^{n} A^{\alpha-1}_{n-v} v a_v,
\end{align*}
\]

where

\[
A^\alpha_n = O(n^\alpha), \quad A^\alpha_0 = 1 \quad \text{and} \quad A^\alpha_{-n} = 0.
\]

The series \(\sum a_n\) is said to be \(|C, \alpha|_k\) summable, \(k \geq 1\), if (see [3])

\[
\sum_{n=1}^{\infty} n^{k-1} |u^\alpha_n - u^\alpha_{n-1}|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t^\alpha_n|^k < \infty.
\]

If we take \(\alpha = 1\), then \(|C, 1|_k\) summability reduces to \(|C, 1|_k\) summability.

Let \((p_n)\) be a sequence of constants, real or complex, and let us write

\[
P_n = p_0 + p_1 + p_2 + \cdots + p_n \neq 0, \quad (n \geq 0).
\]

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The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{p_n} \sum_{t=0}^{n} p_{n-t} s_t$$  \hspace{1cm} (6)

defines the sequence $\{\sigma_n\}$ of the Nörlund mean of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be $[N, p_n]_k$ summable, $k \geq 1$ if (see [4])

$$\sum_{n=1}^{\infty} n^{-k} |\sigma_n - \sigma_{n-1}|^k < \infty.$$  \hspace{1cm} (7)

In the special case when

$$p_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha) \Gamma(n+1)}, \hspace{1cm} \alpha \geq 0$$  \hspace{1cm} (8)

the Nörlund mean reduces to the $(C, \alpha)$ mean and $[N, p_n]_k$ summability becomes $[C, \alpha]_k$ summability. For $p_n = 1$, we get the $(C, 1)$ mean and then $[N, p_n]_k$ summability becomes $[C, 1]_k$ summability. Also, if we take $k = 1$, then we get $[N, p_n]$ summability. For any sequence $(\lambda_n)$, we write $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$.

The known result. Quite recently, Bor [5] proved the following theorem dealing with absolute Nörlund summability factors of infinite series.

**Theorem A.** Suppose $p_0 > 0$, $p_n \geq 0$ and $(p_n)$ is a non-increasing sequence. If

$$\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m) \hspace{1cm} \text{as} \hspace{1cm} m \to \infty,$$  \hspace{1cm} (9)

where $(t_n)$ is the $n$th $(C, 1)$ mean of the sequence $(n a_n)$, $(X_n)$ is an almost increasing sequence such that $|\Delta X_n| = O(X_n/n)$ and $(\lambda_n)$ is a sequence such that

$$\sum_{n=1}^{\infty} n |\Delta^2 \lambda_n| X_n < \infty,$$  \hspace{1cm} (10)

$$|\lambda_n| X_n = O(1) \hspace{1cm} \text{as} \hspace{1cm} n \to \infty,$$  \hspace{1cm} (11)

then the series $\sum a_n p_n \lambda_n (n+1)^{-1}$ is $[N, p_n]_k$ summable, $k \geq 1$.

2. The main result

The aim of this paper is to generalize and further relax the conditions of Theorem A. We shall prove the following form.

**Theorem.** Let $(p_n)$ be as in Theorem A and let $(X_n)$ be an almost increasing sequence such that $|\Delta X_n| = O(X_n/n)$. Suppose also that there exists a sequence of numbers $(A_n)$ such that it is $\delta$-quasi-monotone with $\sum n \delta_n X_n < \infty$, $\sum A_n X_n$ is convergent and $|\Delta \lambda_n| \leq A_n$ for all $n$. If the condition (11) of Theorem A is satisfied and the sequence $(w_{n}^{\alpha})$ defined by (see [6])

$$w_{n}^{\alpha} = |t_{n}^{\alpha}|, \hspace{1cm} \alpha = 1$$

$$w_{n}^{\alpha} = \max_{1 \leq \rho \leq n} |t_{\rho}^{\alpha}|, \hspace{1cm} 0 < \alpha < 1$$

satisfies the condition

$$\sum_{n=1}^{m} \frac{1}{n} (w_{n}^{\alpha})^k = O(X_m) \hspace{1cm} \text{as} \hspace{1cm} m \to \infty,$$  \hspace{1cm} (14)

then the series $\sum a_n p_n \lambda_n (n+1)^{-1}$ is $[N, p_n]_k$ summable, $k \geq 1$.

**Remark.** To show that condition (10) implies the condition “$\sum A_n X_n$ is convergent” of our theorem we can take, for example, $A_k = \sum_{n=k}^{\infty} |\Delta^2 \lambda_n|$.

We need the following lemmas for the proof of our theorem.
Lemma 1 ([7]). If $0 < \alpha \leq 1$ and $1 \leq v \leq n$, then
\begin{equation}
\left| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{m} A_{n-p}^{\alpha-1} a_p \right|.
\end{equation}

Lemma 2 ([8]). If $-1 < \alpha \leq \beta$, $k > 1$ and the series $\sum a_n$ is $|C, \alpha|_k$ summable, then it is also $|C, \beta|_k$ summable. The case $k = 1$ of this lemma is due to Kogbetliantz [9]. The case $k > 1$ is a special case of a theorem of Flett [3, Theorem 1].

Lemma 3 ([10]). Let $(X_n)$ be an almost increasing sequence such that $n|\Delta X_n| = O(X_n)$. If $(A_n)$ is $\delta$-quasi-monotone with $\sum n \delta X_n < \infty$, and $\sum A_n X_n$ is convergent, then
\begin{equation}
n A_n X_n = O(1) \quad \text{as} \quad n \to \infty,
\end{equation}
\begin{equation}
\sum_{n=1}^{\infty} n X_n |\Delta A_n| < \infty.
\end{equation}

Lemma 4 ([11]). Suppose $p_0 > 0$, $p_n \geq 0$ and $(p_n)$ is a non-increasing sequence. If the series $\sum a_n$ is $|C, 1|_k$ summable, then the series $\sum a_n p_n (n+1)^{-1}$ is $|N, p_n|_k$ summable, $k \geq 1$.

3. Proof of the theorem

In order to prove the theorem, we need consider only the special case in which $(N, p_n)$ is $(C, \alpha)$, that is, we shall prove that $\sum a_n \lambda_n$ is $|C, \alpha|_k$ summable. Our theorem will then follow by means of Lemma 2 (for $\beta = 1$) and Lemma 4. Let $(T^\alpha_n)$ be the $n$th $(C, \alpha)$ (with $0 < \alpha \leq 1$) mean of the sequence $(n a_n \lambda_n)$. Then, by (2), we have
\begin{equation}
T^\alpha_n = \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v.
\end{equation}

First applying Abel's transformation and then using Lemma 1 we have that
\begin{align*}
T^\alpha_n &= \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \\
|T^\alpha_n| &\leq \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} |\Delta \lambda_v| \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p |a_p| + \frac{\lambda_n}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v |a_v| \\
&\leq \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} A_v^{\alpha} |\Delta \lambda_v| + |\lambda_n| w_n^{\alpha} \\
&= T^\alpha_{n,1} + T^\alpha_{n,2}.
\end{align*}

Since
\begin{equation}
T^\alpha_{n,1} + T^\alpha_{n,2} \leq 2k(T^\alpha_{n,1} \leq k)^k + (T^\alpha_{n,2} \leq k)^k,
\end{equation}
in order to complete the proof of the theorem, by using (4), it is sufficient to show that
\begin{equation}
\sum_{n=1}^{\infty} n^{-1} |T^\alpha_{n,r}|^k < \infty \quad \text{for} \quad r = 1, 2.
\end{equation}

Whenever $k > 1$, we can apply Hölder's inequality with indices $k$ and $k'$, where $\frac{1}{k} + \frac{1}{k'} = 1$; we get that
\begin{align*}
\sum_{n=2}^{m+1} n^{-1} |T^\alpha_{n,r}|^k &\leq \sum_{n=2}^{m+1} n^{-1} (A_n^{\alpha})^{-k} \left( \sum_{v=1}^{n-1} A_v^{\alpha} w_v^{\alpha} |\Delta \lambda_v| \right)^k \\
&\leq \sum_{n=2}^{m} n^{-1} n^{-\delta k} \left( \sum_{v=1}^{n-1} v^{\delta k} |w_v^{\alpha}| A_v \right) \times \left( \sum_{v=1}^{n-1} A_v \right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{\delta k} |w_v^{\alpha}| A_v \sum_{n=v+1}^{m+1} \frac{1}{n^{\delta k+1}}.
\end{align*}
\[\sum_{n=1}^{m} n^{-1} |T_{n,2}^{\alpha}|^k = O(1) \quad \text{as } m \to \infty, \quad \text{for } r = 1, 2.\]

This completes the proof of the theorem. If we take \(\alpha = 1\), then we get a new result for absolute Nörlund summability factors of infinite series.

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**References**