Intersection graphs of Helly families of subtrees

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Abstract

A graph is called neighborhood chordal if the neighborhood of every vertex is chordal. A family of subtrees of a graph is called 2-acyclic if the union of any two subtrees is acyclic. In the present paper we prove that every graph is an intersection graph of a Helly family of subtrees of a graph without triangles. In particular, we prove that a graph is an intersection graph of a Helly 2-acyclic family of subtrees of a graph iff it is neighborhood chordal, in which case we present a simple greedy algorithm to construct the corresponding family of subtrees. In addition, we describe polynomial-time recognition algorithms for the intersection graphs and for the perfect intersection graphs of Helly families of subtrees in cacti graphs.

Keywords: Intersection graph of subtrees; Helly family of subtrees; Neighborhood chordal graph; Cactus subtree graph; Circular-arc graph

1. Introduction

In the present paper we consider only finite undirected graphs \( G(V, E) \) with no parallel edges and no self-loops, where \( V \) is the set of vertices, \( E \) the set of edges and \( |V| = n \). An edge between \( u \) and \( v \) is denoted \( (u, v) \). The neighborhood \( N(v) \) of a vertex \( v \) is the set containing \( v \) and its adjacent vertices. By a path \( p \) we always mean a simple path; the set of its vertices is also denoted by \( p \). A subset of \( V \) is called a clique if every two of its vertices are adjacent. A clique is maximal if it is not properly contained in a larger clique. We denote by \( \beta(G) \) the size of a maximum clique of \( G \), by \( C_G \) the family of maximal cliques of \( G \) and by \( C_{G,v} \) the subfamily of maximal cliques containing a vertex \( v \). The clique matrix \( M(G) \) of \( G \) is the matrix whose columns correspond to the vertices \( v \) of \( G \), the rows correspond to the maximal cliques \( c \) of \( G \) and an entry \( (j, i) \) is one iff \( v \in c \). \( M(G) \) has the consecutive (circular) ones property if there is a permutation of its rows such that the ones in every column appear in a consecutive (circular consecutive) order. Given two subgraphs \( G(V_1, E_1) \), \( G(V_2, E_2) \) of \( G \), their union is the subgraph \( G(V_1 \cup V_2, E_1 \cup E_2) \) and their intersection is the subgraph \( G(V_1 \cap V_2, E_1 \cap E_2) \).
The clique graph $Q_G$ of $G$ is a weighted graph with vertex set $C_G$, two vertices $c_i, c_j$ being adjacent iff $c_i \cap c_j \neq \emptyset$. The weight of the edge $(c_i, c_j)$ is $|c_i \cap c_j|$. We denote by $Q_{G,v}$ the vertex subgraph $Q_G(C_{G,v})$. It is easy to see that $C_{G,v}$ is exactly the set of maximal cliques of $G(T_v)$ and $Q_{G,v}$ is the clique graph of $G(T_v)$.

A clique $c$ of $G$ is called a clique cut-set if $G(V - c)$ is not connected; let $G(\mathcal{V}_1), \ldots, G(\mathcal{V}_k)$ be its connected components. The subgraphs $G(V_1 \cup c), \ldots, G(V_k \cup c)$ are called the leaves of $c$ in $G$; if $c$ is a maximal clique of $G$, it is called a maximal clique cut-set.

A hole is a chordless cycle with four or more vertices. An antihole is the complement of a hole. A graph $G$ is perfect if the vertices of every vertex subgraph $H$ of $G$ can be colored with $\beta(H)$ colors such that no two adjacent vertices have the same color. The well-known Strong Perfect Graph Conjecture (SPGC) of Berge states that a graph is perfect if and only if it has no odd holes or antiholes.

A graph is called chordal if it has no holes; these graphs were discussed in [4, 12, 13]. A graph $G$ is called neighborhood chordal if for every vertex $v$ of $G$ is chordal. Let $K_{3,3}^-$ denote the graph with vertex set $\{u_1, u_2, u_3, v_1, v_2, v_3\}$ having edges between every two vertices except for the pairs $u_i, v_i$, $i = 1, 2, 3$. The complement of $K_{3,3}^-$ contains exactly three edges which are mutually nonincident. A hole with nine or more vertices contains a vertex subgraph with exactly three edges which are mutually nonincident. Therefore, any graph which does not contain $K_{3,3}^-$ as a vertex subgraph has no antiholes with nine or more vertices. The graph $K_{3,3}^-$ is not neighborhood chordal since every $K_{3,3}^-(T_v)$ contains a hole with four vertices. A cactus is a graph in which every biconnected component is a single vertex, a single edge or a chordless cycle.

A subtree of a graph $H$ is an acyclic connected edge subgraph of $H$. Two subtrees have a nonempty intersection if they have at least one vertex in common. Let $S$ be a family of subtrees of $H$: $S$ is called Helly if every subfamily of mutually intersecting subtrees has a nonempty intersection; $S$ is called 2-acyclic if the union of any two subtrees is acyclic. It is easy to prove that the union of two subtrees is acyclic iff their intersection is connected. A graph $G$ is an intersection graph of a family $S$ of subtrees of a graph if there is a one-to-one correspondence between the vertices of $G$ and the subtrees in $S$ such that two subtrees have a nonempty intersection iff their corresponding vertices in $G$ are adjacent. The subtree in $S$ corresponding to a vertex $v$ of $G$ is denoted $\ell_v$. Intersection graphs are of great interest in various domains such as computer science, genetics, archaeology and ecology. Ref. [11] surveys the main problems and applications of different families of intersection graphs. Many types of intersection graphs of subtrees in trees were discussed in literature. A unified discussion of these graphs can be found in [10]. As proved in [4] a graph $G$ is chordal iff it is the intersection graph of a family of subtrees of a tree $T$ with vertex set $C_G$ such that for every vertex $v$ of $G$ its corresponding subtree is $\ell_v = T(C_G,v)$. Such a tree $T$ is called a clique tree of $G$. By this definition, if $T$ is a clique tree of a chordal graph $G$, then for every vertex $v$ of $G$, $\ell_v = T(C_G,v)$ is a subtree of $T$ and hence it is connected. The clique trees of a chordal graph $G$ are exactly the maximum spanning trees of $Q_G[5, 13]$. Intersection graphs of families of arcs on a circle are called circular-arc...
graphs. Ref. [3] contains a polynomial-time recognition algorithm for the intersection graphs of Helly families of arcs on a circle. Everything about circular-arc graphs remains true if we replace "circle" and "arcs" by "cycle" and "paths". Intersection graphs of families of subtrees on cacti are called cactus subtree graphs.

In the present paper we characterize additional families of intersection graphs of subtrees. In Section 2 we prove that every graph is an intersection graph of a Helly family of subtrees of a graph without triangles. In particular, we prove that a graph is an intersection graph of a Helly 2-acyclic family of subtrees of a graph iff it is neighborhood chordal, in which case we present a simple greedy algorithm to construct the corresponding family of subtrees. In Section 3 we characterize the Helly cactus subtree graphs and describe a polynomial-time recognition algorithm. In Section 4, we present algorithms for recognition and for minimum coloring of perfect Helly cactus subtree graphs.

2. Intersection graphs of Helly families of subtrees

In this section we characterize the intersection graphs of Helly families and Helly 2-acyclic families of subtrees of graphs.

Theorem 1. Every graph G is an intersection graph of a Helly family of subtrees of a graph without triangles.

Proof. Consider a graph G(V, E) and its clique graph QG. For every vertex v of G let \( \ell_v \) be a spanning tree of QG, v and let \( S = \{ \ell_v | v \in V \} \). Consider two vertices u, v of G. If u, v are adjacent then there exists a maximal clique c containing both u, v thus \( \ell_u \cap \ell_v \neq \emptyset \). Conversely, if \( \ell_u \cap \ell_v \neq \emptyset \), then there exists a vertex c of QG such that \( c \in \ell_u \cap \ell_v \), hence \( u, v \in c \), implying that u and v are adjacent. Therefore G is an intersection graph of S and S is Helly.

Let \( H = \bigcup_{v \in V} \ell_v \). Consider in H a triangle \( R = \{(u_1, u_2), (u_2, u_3), (u_3, u_1)\} \). We construct \( H' \) by adding to H a new vertex x with edges \( (x, u_1), (x, u_2), (x, u_3) \) and by deleting the edges of R; R disappears from \( H' \) and no new triangles appear. For every subtree \( \ell \) in S, define a subgraph \( \ell' \) of \( H' \) as follows: if \( \ell \) has no edge of R, let \( \ell' = \ell \); if \( \ell \) has just one edge \( (u_i, u_j) \) of R, \( \ell' \) is obtained from \( \ell \) by replacing \( (u_i, u_j) \) with \( (u_i, x) \) and \( (u_j, x) \); if \( \ell \) has two edges \( (u_i, u_j) \) and \( (u_j, u_k) \) of R, \( \ell' \) is obtained from \( \ell \) by replacing \( (u_i, u_j) \) and \( (u_j, u_k) \) with \( (u_i, x), (u_j, x) \) and \( (u_k, x) \). Assume that \( \ell' \) contains a cycle CY. Clearly x ∈ CY (otherwise CY is contained in \( \ell \)) and CY contains at least two vertices \( u_i, u_j \) of R. If \( (u_i, u_j) \in \ell \) then by replacing \( (u_i, x), (x, u_j) \) with \( (u_i, u_j) \) in CY we obtain a cycle in \( \ell \) contradicting its being a tree. If \( (u_i, u_j) \notin \ell \) then \( (u_i, u_k) \) and \( (u_j, u_k) \in \ell \) and again a cycle appears in \( \ell \) contradicting its being a tree. Therefore \( \ell' \) is a tree. Let \( S' \) be the family of the above subtrees \( \ell' \) of \( H' \). Clearly, if two trees \( \ell', f \in S \) have a common vertex then \( \ell', f \) have the same vertex in common. Conversely, assume that \( \ell', f \) have a vertex in common. If this vertex is not x then \( \ell', f \) have the same vertex in common. If
this vertex is \( x \), then \( \ell', \ell' \) also have in common one of the vertices of \( R \) and so do \( \ell, \ell' \).

Therefore, the intersection relationship in \( S' \) is the same as in \( S \).

Let \( F' \) be a subfamily of mutually intersecting subtrees of \( S' \). Its corresponding subfamily \( F \) of \( S \) is also a family of mutually intersecting subtrees. Thus, by the Helly property all its elements contain an old vertex \( v \). Clearly, \( v \) is contained also in all the elements of \( F' \). Therefore \( S' \) is Helly. We continue in the same way on \( H' \) with another triangle, and so on, until we obtain a graph without triangles. \( \square \)

Theorem 1 is a complete characterization of the intersection graphs of Helly families of subtrees, yet it does not imply a polynomial-time algorithm for the construction of the family of subtrees itself, since a graph can have an exponential number of maximal cliques. As we shall see, when we restrict the problem to intersection graphs \( G \) of Helly 2-acyclic families \( S \) of subtrees, a similar characterization exists, but now \( G \) has at most \( O(n^2) \) maximal cliques, making possible the construction of \( S \) by a simple polynomial-time greedy algorithm. Before that, we need some preparatory lemmas and theorems.

Let \( S \) be a family of subtrees of a graph \( H \) such that every three mutually intersecting elements of \( S \) have a vertex in common. This does not imply that \( S \) is Helly as for example happens when \( H \) is a cycle \( \{(u, v), (v, w), (w, z), (z, u)\} \) and every subtree in \( S \) is composed by two consecutive edges. This does imply that \( S \) is Helly when \( S \) is 2-acyclic (see Theorem 2 below).

**Lemma 1.** Let \( S \) be a 2-acyclic family of subtrees of a graph \( H \) having a common vertex \( x \). Then the union \( \ell \) of the elements of \( S \) is a subtree of \( H \).

**Proof.** The proof is by induction on \( |S| \). If \( |S| \leq 2 \) the lemma is true. Assume that \( |S| > 2 \) and let \( \ell \) be any element of \( S \). By induction, the union \( \ell' \) of the elements of \( S-\{\ell\} \) is a tree. Let us assume that \( \ell' \) contains a cycle, that is \( \ell \cap \ell' \) contains two vertices \( y, z \) connected in \( \ell' \) by a subpath \( p \) whose edges are contained in \( \ell - \ell' \). The path in \( \ell \) from \( x \) to \( y \) or from \( x \) to \( z \) contains edges of \( p \), for otherwise \( \ell \) contains a cycle. So assume w.l.o.g. that the path \( q \) in \( \ell \) from \( x \) to \( y \) contains edges of \( p \). Let \( \ell_1 \in S-\{\ell\} \) be a subtree containing \( y \). Let \( r \) be a path in \( \ell_1 \) from \( x \) to \( y \). Then \( q \cup r \) is a closed walk in \( \ell \cup \ell_1 \) from \( x \) to \( y \) and back to \( x \). Since \( r \) has no edges of \( p \) while \( q \) has edges of \( p \), it follows that \( q \cup r \) contains a cycle, so \( \ell \cup \ell_1 \) is not acyclic, contradicting the assumption that \( S \) is 2-acyclic. \( \square \)

**Theorem 2.** A 2-acyclic family \( S \) of subtrees of a graph \( H \) is Helly iff the intersection of every three mutually intersecting subtrees is not empty.

**Proof.** We prove the “if” part by induction on \( |S| \). If \( |S| \leq 2 \), the Theorem is true. Assume that \( |S| > 2 \) and it has a subfamily \( S' \) of mutually intersecting subtrees having an empty intersection. By the induction hypothesis \( S' = S \). Let \( \ell \) be a subtree in \( S \). By induction, \( S-\{\ell\} \) is Helly and its elements have a common vertex \( x \). By Lemma 1, the
union of the elements in $S\setminus\{\ell\}$ form a subtree $\ell'$; $\ell$ is not a subtree of $\ell'$ otherwise 
$S$ would be Helly [4]. Therefore $x \notin \ell$ and $\ell = \ell \cup \ell'$ contains two vertices $y, z$ connected 
by a subpath $p$ whose edges are in $\ell - \ell'$. Let $\ell_1, \ell_2 \in S\setminus\{\ell\}$ be subtrees containing 
y, z and $p_1, p_2$ be the paths in $\ell_1, \ell_2$ from $y, z$, respectively, to $x$. Since $x$ is not in $\ell$,
$p_1 \cup p_2 \cup p$ contains a cycle $CY$. Since $\ell_1, \ell_2, \ell$ are mutually intersecting, it follows that
$\ell_1 \cap \ell_2 \cap \ell \neq \emptyset$. But then, by Lemma 1, $\ell_1 \cup \ell_2 \cup \ell$ is a tree contradicting the existence of
CY.

The converse is trivial. □

**Theorem 3.** A graph $H$ is the union of a Helly 2-acyclic family $S$ of subtrees iff $H$ has no triangles.

**Proof.** Let $H$ be a graph without triangles and let every edge be considered a subtree in $S$. Since $H$ has no parallel edges, $S$ is 2-acyclic. Since $H$ has no triangles, $S$ is Helly. The intersection graph of $S$ is exactly the line graph of $H$.

Conversely, let $H$ be the union of a Helly 2-acyclic family $S$ of subtrees, and assume
that $H$ has a triangle with edges $e_1, e_2, e_3$. Let $\ell_1, \ell_2, \ell_3$ be subtrees of $S$ containing 
e_1, e_2, e_3, respectively; $\ell_1, \ell_2, \ell_3$ are distinct since $S$ is a 2-acyclic. By the Helly
property, there is a vertex $x \in \ell_1 \cap \ell_2 \cap \ell_3$. But then, by Lemma 1, $t = \ell_1 \cup \ell_2 \cup \ell_3$ is
a subtree of $H$, contradicting the fact that $t$ contains a triangle. Therefore $H$ has no triangles. □

**Theorem 4.** A graph $G$ is the intersection graph of a Helly 2-acyclic family of subtrees of
a graph iff $G$ is the intersection graph of a Helly 2-acyclic family of subtrees of a graph $H$ with vertex set $C_G$, such that for every vertex $v$ of $G$ $\ell_v = H(C_G, v)$.

**Proof.** Consider an intersection graph $G(V, E)$ of a Helly 2-acyclic family $S$ of
subtrees of a graph $H$. For every vertex $x$ of $H$ the set $V(x) = \{v \mid v \in V, x \in \ell_v\}$ is
a clique of $G$. Conversely, since $S$ is Helly, every maximal clique of $G$ has the form $V(x)$
for some vertex $x$ of $H$. Consider two vertices $x, y$ of $H$ such that $V(x)$ is contained in
$V(y)$. There exists a vertex $z$ of $H$ adjacent to $x$ such that $V(x)$ is contained in $V(z)$,
otherwise $V(x)$ contains two subtrees whose union contains a cycle. Therefore, by
collapsing $x$ and $z$ into one vertex in $H$ and in the subtrees of $S$, we obtain that $G$
remains the intersection graph of the subtrees on the new graph; the new $S$ remains Helly
and 2-acyclic. Continuing until no more possible, we obtain a graph $H$ in which every
$V(x)$ is a maximal clique of $G$. We rename every vertex $x$ by the maximal clique $V(x)$. By
the construction, for every vertex $v$ of $G$ $\ell_v = H(C_G, v)$ and $\ell_v$ is a clique tree of $G(\Gamma v)$. □

**Lemma 2.** Let $G$ be a neighborhood chordal graph. Then
(a) $G$ has at most $n^2$ maximal cliques which can be found in time $O(n^3)$;
(b) for every two nonadjacent vertices $u, v$ of $G$, the connected components of
$G(\Gamma u \cap \Gamma u)$ are cliques;
(c) $G$ has no antiholes with nine or more vertices.
Proof. A set \( c \) is a maximal clique of \( G \) iff it is a maximal clique of some \( G(\Gamma v) \). As known [4], every chordal graph \( G(\Gamma v) \) has at most \( n \) maximal cliques which can be found in time \( O(n^2) \). Thus, \( G \) has at most \( n^2 \) maximal cliques which can be found in time \( O(n^3) \).

Consider two nonadjacent vertices \( u, v \) of \( G \) and assume that \( G(\Gamma u \cap \Gamma v) \) has a connected component \( A \) which is not a clique, i.e., there are three vertices \( w_1, w_2, w_3 \in \Gamma u \cap \Gamma v \) such that \( w_2 \) is adjacent to both \( w_1, w_3 \) while \( w_1, w_3 \) are not adjacent. But then \((u, w_1), (w_1, v), (v, w_3), (w_3, u)\) form a hole in \( G(\Gamma w_2) \) contradicting the fact that \( G(\Gamma w_2) \) is chordal. Therefore, (b) is true.

A neighborhood chordal graph \( G \) does not contain \( K_{3,3} \) as a vertex subgraph; therefore, as mentioned in the Introduction \( G \) has no antiholes with nine or more vertices. \( \square \)

Theorem 5. A graph \( G \) is the intersection graph of a Helly 2-acyclic family of subtrees of a graph \( H \) iff \( G \) is neighborhood chordal.

Proof. Assume that \( G \) is an intersection graph of a Helly 2-acyclic family of subtrees of a graph \( H \). By Theorem 4, we can assume that the vertex set of \( H \) is \( C_G \) and for every vertex \( v \) \( \ell_v = H(C_{G,v}) \). Two vertices \( u, w \in \Gamma v \) are adjacent iff \( \ell_u \cap \ell_w \neq \emptyset \) iff \((\ell_u \cap \ell_v) \cap (\ell_w \cap \ell_v) \neq \emptyset \), by the Helly property. In addition, for every \( u \in \Gamma v \) the intersection of \( \ell_u \) and \( \ell_v \) is connected, otherwise \( \ell_u \cup \ell_v \) would contain a cycle. Therefore \( G(\Gamma v) \) is the intersection graph of the family of subtrees \( \{\ell_u \setminus \ell_v \mid u \in \Gamma v\} \) of the tree \( \ell_v \) and by [4] \( G(\Gamma v) \) is chordal.

Conversely, assume that \( G \) is neighborhood chordal; thus it has at most \( n^2 \) maximal cliques. By [4], every \( G(\Gamma v) \) is an intersection graph of a family of subtrees of a clique tree. By [5, 13], the clique trees of \( G(\Gamma v) \) are exactly the maximum spanning trees of the weighted graph \( Q_{G,v} \). Such a maximum spanning tree can be constructed by Kruskal’s algorithm: start with a tree \( \ell \) containing only an edge of maximum weight, and add to it edges in decreasing order of weight such that \( \ell \) remains acyclic; after all the edges have been considered, \( \ell \) will be a maximum spanning tree.

Our purpose is to construct a spanning graph \( H \) of \( Q_G \) such that every \( \ell_v = H(C_{G,v}) \) is a maximum spanning tree of \( Q_{G,v} \), that is a clique tree of \( G(\Gamma v) \). For such a graph \( H \) let \( S = \{\ell_v \mid \ell_v = H(C_{G,v})\} \); every \( \ell_v \in S \) is a clique tree of \( G(\Gamma v) \), hence for any \( u \in \Gamma v \) \( \ell_u \cap \ell_v \) is a subtree of \( \ell_v \) (\( \ell_v \) being a clique tree of \( G(\Gamma v) \)) and is thus connected. Therefore every \( \ell_u \cup \ell_v \) is acyclic and \( S \) is 2-acyclic. In addition, for every three mutually intersecting subtrees \( \ell_u, \ell_v, \ell_w \) we have \( \ell_u \cap \ell_v \cap \ell_w \neq \emptyset \) since there exists a maximal clique of \( G \) containing \( u, v, w \). Therefore \( S \) is Helly and \( H \) has no triangles.

We construct \( H \) by a greedy algorithm on a given sequence \( e_1, e_2, \ldots, e_m \) of the edges of \( Q_G \) in order of decreasing weight. We start with \( H_0(C_G) \) containing no edges and add to it the edge \( e_1 \) of maximum weight to obtain \( H_1(C_G) \). At stage \( i \), let \( P \) be the graph obtained by adding \( e_i \) to \( H_{i-1}(C_G) \); if every \( P(C_{G,v}) \) remains acyclic then \( H_i(C_G) = P \), otherwise \( H_i(C_G) = H_{i-1}(C_G) \). Clearly, every \( H_i(C_{G,v}) \) is a forest denoted \( \ell_{v,i} \). When \( i = 0 \), every \( \ell_{v,0} \) is contained in a maximum spanning tree of \( Q_{G,v} \). Assume
that this is true up to $i$, and let $e_{i+1} = (c_1, c_2)$ be the next edge to be considered for addition to $H_i$. If for every vertex $v$, $\ell_{v,i} \cup \{e_{i+1}\}$ is acyclic, then adding $e_{i+1}$ to $H_i(C_G)$ is equivalent to adding $e_{i+1}$ to every $\ell_{v,u}$, $v \in c_1 \cap c_2$, by applying Kruskal's algorithm to every $Q_{G,v}$, since $e_{i+1}$ is the next edge in order of weight for every $v \in c_1 \cap c_2$.

Assume that for some $v$, $\ell_{v,i} \cup \{e_{i+1}\}$ contains a cycle $CY$. Let $T_v$ be the maximum spanning subtree of $Q_{G,v}$ which contains $\ell_{v,i}$. So $T_v$ does not contain $e_{i+1}$. Thus $T_v \cup \{e_{i+1}\}$ contains the cycle $CY$. Since $T_v$ is a clique tree of $G(\Gamma v)$, for every vertex $u \in c_1 \cap c_2$, $T_v(C_G \cap C_{G,u})$ is a subtree $\ell_u$ of $T_v$. The path connecting $c_1, c_2$ in $\ell_u$ together with the edge $e_{i+1}$ form a cycle in $T_v \cup \{e_{i+1}\}$ which is exactly $CY$ and $CY$ is contained in $\ell_u \cup \{e_{i+1}\}$. Therefore, $e_{i+1}$ is not contained in any maximum spanning tree $T_u$ of $Q_{G,v}$ containing $\ell_{u,i}$ and Kruskal's algorithm applied to every $Q_{G,v}$ will discard $e_{i+1}$.

In conclusion, the above greedy algorithm is equivalent to applying Kruskal's algorithm independently on every $Q_{G,v}$, given that the edges are in the same order of decreasing weight. Thus, the final result $H(C_G) = H_m$ is a graph in which for every vertex $v$ $H(C_{G,v})$ is a maximum spanning tree of $Q_{G,v}$.

To recognize that a given graph $G$ is an intersection graph of a Helly 2-acyclic family of subtrees of a graph $H$, we check that every $G(\Gamma v)$ is chordal and construct the maximal cliques of $G$ in time $O(n^3)$. Then, we construct the weighted clique graph $Q_G$ and apply the greedy algorithm described in the second part of the proof of Theorem 5 to construct $H$ and the family of subtrees. The entire algorithm works in time $O(n^3)$.

3. Intersection graphs of Helly cactus subtree graphs

In this section we discuss intersection graphs of Helly families of subtrees on cacti graphs.

**Theorem 6.** A graph $G$ is a Helly cactus subtree graph iff it is the intersection graph of a Helly family of subtrees of a cactus with vertex set $C_G$.

**Proof.** Assume that $G$ is an intersection graph of a Helly family $S$ of subtrees of a cactus $CT$. For each vertex $x$ of $CT$, the subtrees from $S$ that contain $x$ correspond to a clique $V(x)$ of $G$, and since $S$ is Helly, every clique of $G$ is contained in some $V(x)$. Consider two vertices $x, y \in CT$ such that $V(x)$ is contained in $V(y)$. Let $C_{I_1}, \ldots, C_{I_k}$ be the connected components of $CT - \{x\}$, and let $CT_y$ be the connected component which contains $y$. In $CT$, $x$ is adjacent either to one vertex $z_r$ or to two vertices $z_r, w_r$ of $CT_y$. We delete $x$ and its incident edges from $CT$ and for every vertex $z \in CT_i$, $i \neq r$, adjacent to $x$ we connect it by an edge to $y$. In addition, if $x$ is adjacent to $z_r, w_r$, we connect $z_r, w_r$ by an edge. We do the same for every subtree $\ell_v$ in $S$. The result is a cactus with less vertices, while $G$ remains the intersection graph of the new Helly
family of subtrees. Continuing in this way until no more possible, the vertex set of the
final cactus CT is in a one-to-one correspondence with $C_G$ and can be identified with
$C_G$ and for every vertex $v$ of $G$, $\ell_v = CT(C_{G,v})$.

The converse is trivial. □

Theorem 6 implies that when $G$ is a Helly cactus subtree graph on a cactus $CT(C_G)$
and CT is not a cycle, a vertex or an edge, then $G$ has a maximal clique cut-set.

**Theorem 7.** A connected graph $G$ is a Helly cactus subtree graph iff one of the following
statements is true:

(a) $G$ is a Helly circular-arc graph;
(b) $G$ has a maximal clique cut-set whose leaves are Helly cactus subtree graphs.

**Proof.** Assume that $G$ is a Helly cactus subtree graph on a cactus $CT(C_G)$ and $G$ is not
a Helly circular-arc graph. Then CT has a cut-vertex $x$ which is a maximal clique
cut-set of $G$; the leaves of $x$ in $G$ are Helly cactus subtree graphs on the leaves of $x$ in
CT.

Conversely, assume that $G$ has a maximal clique cut-set $x$ such that the leaves $G_i$,
$1 \leq i \leq k$, of $x$ in $G$ are cactus subtree graphs on the cacti $CT_i(C_{G_i})$. The maximal
clique $x$ is represented by a vertex in every $CT_i$. We construct CT by collapsing these
vertices into one vertex denoted $x$. □

**Corollary 1.** A connected graph $G$ is a Helly cactus subtree graph iff it can be
decomposed by a sequence of maximal clique cut-sets into a family of leaves which are
Helly circular-arc graphs.

**Lemma 3.** A Helly cactus subtree graph $G$ has at most $n^2$ maximal cliques.

**Proof.** Assume that $G$ is a Helly cactus subtree graph on $CT(C_G)$. For every cyclic
biconnected component $B$ of CT we collapse $B$ and every $\ell_v \cap B$ into a vertex. The
result is a tree $T_G$ and a family $S'$ of subtrees of $T_G$. We construct a graph $G'$ from $G$ by
adding an edge between two vertices $u, v$ iff for some cyclic biconnected component
$B$ of $CT \ell_u \cap B \neq \emptyset, \ell_v \cap B \neq \emptyset$. It is easy to see that $G'$ is the intersection graph of $S'$ in
$T_G$. So $G'$ is chordal with at most $n$ maximal cliques, and $T_G$ is a clique-tree of $G'$.
Thus, $T_G$ has at most $n$ vertices, implying that CT has at most $n - 1$ cut-vertices. By
[3], a Helly circular-arc graph has at most $n$ maximal cliques. Therefore $G$ has at most
$n^2$ maximal cliques. □

A recognition algorithm for the Helly cactus subtree graphs is immediately implied
by Theorem 7. Given a graph $G$, we construct its at most $n^2$ maximal cliques in time
$O(n^4)$, using the algorithm in [14]. We decompose $G$ by a sequence of maximal clique
cut-sets into a family of leaves which have no maximal clique cut-sets and we check
that each such leaf is a Helly circular-arc graph. Then, we construct the cactus CT as in the proof of Theorem 7.

A simple recognition algorithm for Helly circular-arc graphs is presented in [3]. Given a graph $G(V, E)$ and its maximal cliques, we construct the clique matrix $M(G)$ of $G$. By Theorem 6, $G$ is a Helly circular-arc graph iff it is the intersection graph of a family of arcs on a cycle with vertex set $C_G$. Thus, the graph $G$ is a Helly circular-arc graph iff $M(G)$ has the circular ones property (the rows of $M(G)$ in a circular consecutive order can be seen as vertices of a cycle). The circular ones property of $M(G)$ can be tested as described in [15] by taking a row $c$ with a minimum number of ones of $M(G)$ and constructing a matrix $M(G, c)$ as follows: interchange $c$ with the first row of $M(G)$ to make $c$ its first row and for every column $v$ having a one in $c$ replace the zeroes by ones and the ones by zeroes. Then [15], $M(G)$ has the circular ones property iff $M(G, c)$ has the consecutive ones property. $M(G, c)$ can be tested for the consecutive ones property in time $O(n^2)$ using PQ-trees as described in [6].

The above recognition algorithm can be used for testing whether two Helly circular-arc graphs $G_1, G_2$ are isomorphic. We take a row $u$ of $M(G_1)$ and construct the labelled PQ-tree $T(G_1, u)$ of $M(G_1, u)$. For every row $v$ of $M(G_2)$ we construct the labelled PQ-tree $T(G_2, v)$ of $M(G_2, v)$. Then, $G_1$ and $G_2$ are isomorphic iff for some vertex $v$ of $G_2$ the labelled trees $T(G_1, u)$ and $T(G_2, v)$ are isomorphic. We test their isomorphism as described in [7]. This algorithm works in time $O(n^3)$.

We remark that the isomorphism problem for Helly cactus subtree graphs is $I$-complete being so for chordal graphs.

4. Recognition of perfect Helly cactus subtree graphs

The circular-arc graphs fulfill the SPGC [16]. Yet, no polynomial-time algorithms were known for recognizing when they are perfect. [1, 9] present efficient algorithms to find a chordless path of a given parity in an interval graph. We can use these algorithms to find an odd hole in a circular-arc graph by finding for every vertex $v$ whether between two nonadjacent vertices $u, w \in V$ there is a chordless path with an even number of vertices in $G((V - \Gamma v) \cup \{u, w\})$. Unfortunately, it seems more difficult to find whether the complement of a circular-arc graph contains an odd hole. But, we can do this for Helly circular-arc graphs using an observation by Manacher [8] that $K_{3,3}$ is not a Helly circular-arc graph: in any paths representation of $K_{3,3}$ on a cycle, the paths corresponding to the vertices of the hole $\{u_1, v_2, v_2, v_1, v_1, u_2, u_2, u_1\}$ must cover the cycle, while the disjoint paths $\ell_{v_1}, \ell_{v_2}$ must each intersect all the paths $\ell_{u_1}, \ell_{u_2}$: if for example the endpoints of $\ell_{u_2}$, $\ell_{v_2}$ are in $\ell_{u_1} \cap \ell_{u_2}$ and in $\ell_{v_1} \cap \ell_{v_2}$ then we obtain three mutually intersecting paths $\ell_{v_1}, \ell_{u_2}, \ell_{u_1}$ with an empty intersection, contradicting the Helly property. Since a Helly circular-arc graph cannot contain $K_{3,3}$ as a vertex subgraph, it follows that it has no antiholes with nine or more vertices. Thus, for recognizing if a Helly circular-arc graph $G$ is perfect it is enough to
check that \( G \) has no odd holes and that its complement has no holes with seven vertices. We remark that \( K_{3,3} \) is a circular arc graph represented by a family of arcs whose chords form a Star-of-David [8].

Consider a graph \( G \) having a clique cut-set: \( G \) is perfect iff all its leaves are perfect. Thus, a cactus subtree graph is perfect iff the circular-arc graphs in which it can be decomposed are perfect. This implies that the cactus subtree graphs fulfill the SPGC. Moreover, since \( K_{3,3} \) has no clique cut-sets it follows that it is not a Helly cactus subtree graph implying that a Helly cactus subtree graph has no antiholes with nine or more vertices. Thus, a Helly cactus subtree graph is tested for perfectness by decomposing it into Helly circular-arc graphs (using maximal clique cut-sets) and testing them for perfectness.

The k-coloring problem for circular-arc graphs is NP-complete [2].

Theorem 8. The k-coloring problem for circular-arc graphs is reducible to the k-coloring problem for Helly circular-arc graphs. Thus, this problem is also NP-complete.

Proof. Consider an intersection graph \( G \) of a family \( S \) of paths on a cycle \( CY \). For every two paths \( \ell_u, \ell_v \) covering \( CY \) we subdivide an edge \((y_u, z_u)\) of \( \ell_u - \ell_v \) by a vertex \( x_u \), we delete \( x_u \) and its incident edges from \( \ell_u \) and replace \( \ell_u \) by two paths \( \ell_{u1}, \ell_{u2} \). For every three mutually intersecting paths \( \ell_u, \ell_v, \ell_w \) with an empty intersection, we subdivide an edge \((y_u, z_u)\) of \( \ell_u - (\ell_v \cup \ell_w) \) by a vertex \( x_u \). We delete \( x_u \) and its incident edges from \( \ell_u \) and replace \( \ell_u \) by two paths \( \ell_{u1}, \ell_{u2} \). For every added vertex \( x_u \), if \( x_u \) is contained in \( r \) paths, we add to \( S \) \( k - r - 1 \) identical paths of the form \( \{(y_u, x_u), (x_u, z_u)\} \). It is easy to see that the new family of arcs \( S_1 \) has no two and no three arcs covering \( CY \). Thus, by Theorem 2, \( S_1 \) is a Helly family of arcs. Let \( G_1 \) be its intersection graph. Any k-coloring of \( G \) can be transformed into a k-coloring of \( G_1 \) by assigning the color of \( u \) in \( G \) to both \( u_1, u_2 \) in \( G_1 \), while the vertices corresponding to the new paths \( \{(y_u, x_u), (x_u, z_u)\} \) get the remaining \( k - r - 1 \) colors.

Conversely, consider a k-coloring of \( G_1 \). For every two paths \( \ell_{u1}, \ell_{u2} \) obtained from a path \( \ell_u \), the vertices \( u_1, u_2 \) must have the same color because of the paths \( \{(y_u, x_u), (x_u, z_u)\} \). Thus, we obtain a k-coloring of \( G \).  

In contrast to the above situation we will describe a polynomial time algorithm for finding minimum colorings of perfect circular-arc graphs. Consider a perfect circular-arc graph \( G \) represented by a family of paths \( S \) on a cycle \( CY \). Such a graph may have cliques whose corresponding paths have an empty intersection. The algorithm works as follows: we find an independent set \( I_1 \) intersecting all its maximum cliques (such an independent set always exists for perfect graphs), we make \( I_1 \) the first color and we delete the vertices of \( I_1 \) from \( G \). The remaining graph is also perfect and we continue in the same way. When the remaining graph is empty, the colors \( I_1, \ldots, I_k \) form a minimum coloring of \( G \).

It remains to show how to find in \( G \) an independent set \( I \) which intersects all its maximum cliques. For two adjacent vertices \( x, y \) of \( CY \), if \( V(x) \) is contained in \( V(y) \) we collapse the edge \((x, y)\) into one vertex in \( CY \) and in every path of \( S \). Continuing until
no more possible, we obtain a cycle $\text{CY}$ in which for every two adjacent vertices $x, y$, $V(x) \neq \emptyset$ and $V(y) \neq \emptyset$. We divide every edge of CY (and of every path of $S$) into two edges by inserting a new vertex in its middle; we denote the new cycle by CY'. By the algorithm in [3] we find the size $\beta(G)$ of a maximum clique of $G$ and we mark the vertices $z$ of CY' having $|V(z)| = \beta(G)$. No vertex $z$ of CY' obtained by dividing an edge $(x, y)$ of CY will be marked, since $V(x) - V(y) \neq \emptyset$, $V(y) - V(x) \neq \emptyset$ and thus $|V(z)| \leq |V(x) \cap V(y)| < \beta(G)$. The paths corresponding to the independent set $I$ do not cover all the edges of CY and thus it will not cover some unmarked vertex $x$ of CY'. Our purpose is thus to find an unmarked vertex $x$ for which there exists an independent set $I_x$ of $G(V - V(x))$ which intersects all the maximum cliques of $G$. Let $x$ be an unmarked vertex. We find $I_x$ as follows: for any maximum clique $c$ of $G$ having $\bigcap_{u \in c} \ell_u = \emptyset$, the mutually intersecting arcs in $\{\ell_u | u \in c - V(x)\}$ do not cover CY' and must have a nonempty intersection containing a vertex $y$; thus $V(x) \cup V(y)$ contains $c$. On the other hand, consider a vertex $y$ such that $G(V(x) \cup V(y))$ contains a maximum clique of size $\beta(G)$. The independent set $I_x$ must fulfill $V(x) \cap I_x = \emptyset$, $|V(y) \cap I_x| \leq 1$ since $V(y)$ is a clique, and it must intersect all the maximum cliques of $G(V(x) \cup V(y))$. Thus $|V(y) \cap I_x| = 1$ and $I_x$ must cover $y$. Hence, for every $y$ such that $\beta(G(V(x) \cup V(y)))) = \beta(G)$, the independent set $I_x$ must contain an element of $V(y)$ which is contained in all the maximum cliques of $G(V(x) \cup V(y))$. For every such $y$, we delete from $G$ the vertices $v$ of $V(y)$ which are not contained in all the maximum cliques of $G(V(x) \cup V(y))$ and mark the vertex $y$, since $I_x$ must cover $y$. We can identify the vertices $v$ by finding in polynomial time whether $\beta(G(V(x) \cup V(y)) - \{v\}) < \beta(G)$, $G(V(x) \cup V(y)) - \{v\}$ being the complement of a bipartite graph. Finally, we delete the vertices of $V(x)$ and denote the remaining graph by $G_1(V_1, E_1)$; $G_1$ is an interval graph on CY - $\{x\}$.

It remains to find in $G_1$ an independent set $I_x$ whose corresponding paths cover all the marked vertices. Let the marked vertices appear clockwise, or equivalently left to right on CY - $\{x\}$ in the order $x_1, \ldots, x_k$. We define a graph $H$ on $V_1$ as follows: for every $1 \leq i < k$, we insert edges $(u, v)$ between vertices $u, v$ fulfilling $\ell_u \cap \ell_v = \emptyset$ and both the right endvertex of $\ell_u$ and the left endvertex of $\ell_v$ are between $x_i$ and $x_{i+1}$. Thus, a requested independent set $I_x$ corresponds to a path in $H$ from a vertex $u$ to a vertex $v$ such that $x_i \in \ell_u$, $x_k \in \ell_v$. Such a path can be found in linear time directly from the family of paths on CY - $\{x\}$ using pointers and then backtracking. When $G$ is Helly, we take an unmarked vertex $x$ and directly construct $G_1(V_1, E_1)$.

The above algorithm implies a minimum coloring algorithm for perfect Helly cactus subtree graphs using the decomposition by maximal clique cut-sets into circular-arc graphs.

References