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# The restricted three body problem revisited ☆

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ABSTRACT

We present a new computation of the Birkhoff normal form for the Hamiltonian of the restricted three body problem near the Lagrangian libration points. This leads to a new proof of the Lyapunov stability of these points.



## 1. Introduction

The restricted three body problem is defined by the following time dependent Hamiltonian with two degrees of freedom

$$\frac{1}{2}\left(p_x^2+p_y^2\right)+V(x,y,t), \quad V=-\frac{1-\mu}{\rho_1}-\frac{\mu}{\rho_2},$$

where $\rho_1(x,y,t)$ and $\rho_2(x,y,t)$ are the distances of the point $A=(x,y)$ to the points $J=((1-\mu)\cos t,(1-\mu)\sin t)$ and $S=(-\mu\cos t,-\mu\sin t)$ respectively. Here $J$ (Jupiter) and $S$ (Sun) are interpreted as positions of two bodies rotating in an invariant plane about their center of mass and $A$ is a position of the third body (Asteroid) with mass so small that it does not influence the motion of the $S-J$ system. The parameter $\mu=\text{mass}(J)/(\text{mass}(J)+\text{mass}(S))$.

It turns out that the position of $A$ at the third vertex of one of the equilateral triangles with base $SJ$ is a point of relative equilibrium, so-called triangular libration point. After applying the rotation

$$\begin{pmatrix}x\\y\end{pmatrix}\to X(t)\begin{pmatrix}x\\y\end{pmatrix}, \qquad \begin{pmatrix}p_x\\p_y\end{pmatrix}\to X(t)\begin{pmatrix}p_x\\p_y\end{pmatrix}, \quad X=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix},$$

and moving the origin to the triangular libration point one arrives at a time independent system with the following Hamiltonian

$$H(q,p)=H_2+H_3+H_4+\cdots \tag{1.1}$$

where

$$H_2=\frac{1}{2}p_1^2+\frac{1}{2}p_2^2+p_1q_1-p_2q_2+\frac{1}{8}q_1^2-\zeta q_1q_2-\frac{5}{8}q_2^2, \tag{1.2}$$

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$$H_3 = -\frac{7\sqrt{3}\zeta}{36}q_1^3 + \frac{3\sqrt{3}}{16}q_1^2 q_2 + \frac{11\sqrt{3}\zeta}{12}q_1 q_2^2 + \frac{3\sqrt{3}}{16}q_2^3, \tag{1.3}$$

$$H_4 = \frac{37}{128}q_1^4 + \frac{25\zeta}{24}q_1^3 q_2 - \frac{123}{64}q_1^2 q_2^2 - \frac{15\zeta}{8}q_1 q_2^3 - \frac{3}{128}q_2^4 \tag{1.4}$$

and

$$\zeta = \frac{3\sqrt{3}}{4}(1-2\mu). \tag{1.5}$$

These formulas are obtained from the Taylor expansions of $\rho_1^{-1}$ and $\rho_2^{-1}$, where $\rho_{1,2}^2 = 1 \pm q_1 + \sqrt{3}q_2 + q_1^2 + q_2^2$ (compare also [7,8]). In what follows the parameter $\zeta$, which replaces $\mu$, will be principal.

One checks that when

$$\frac{\sqrt{23}}{4} < \zeta < \frac{3\sqrt{3}}{4}, \tag{1.6}$$

i.e. $0 < \mu < \mu_1 := \frac{1}{2}(1 - \sqrt{23/27})$, the linear part of the system, i.e. defined by $H_2$, has purely imaginary eigenvalues $\lambda_{1,2} = \mp i\omega_1$, $\lambda_{3,4} = \pm i\omega_2$, where $\omega_{1,2} > 0$ are defined by

$$\omega_{1,2}^2 = \frac{1}{2}(1 \pm \sqrt{\Delta}), \quad \Delta = 4\zeta^2 - \frac{23}{4}.$$

Therefore the corresponding linear system is Lyapunov stable.

In order to prove the genuine Lyapunov stability authors use the Kolmogorov–Arnold–Moser theory (see [1,2,9–11]). To this aim one reduces the Hamiltonian $H(q,p)$ to the following restricted Birkhoff normal form

$$H(q,p) = F(R,S) = \omega_1 I_1 - \omega_2 I_2 + 4\left(A I_1^2 + B I_1 I_2 + C I_2^2\right) + \cdots \tag{1.7}$$

where

$$I_{1,2} = \frac{1}{2}\left(R_{1,2}^2 + S_{1,2}^2\right) \tag{1.8}$$

and $F_0 = \omega_1 I_1 - \omega_2 I_2 + 4(A I_1^2 + B I_1 I_2 + C I_2^2)$ is treated as an unperturbed Hamiltonian. This reduction holds in the absence of order 3 and 4 resonances between the frequencies $\omega_1$ and $\omega_2$; there are only two values $\mu_2$ and $\mu_3$ values of the parameter $\mu$ for which these resonances occur.

Finally, the condition

$$\Gamma(\zeta) := A\omega_2^2 + B\omega_1\omega_2 + C\omega_1^2 \neq 0 \tag{1.9}$$

is sufficient to conclude the existence of invariant tori on the energy hypersurfaces $K = \mathrm{const}$ (the KAM theorem), which implies the Lyapunov stability.

The geometrical meaning of the condition (1.9) is following. In the angle-action variables $(\varphi, I)$ we have $\dot\varphi_1 = \omega_1 + 8AI_1 + 4BI_2$, $\dot\varphi_2 = -\omega_2 + 4BI_1 + 8CI_2$ for the unperturbed system. The derivative of the ratio $\frac{\dot\varphi_1(I)}{\dot\varphi_2(I)}$ in the direction of the vector $\omega_2\frac{\partial}{\partial I_1} + \omega_1\frac{\partial}{\partial I_2}$, tangent to the hypersurface $F = \mathrm{const}$, is proportional to $\Gamma$. Therefore the rotation number of the return map on $F = \mathrm{const}$ varies with the change of the radius of invariant circle. This is sufficient to prove the existence of invariant circles for the two-dimensional return map.

A.M. Leontovich in 1962 [7] proved that the algebraic function $\Gamma(\zeta)$ is non-constant, which implies that the Lyapunov stability takes place for all but discrete values of the parameter $\zeta$ satisfying (1.6).

In 1967 A. Deprit and A. Deprit-Bartholomé [5] gave the following explicit 'formula' for the function $\Gamma$:

$$\frac{36 - 541\omega_1^2\omega_2^2 + 644\omega_1^4\omega_2^4}{(1-4\omega_1^2\omega_2^2)(4-25\omega_1^2\omega_2^2)}. \tag{1.10}$$

(In [5] the authors use the notation $D = -\frac{1}{8}\Gamma$.) We shall denote the function (1.10) by $\Theta(\zeta)$. Since the function (1.10) vanishes for only one value of $\zeta$, which corresponds to the value $\mu = \mu_c \approx 0.0109$, the authors concluded that only three values of the parameter are bad for the Lyapunov stability. Formula (1.10) and the value $\mu_c$ are cited (without proofs) in many classical sources, like [3,10,11,8,6].

In fact, the additional bad value of the parameter can be computed explicitly, $\mu_c := \frac{1}{2} - \frac{2\sqrt{3}}{9}\sqrt{\frac{27}{16} - \frac{541}{1288} + \frac{\sqrt{199{,}945}}{1288}}$. This follows from the formulas given below

$$\omega_{1,2}^4 = \omega_{1,2}^2 + \zeta^2 - \frac{27}{16}, \tag{1.11}$$

$$\omega_1^2 + \omega_2^2 = 1, \tag{1.12}$$

$$\omega_1^2\omega_2^2 = \frac{27}{16} - \zeta^2. \tag{1.13}$$

It follows that the critical value for $\zeta$ equals $\zeta_c \approx 1.2707$.

Recently we looked closely into the restricted three body problem and tried to derive formula (1.10) independently. With some surprise we discovered that we cannot understand the Deprit-Bartholomé's argument. According to our calculations, which we present below, this formula should be much more complicated.

Also the Leontovich's analysis demonstrates that the formula (1.10) must be wrong. Leontovich analyzed the behavior of $\Gamma(\zeta)$ near the point where $\omega_2 = 0$. The paper [7] appeared much earlier than [5], but the contradiction is not discussed in [5].

We were not able to compute the complete formula for the algebraic function $\Gamma(\zeta)$. We study only its behavior near three of its singular points: where $\zeta \to \infty$, near the resonance $\omega_1 = 2\omega_2$ and near the Leontovich's point $\omega_2 = 0$.

We find that the behavior of $\Gamma$ near these points is definitely different than predicted by formula (1.10) (see Section 5). The main result is analogous to the Leontovich's theorem (but with somewhat new proof).

**Theorem 1.1.** *The function $\Gamma \not\equiv 0$. It means that there exists only finite set of bad values $\mu_j$, such that if $\mu \neq \mu_j$ then the Lagrangian points are Lyapunov stable.*

## 2. Reduction of $H_2$

The linear part of the Hamiltonian system takes the form $\dot{x} = Lx$, where $x = (q_1, p_1, q_2, p_2)^\top$ and

$$L = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1/4 & 0 & \zeta & 1 \\ -1 & 0 & 0 & 1 \\ \zeta & -1 & 5/4 & 0 \end{pmatrix}. \tag{2.1}$$

We apply a change of the form $x = Dy$, where the complex vector $y = (y_1, \bar{y}_1, y_2, \bar{y}_2)^\top$ satisfies the equations

$$\dot{y}_1 = -i\omega_1 y_1, \qquad \dot{y}_2 = i\omega_2 y_2, \tag{2.2}$$

i.e. $\dot{y} = \Lambda y$ where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is a suitable diagonal matrix. It follows that $LD = D\Lambda$, i.e. the columns $D_j$ of $D$ are eigenvectors of $A$ with the eigenvalues $\lambda_j$. These columns are defined modulo constant multipliers; these multipliers are related with the normalization $y_1 = M_1 z_1$, $y_2 = M_2 z_2$, where the real constants $M_{1,2}$ are determined from the condition that the corresponding change is symplectic. Namely the new symplectic coordinates are $(r_1, s_1, r_2, s_2)$ such that

$$z_{1,2} = r_{1,2} + is_{1,2}$$

and $dq_1 \wedge dp_1 + dq_2 \wedge dp_2 = dr_1 \wedge ds_1 + dr_2 \wedge ds_2$.

Elementary calculations show that the following choice is admissible

$$D_1 = \begin{pmatrix} -\zeta + 2i\omega_1 \\ \omega_1^2 - \frac{3}{4} + i\zeta\omega_1 \\ \omega_1^2 + \frac{3}{4} \\ -\zeta + i\omega_1(\frac{5}{4} - \omega_1^2) \end{pmatrix}, \qquad D_3 = \begin{pmatrix} -\zeta - 2i\omega_2 \\ \omega_2^2 - \frac{3}{4} - i\zeta\omega_2 \\ \omega_2^2 + \frac{3}{4} \\ -\zeta - i\omega_2(\frac{5}{4} - \omega_2^2) \end{pmatrix},$$

$$D_2 = \bar{D}_1, \qquad D_4 = \bar{D}_3. \tag{2.3}$$

Therefore

$$\begin{aligned} q_1 &= (-\zeta + 2i\omega_1)M_1 z_1 + (-\zeta - 2i\omega_1)M_1\bar{z}_1 + (-\zeta - 2i\omega_2)M_2 z_2 + (-\zeta + 2i\omega_2)M_2\bar{z}_2, \\ p_1 &= \left(\omega_1^2 - \frac{3}{4} + i\zeta\omega_1\right)M_1 z_1 + \left(\omega_1^2 - \frac{3}{4} - i\zeta\omega_1\right)M_1\bar{z}_1 + \left(\omega_2^2 - \frac{3}{4} - i\zeta\omega_2\right)M_2 z_2 + \left(\omega_2^2 - \frac{3}{4} + i\zeta\omega_2\right)M_2\bar{z}_2, \\ q_2 &= \left(\omega_1^2 + \frac{3}{4}\right)(M_1 z_1 + M_1\bar{z}_1) + \left(\omega_2^2 + \frac{3}{4}\right)(M_2 z_2 + M_2\bar{z}_2), \\ p_2 &= \left(-\zeta + i\omega_1\left(\frac{5}{4} - \omega_1^2\right)\right)M_1 z_1 + \left(-\zeta - i\omega_1\left(\frac{5}{4} - \omega_1^2\right)\right)M_1\bar{z}_1 + \left(-\zeta - i\omega_2\left(\frac{5}{4} - \omega_2^2\right)\right)M_2 z_2 \\ &\quad + \left(-\zeta + i\omega_2\left(\frac{5}{4} - \omega_2^2\right)\right)M_2\bar{z}_2. \end{aligned} \tag{2.4}$$

The condition of symplectomorphism, i.e. the compatibility of the following Poisson brackets relations $\{q_{1,2}, p_{1,2}\} = 1$, $\{z_{1,2}, \bar{z}_{1,2}\} = -2i$ (and vanishing of other brackets), implies that

$$M_{1,2}^2 = \frac{1}{4\omega_{1,2}(3/4 + \omega_{1,2}^2)\sqrt{\Delta}}. \tag{2.5}$$

Of course, the variables $z_1$ and $z_2$ satisfy the same Eqs. (2.2) as the variables $y_1$ and $y_2$. So we get

$$\dot{r}_1 = \omega_1 s_1, \qquad \dot{s}_1 = -\omega_1 r_1, \qquad \dot{r}_2 = -\omega_2 s_2, \qquad \dot{s}_2 = \omega_2 r_2,$$

which is a Hamiltonian system with the Hamiltonian

$$K_2(r, s) = \frac{1}{2}\omega_1 z_1 \bar{z}_1 - \frac{1}{2}\omega_2 z_2 \bar{z}_2. \tag{2.6}$$

We finish this section by a discussion about the dependence of the introduced quantities on the parameter $\zeta$. Of course, $\sqrt{\Delta}$, $\omega_1$, $\omega_2$, $M_{1,2}$ are algebraic functions of $\zeta$. Also the entries of the matrix $C$ are algebraic functions of $\zeta$. So they can be treated as multivalued functions of the complex parameter $\zeta$. We shall be interested in the behavior of these functions as $\zeta \to \infty$, as well as when $\omega_1 \approx 2\omega_2$. Of course, a suitable branch of a given algebraic function must be chosen. Below the symbol $\sim$ denotes the leading term in asymptotic expansion of a function and the symbol $\ldots$ denotes further terms in this expansion.

**Lemma 2.1.** *As $\zeta \to \infty$ we have*

$$\begin{aligned}
&\sqrt{\Delta} \sim 2\zeta, \qquad \omega_{1,2}^2 \sim \pm\zeta,\\
&\omega_1 \sim \zeta^{1/2}, \qquad \omega_2 \sim i\zeta^{1/2},\\
&M_1 \sim 2^{-3/2}\zeta^{-5/4}, \qquad M_2 \sim \sqrt{i}2^{-3/2}\zeta^{-5/4},\\
&\Theta \sim \frac{161}{25}.
\end{aligned}$$

*Moreover,*

$$\begin{aligned}
q_1 &= \frac{-\zeta^{-1/4}}{2^{3/2}}\left\{z_1 + \bar{z}_1 + \sqrt{i}(z_2 + \bar{z}_2)\right\} + \cdots,\\
q_2 &= \frac{\zeta^{-1/4}}{2^{3/2}}\left\{z_1 + \bar{z}_1 - \sqrt{i}(z_2 + \bar{z}_2)\right\} + \cdots.
\end{aligned}$$

*Here $i = e^{i\pi/2}$, $\sqrt{i} = e^{i\pi/4} = (1+i)/\sqrt{2}$ and $\Theta$ is defined by* (1.10).

**Lemma 2.2.** *The resonance $\omega_1 : \omega_2 = 2 : 1$ of order* 3 *occurs for*

$$\zeta = \zeta_2 = \frac{\sqrt{611}}{20}.$$

*In this case we have*

$$\begin{aligned}
&\sqrt{\Delta} = \frac{3}{5} + \frac{10}{3}\left(\zeta^2 - \zeta_2^2\right) + \cdots,\\
&\omega_1^2 = \frac{4}{5} + \frac{5}{3}\left(\zeta^2 - \zeta_2^2\right) + \cdots, \qquad \omega_2^2 = \frac{1}{5} - \frac{5}{3}\left(\zeta^2 - \zeta_2^2\right) + \cdots,\\
&\omega_1 \sim \frac{2}{\sqrt{5}}, \qquad \omega_2 \sim \frac{1}{\sqrt{5}},\\
&M_1 \sim \frac{5\sqrt[4]{5}}{\sqrt{6\cdot 31}}, \qquad M_2 \sim \frac{5\sqrt[4]{5}}{\sqrt{3\cdot 19}},\\
&\Theta = -\frac{21\,296}{5625\cdot(\zeta^2 - \zeta_2^2)} + \cdots,\\
&q_1 \sim \left\{\left(-\zeta_2 + \frac{4i}{\sqrt{5}}\right)M_1 z_1 + \mathrm{conj}\right\} + \left\{\left(-\zeta_2 - \frac{2i}{\sqrt{5}}\right)M_2 z_2 + \mathrm{conj}\right\},\\
&q_2 \sim \frac{31}{20}M_1(z_1 + \bar{z}_1) + \frac{19}{20}M_2(z_2 + \bar{z}_2).
\end{aligned}$$

*Here* conj *means the conjugated term.*

**Lemma 2.3.** *Near the point $\zeta = \zeta_0 := \frac{3\sqrt{3}}{4}$, i.e. where $\omega_2 = 0$, we have*

$$\sqrt{\Delta} = 1 - 2\nu + \cdots,$$
$$\omega_1 = 1 + O(\nu), \qquad \omega_2 \sim \nu^{1/2} + O\big(\nu^{3/2}\big),$$
$$M_1 = \frac{1}{\sqrt{7}} + O(\nu), \qquad M_2 = \frac{1}{\sqrt{3}}\nu^{-1/4} + O\big(\nu^{3/4}\big),$$
$$\Theta \sim 9,$$

*where*

$$\nu := \frac{3\sqrt{3}}{2}(\zeta_0 - \zeta) \to 0.$$

*Moreover,*

$$q_1 = -\frac{3}{4}\nu^{-1/4}(z_2 + \bar{z}_2) + \frac{1}{\sqrt{7}}\big[(-\zeta_0 + 2i)z_1 - (\zeta_0 + 2i)\bar{z}_1\big] - \frac{2i}{\sqrt{3}}\nu^{1/4}(z_2 - \bar{z}_2) + O\big(\nu^{3/4}\big),$$
$$q_2 = \frac{\sqrt{3}}{4}\nu^{-1/4}(z_2 + \bar{z}_2) + \frac{1}{4\sqrt{7}}(z_1 + \bar{z}_1) + O\big(\nu^{3/4}\big).$$

## 3. Cubic and quartic terms

Let us rewrite the parts $H_3$ and $H_4$ in the variables $z_{1,2}$. The complete formulas are highly complicated. So we begin with some simplifications.

Note that the quadratic part $H_2 = K_2$ is invariant with respect to the following action of the torus $\mathbb{S}^1 \times \mathbb{S}^1 = \{(e^{i\alpha}, e^{i\beta})\}$:

$$(z_1, z_2) \to \big(e^{i\alpha} z_1, e^{i\beta} z_2\big). \tag{3.1}$$

These changes are symplectic and represent the only non-uniqueness in the reduction of the linear Hamiltonian system to its normal form.

We divide the terms in $H_3(q, p) = K_3(z, \bar{z})$ into semi-invariants with respect to this action. The weights are of the form $(m, n)$, where $m, n = -3, -2, -1, 0, 1, 2, 3$ but not all pairs are admissible.

Thus

$$\begin{aligned} K_3 = {} & k_{3,0}z_1^3 + k_{1,0}z_1 + k_{-1,0}\bar{z}_1 + k_{-3,0}\bar{z}^3 + k_{0,3}z_2^3 + k_{0,1}z_2 + k_{0,-1}\bar{z}_2 + k_{0,-3}\bar{z}^3 + k_{2,1}z_1^2 z_2 \\ & + k_{2,-1}z_1^2\bar{z}_2 + k_{-2,1}\bar{z}_1^2 z_2 + k_{-2,-1}\bar{z}_1^2\bar{z}_2 + k_{1,2}z_1 z_2^2 + k_{1,-2}z_1\bar{z}_2^2 + k_{-1,2}\bar{z}_1 z_2^2 + k_{-1,-2}\bar{z}_1\bar{z}_2^2, \end{aligned} \tag{3.2}$$

where the coefficients $k_{1,0}, k_{-1,0}, k_{0,1}, k_{0,-1}$ are homogeneous quadratic polynomials of the form $a|z_1|^2 + b|z_2|^2$; we write

$$k_{1,0} = k'_{1,0}|z_1|^2 + k''_{1,0}|z_2|^2,$$

etc. Of course, the reality of $H_3$ implies that

$$k_{-m,-n} = \bar{k}_{m,n}.$$

Like in the end of the previous section we treat the coefficients $k_{m,n}$ as algebraic functions of $\zeta$.

**Remark 3.1.** The above reality condition holds true in the domain (1.6), where $\omega_1$ and $\omega_2$ are positive. When $\zeta$ moves from this interval into the complex domain the coefficients $k_{m,n}(\zeta)$ must be treated as multivalued algebraic functions and there is no reason for such reality conditions.

In the next section we shall use notations, like $|k_{3,0}|^2$. In the real domain it equals $k_{3,0}k_{-3,0}$. In the complex domain we should use rather the second notation. For the sake of simplicity of notations we shall still use $|k_{3,0}|^2$, with the agreement that it is $k_{3,0} \cdot k_{-3,0}$, when continued for complex values of $\zeta$. This agreement applies also to other coefficients $k_{-m,-n}$.

**Lemma 3.1.** *As $\zeta \to \infty$ we have $k_{m,n} \sim \frac{-\zeta^{1/4}}{2^{11/2}3^{3/2}} \cdot \tilde{k}_{m,n}$, where*

$$\tilde{k}_{3,0} = \tilde{k}_{-3,0} = 13, \qquad \tilde{k}'_{1,0} = \tilde{k}'_{-1,0} = 39,$$
$$\tilde{k}''_{1,0} = \tilde{k}''_{-1,0} = -54i, \qquad \tilde{k}_{0,3} = \tilde{k}_{0,-3} = 13\frac{1-i}{\sqrt{2}},$$

$$\tilde{k}'_{0,1} = \tilde{k}'_{0,-1} = -54\frac{1+i}{\sqrt{2}}, \qquad \tilde{k}''_{0,1} = \tilde{k}''_{0,-1} = 39\frac{1-i}{\sqrt{2}},$$
$$\tilde{k}_{2,1} = \tilde{k}_{2,-1} = \tilde{k}_{-2,1} = \tilde{k}_{-2,-1} = -27\frac{1+i}{\sqrt{2}},$$
$$\tilde{k}_{1,2} = \tilde{k}_{1,-2} = \tilde{k}_{-1,2} = \tilde{k}_{-1,-2} = -27i.$$

**Proof.** From Eq. (1.3) and Lemma 2.1 we get

$$K_3 = \frac{\sqrt{3}\zeta^{1/4}}{36\cdot 2^{9/2}} \times \left\{7\left(z_1+\bar{z}_1+\frac{1+i}{\sqrt{2}}(z_2+\bar{z}_2)\right)^3 - 33\left(z_1+\bar{z}_1+\frac{1+i}{\sqrt{2}}(z_2+\bar{z}_2)\right)\left(z_1+\bar{z}_1-\frac{1+i}{\sqrt{2}}(z_2+\bar{z}_2)\right)^2\right\}.$$

From this the lemma follows. □

**Lemma 3.2.** *At $\zeta = \zeta_2$ we have*

$$k_{1,2} = \frac{5^{3+3/4}}{228\cdot\sqrt{62}}\left(\frac{3371}{3750} + i\frac{17}{375}\sqrt{\frac{611}{5}}\right)$$

*and*

$$|k_{1,2}|^2 = \frac{1331}{51\,840}\sqrt{5}.$$

**Proof.** Using formula (1.3) and Lemma 2.2 we find

$$\begin{aligned} k_{1,2} = \frac{5^{3+3/4}}{3\cdot 4\cdot\sqrt{2\cdot 31}\cdot 19}\Bigg\{&\frac{7\zeta_2}{9}\cdot 3\cdot\left(\zeta_2-\frac{4i}{\sqrt{5}}\right)\left(\zeta_2+\frac{2i}{\sqrt{5}}\right)^2 \\ &+\frac{3}{4}\cdot\left[2\cdot\left(-\zeta_2+\frac{4i}{\sqrt{5}}\right)\cdot\left(-\zeta_2-\frac{2i}{\sqrt{5}}\right)\cdot\frac{19}{20}+\left(\zeta_2+\frac{2i}{\sqrt{5}}\right)^2\cdot\frac{31}{20}\right] \\ &+\frac{11\zeta_2}{3}\left[\left(-\zeta_2+\frac{4i}{\sqrt{5}}\right)\cdot\frac{19^2}{20^2}+2\cdot\left(-\zeta_2-\frac{2i}{\sqrt{5}}\right)\cdot\frac{19}{20}\cdot\frac{31}{20}\right]+\frac{3}{4}\cdot 3\cdot\frac{31}{20}\cdot\frac{19^2}{20^2}\Bigg\}. \end{aligned}$$

After collection of similar terms we arrive to the above formula. □

**Lemma 3.3.** *Near $\zeta = \zeta_0$ we have*

$$K_3 = \frac{9}{2^5\sqrt{7}}\nu^{-1/2}\cdot\left[(\sqrt{3}-4i)z_1+(\sqrt{3}+4i)\bar{z}_1\right]\cdot(z_2+\bar{z}_2)^2+\frac{3\sqrt{3}i}{8}\cdot\nu^{-1/4}\cdot\left(z_2^2-\bar{z}_2^2\right)\cdot(z_2+\bar{z}_2) + \nu^{-1/4}(z_2+\bar{z}_2)P(z_1,\bar{z}_1)+O(1),$$

*where $P$ is a quadratic polynomial and $\nu = \frac{3\sqrt{3}}{2}(\zeta_0-\zeta)\to 0$.*

**Proof.** Using (1.3) and Lemma 2.3 we find that $K_3$ is of the form

$$a_1(z_2+\bar{z}_2)^3\nu^{-3/4}+(a_2z_1+\bar{a}_2\bar{z}_1)(z_2+\bar{z}_2)^2\nu^{-1/2}+ia_3(z_2-\bar{z}_2)(z_2+\bar{z}_2)^2\nu^{-1/4}+O(1).$$

Calculations show that $a_1 = 0$ and $a_2$, $a_3$ are like in the thesis of the lemma. □

We can expand $K_4(z,\bar{z}) = H_4(q,p)$ in a form similar to (3.2). But for us only one term turns out interesting, namely

$$k_{0,0}\cdot 1 = k'_{0,0}|z_1|^4 + k''_{0,0}|z_1|^2|z_2|^2 + k'''_{0,0}|z_2|^4.$$

**Lemma 3.4.** *As $\zeta\to\infty$ we have*

$$k_{0,0}\sim -\frac{5}{2^6}\left(|z_1|^4+|z_2|^4\right).$$

**Proof.** By Eq. (1.4) and Lemma 2.1 we have

$$K_4 = \frac{5}{3\cdot 2^9}\left\{5\left(z_1+\bar z_1+\sqrt{i}(z_2+\bar z_2)\right)^2 - 9\left(z_1+\bar z_1-\sqrt{i}(z_2+\bar z_2)\right)^2\right\}$$
$$\times \left(z_1+\bar z_1+\sqrt{i}(z_2+\bar z_2)\right)\left(z_1+\bar z_1-\sqrt{i}(z_2+\bar z_2)\right)+\cdots.$$

Further calculations show that the term with $\zeta^0$ is like above. □

**Lemma 3.5.** *As $\zeta \to \zeta_2$ the quantity $k_{0,0}$ in $K_4$ remains finite.*

**Proof.** This is obvious. □

**Lemma 3.6.** *As $\zeta \to \zeta_0$ we have*

$$k_{0,0} = -\frac{81}{64}\nu^{-1}|z_2|^4+\cdots.$$

**Proof.** Here calculations are similar as in the proof of Lemma 3.3. □

## 4. Nonlinear Birkhoff transformation

The standard canonical transformation $(r,s)\to(R,S)$ which should reduce the cubic terms in the Hamiltonian is defined by the formulas

$$r = R - \frac{\partial\Phi}{\partial S}(r,S), \qquad s = S + \frac{\partial\Phi}{\partial r}(r,S), \tag{4.1}$$

i.e. $s\,dr + R\,dS = d(rS+\Phi(r,S))$. Here the generating function (see [2,4]) $\Phi(r,S)$ is a cubic homogeneous polynomial which soon will be determined. In order to eliminate the dependence of the right-hand sides of (4.1) on $r$ we perform one more iteration:

$$r = R - \frac{\partial\Phi}{\partial S}(R,S) + \frac{\partial^2\Phi}{\partial r\partial S}(R,S)\frac{\partial\Phi}{\partial S}(R,S)+\cdots,$$
$$s = S + \frac{\partial\Phi}{\partial r}(R,S) - \frac{\partial^2\Phi}{\partial r^2}(R,S)\frac{\partial\Phi}{\partial S}(R,S)+\cdots \tag{4.2}$$

where the dots mean terms of order $\geqslant 4$. In terms of the complex variables $z=r+is$ and $Z=R+iS$, with $\Psi(Z,\bar Z)=\Phi(R,S)$, we have

$$z_{1,2} = Z_{1,2} + 2i\frac{\partial\Psi}{\partial\bar Z_{1,2}} + 2\left(\frac{\partial}{\partial Z}+\frac{\partial}{\partial\bar Z}\right)\frac{\partial\Psi}{\partial\bar Z_{1,2}}\cdot\left(\frac{\partial}{\partial Z}-\frac{\partial}{\partial\bar Z}\right)\Psi+\cdots. \tag{4.3}$$

Substituting this into the Hamiltonian and comparison of the homogeneous cubic terms gives

$$\psi_{m,n} = \frac{-i}{m\omega_1 - n\omega_2}k_{m,n}$$

in the expansion

$$\Psi = \psi_{3,0}Z_1^3+\cdots+\psi_{-1,-2}\bar Z_1\bar Z_2^2,$$

analogous to (3.2).

Having determined the form $\Psi$ we should now look at the resonant terms of order four in the Hamiltonian, i.e. $|Z_1|^4$, $|Z_1|^2|Z_2|^2$ and $|Z_2|^4$. There are four sources of such terms:

1. remaining from $K_4(z,\bar z)$,
2. arising from application of the quadratic part of (4.3) to $K_3(z,\bar z)$,
3. arising from application of the cubic part of (4.3) to $K_2$,
4. arising from application of the quadratic part of (4.3) to $K_2$.

We shall deal with all these contributions separately. We denote by $A_i$ (respectively by $B_i$ and $C_i$) the coefficient before $|Z_1|^4$ (respectively before $|Z_1|^2|Z_2|^2$ and $|Z_2|^4$) from the $i$th part.

1. This part gives $A_1 = k'_{0,0}$, $B_1 = k''_{0,0}$, $C_1 = k'''_{0,0}$.

2. This part arises from the expression $2i\frac{\partial K_3}{\partial z}\frac{\partial\Psi}{\partial\bar z} - 2i\frac{\partial K_3}{\partial\bar z}\frac{\partial\Psi}{\partial Z}$.
3. Denote by $A_3'$ and $B_3'$ the coefficients before $Z_1^2\bar Z_1$ and $Z_1|Z_2^2|$, respectively, in the third order part of the change $z_1 = Z_1 + \cdots$, i.e. in $2(\frac{\partial}{\partial Z_1} + \frac{\partial}{\partial\bar Z_1})\frac{\partial\Psi}{\partial\bar Z_1}\cdot(\frac{\partial}{\partial Z_1} - \frac{\partial}{\partial\bar Z_1})\Psi + 2(\frac{\partial}{\partial Z_2} + \frac{\partial}{\partial\bar Z_2})\frac{\partial\Psi}{\partial\bar Z_1}\cdot(\frac{\partial}{\partial Z_2} + \frac{\partial}{\partial\bar Z_2})\Psi$ (see (4.3)). Also denote by $B_3''$ and $C_3''$ the corresponding coefficients before $|Z_1|^2Z_2$ and $Z_2^2\bar Z_2$ in $z_2 = Z_2 + \cdots$. Then we have
$$A_3 = \frac{1}{2}\omega_1\big(A_3' + \bar A_3'\big), \qquad C_3 = -\frac{1}{2}\omega_2\big(C_3'' + \bar C_3''\big),$$
$$B_3 = \frac{1}{2}\omega_1\big(B_3' + \bar B_3'\big) - \frac{1}{2}\omega_2\big(B_3'' + \bar B_3''\big).$$
The computation of $A_3'$, $B_3'$, $B_3''$ and $C_3''$ is standard, but rather tedious.
4. Here we get the resonant terms in $\frac{1}{2}\omega_1|2i\frac{\partial\Psi}{\partial\bar Z_1}|^2 - \frac{1}{2}\omega_2|2i\frac{\partial\Psi}{\partial\bar Z_2}|^2$.

Summing up results of the above computations we get the following result, proved firstly by Leontovich (in [7] one finds $K_2 = \lambda_1 z_1\bar z_1 + \lambda_2 z_2\bar z_2$, so the coefficients are slightly changed).

**Proposition 4.1.** *The coefficients before the resonant terms in $F_4$ are the following*
$$A = k_{0,0}' - 6\frac{|k_{1,0}'|^2}{\omega_1} - 6\frac{|k_{3,0}|^2}{\omega_1} + 2\frac{|k_{0,1}'|^2}{\omega_2} - 2\frac{|k_{2,1}|^2}{2\omega_1 - \omega_2} + 2\frac{|k_{2,-1}|^2}{2\omega_1 + \omega_2},$$
$$B = k_{0,0}'' - 4\frac{k_{1,0}'\bar k_{1,0}'' + \bar k_{1,0}' k_{1,0}''}{\omega_1} + 4\frac{k_{0,1}''\bar k_{0,1}' + \bar k_{0,1}'' k_{0,1}'}{\omega_2} - 8\frac{|k_{2,1}|^2}{2\omega_1 - \omega_2}$$
$$- 8\frac{|k_{2,-1}|^2}{2\omega_1 + \omega_2} - 8\frac{|k_{1,2}|^2}{\omega_1 - 2\omega_2} + 8\frac{|k_{1,-2}|^2}{\omega_1 + 2\omega_2},$$
$$C = k_{0,0}''' + 6\frac{|k_{0,1}''|^2}{\omega_2} + 6\frac{|k_{0,3}|^2}{\omega_2} - 2\frac{|k_{1,0}''|^2}{\omega_1} - 2\frac{|k_{1,2}|^2}{\omega_1 - 2\omega_2} - 2\frac{|k_{1,-2}|^2}{\omega_1 + 2\omega_2}.$$

**Remark 4.1.** The Birkhoff normalization of the quadratic, cubic and quartic terms of the Hamiltonian is unique modulo the changes (3.1), i.e. the torus action. It follows that the coefficients $A$, $B$ and $C$ are defined uniquely; they are invariant with respect to the torus action. Therefore also the function $\Gamma(\zeta)$ is unique.

**Remark 4.2.** In [5] the authors refer to a 1966 thesis of J. Henrard, who: 'has shown how to carry on in a straightforward manner Birkhoff's normalization without introducing generating functions and without inverting power series'. In fact, these changes are of the form $x = f(\varphi, I)$, $y = g(\varphi, I)$, where $f$ and $g$ are functions of the angle-action variables which should satisfy corresponding Poisson brackets relations. The authors say that: 'In this way, a Birkhoff normalizing transformation can be constructed entirely by the method of undetermined coefficients'.

Unfortunately, the calculations of [5] do not include checking the Poisson relations, which are essential for the canonical form of the change.

We note also investigations by E. Grebenikov and his students [6] of the restricted problem of many bodies ($> 3$).

## 5. The algebraic function $\Gamma$

The function $\Gamma(\zeta)$ from Introduction can be represented as the sum of four terms,
$$\Gamma = \Gamma' + \Gamma'',$$
where
$$\Gamma' = k_{0,0}'\omega_2^2 + k_{0,0}''\omega_1\omega_2 + k_{0,0}'''\omega_1^2$$
and $\Gamma'' = \Gamma - \Gamma'$ is calculated using $A'' = A - k_{0,0}'$, $B'' = B - k_{0,0}''$ and $C'' = C - k_{0,0}'''$. Our aim is to compute behavior of $\Gamma(\zeta)$ as $\zeta \to \infty$, as $\zeta \to \zeta_2$ and as $\zeta \to \zeta_0$.

1. $\zeta \to \infty$. By Lemma 2.1 and Lemma 3.4 we have
$$\Gamma' = o(\zeta) \quad \text{as } \zeta \to \infty.$$

In calculations of $\Gamma''$ we use formulas from the previous section, which are given for the case of real $\zeta$ and real positive $\omega_{1,2}$. According to Remark 3.1 we continue these formulas to other values of $\zeta$ with the agreement that $|k_{m,n}|^2$ means $k_{m,n}k_{-m,-n}$.

The calculation of $\Gamma''$ is following:

$$27\cdot 2^{11}\cdot\zeta^{-1}\cdot\omega_2^2 A''\sim\frac{18\,624}{5},$$
$$27\cdot 2^{11}\cdot\zeta^{-1}\cdot\omega_1\omega_2 B''\sim -33\,696,$$
$$27\cdot 2^{11}\cdot\zeta^{-1}\cdot\omega_1^2 C''\sim -\frac{18\,624}{5}.$$

Summing the above we get the following

**Proposition 5.1.** *As $\zeta\to\infty$ the function $\Gamma$ grows linearly,*

$$\Gamma(\zeta)\sim -\frac{39}{64}\cdot\zeta.$$

Of course, this behavior differs from the behavior of the function $\Theta(\zeta)$, see Lemma 2.1.

2. $\zeta\to\zeta_2$. Since the coefficients $B''$ and $C''$ contain terms proportional to $|k_{1,2}|^2/(\omega_1-2\omega_2)$ (see Proposition 4.1) also $\Gamma$ contains such term. It equals

$$\left(-8\cdot\frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}}-2\cdot\left(\frac{2}{\sqrt{5}}\right)^2\right)\cdot\frac{|k_{1,2}|^2}{\omega_1-2\omega_2}.$$

Using $\omega_1-2\omega_2=(\omega_1^2-4\omega_2^2)/(\omega_1+2\omega_2)\sim\frac{25\sqrt{5}}{12}(\zeta^2-\zeta_2^2)$ and Lemma 3.2 we arrive to the following

**Proposition 5.2.** *The function $\Gamma(\zeta)$ has simple order pole at the point $\zeta_2$, corresponding to* 2 : 1 *resonance, of the form*

$$\Gamma(\zeta)=-\frac{1331}{22\,500\cdot(\zeta^2-\zeta_2^2)}+O(1).$$

On the other hand, the expression $\Theta(\zeta)$ given in (1.10) has also first order pole at $\zeta_2$, but with different residuum (see Lemma 2.2).

3. $\zeta\to\zeta_0$. By Lemma 3.6 we get

$$\Gamma'\sim -\frac{81}{64}\nu^{-1},$$

where $\nu=\frac{3\sqrt{3}}{2}(\zeta_0-\zeta)\to 0$. Next, using Lemma 2.3 and formulas for $k_{m,n}$ in Lemma 3.3 we find that: $\omega_2^2 A''$ is of order $O(\nu^{1/4})$, the term $\omega_1\omega_2 B''$ is of order $O(\nu^{-1/2})$ and the last term $\omega_1^2 C''$ is of order $O(\nu^{-1})$. Thus the leading term in $\Gamma''$ arises from

$$6\left(\left|k''_{0,1}\right|^2+|k_{0,3}|^2\right)\nu^{-1/2}-2\left(\left|k''_{1,0}\right|^2+|k_{1,2}|^2+2|k_{1,-2}|^2\right).$$

Taking suitable values from Lemma 3.3 we get

$$\Gamma''\sim\frac{4455}{1792}\cdot\nu^{-1}.$$

Together we get the following result which contradicts the finiteness of the function $\Theta$ (see Lemma 2.3).

**Proposition 5.3.** *As $\zeta\to\zeta_0$ the function $\Gamma(\zeta)$ tends to infinity,*

$$\Gamma\sim\frac{2187}{1792}\cdot\nu^{-1}.$$

**Remark 5.1.** Theoretically it is possible to find an explicit expression for the function $\Gamma(\zeta)$. One should determine its behavior at its singular points corresponding to: $\Delta=0$, $\omega_1=0$, $\omega_2=0$, $\omega_1=2\omega_1$, $\omega_1=-2\omega_2$, $2\omega_1=\omega_2$, $2\omega_1=-\omega_2$, $\omega_1^2=-\frac{3}{4}$, $\omega_2^2=-\frac{3}{4}$ and $\zeta=\infty$.

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