Spectral conditions on Lie and Jordan algebras of compact operators ☆

Matthew Kennedy [*], Heydar Radjavi

Department of Pure Mathematics, University of Waterloo, 200 University Ave West, Waterloo, Ontario, Canada N2L 3G1

Abstract

We investigate the properties of bounded operators which satisfy a certain spectral additivity condition, and use our results to study Lie and Jordan algebras of compact operators. We prove that these algebras have nontrivial invariant subspaces when their elements have sublinear or submultiplicative spectrum, and when they satisfy simple trace conditions. In certain cases we show that these conditions imply that the algebra is (simultaneously) triangularizable.
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1. Introduction

Conditions on the spectrum of an operator have been studied for some time, for instance in the work of Motzkin and Taussky [7], who investigated pairs of $n \times n$ matrices A and B with the property that for every scalar λ, the eigenvalues of $A + \lambda B$ are, subject to a slight technical condition, linear functions of the eigenvalues of A and B. Specifically, they required that the eigenvalues $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$, of A and B respectively, could be expressed as ordered sets $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$, such that for every scalar λ, the eigenvalues of $A + \lambda B$ are precisely $\alpha_i + \lambda \beta_i$, for $i = 1, \ldots, n$. Such matrices are said to have property L ("L" is for "linear").

A pair of (simultaneously) triangularizable matrices clearly has property L, and Motzkin and Taussky were interested in conditions under which the converse of this was true. They showed that a finite group of matrices, with every pair of elements in the group having property L, is not just triangularizable, but diagonalizable. Somewhat later, Wales and Zassenhaus [13], Zassenhaus [14], and Guralnick [2] showed that a multiplicative semigroup of $n \times n$ matrices is triangularizable under the same circumstances.

The following conditions are seemingly much weaker than property L.

Definition 1.1. A pair of bounded operators A and B on a Banach space are said to have *subadditive spectrum* if

$$\sigma(A + B) \subseteq \sigma(A) + \sigma(B),$$

where $\sigma(A) + \sigma(B)$ means the set of all $\alpha + \beta$, with α and β in $\sigma(A)$ and $\sigma(B)$ respectively. Similarly, A and B are said to have *sublinear spectrum* if

$$\sigma(A + \lambda B) \subseteq \sigma(A) + \lambda \sigma(B)$$

for every complex number λ. A family $\mathcal{F}$ of bounded operators on a Banach space is said to have subadditive (resp. sublinear) spectrum if every pair of elements in $\mathcal{F}$ has subadditive (resp. sublinear) spectrum.

Note that for linear spaces, and in particular for Lie and Jordan algebras, subadditivity of the spectrum clearly implies sublinearity.

The conditions of sublinear and subadditive spectrum are, in some sense, even weaker than one might initially suspect. Indeed, as the following example illustrates, there is little hope of obtaining any results about the existence of invariant subspaces for an arbitrary family of operators with subadditive or even sublinear spectrum without imposing a great deal of additional structure.

Example 1.2. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is easy to verify that the linear space $\mathcal{S}$ spanned by A and B consists entirely of nilpotent matrices, so $\mathcal{S}$ has sublinear spectrum, yet $\mathcal{S}$ clearly has no nontrivial invariant subspaces.

It was shown in [8] that the property of sublinear spectrum implies the triangularizability of a semigroup of compact operators, which is an extension of the classical results mentioned above to the context of an infinite-dimensional Banach space.

A natural question is whether this kind of result holds for other types of algebraic structures. In this paper we study Lie and Jordan algebras of compact operators satisfying spectral conditions like sublinearity and submultiplicativity, which have been shown to imply the existence of invariant subspaces for semigroups of compact operators. We will prove that many of the results obtained for semigroups also hold in this nonassociative context.

Shulman and Turovskii obtained several results of this nature in their development of a radical theory for Lie algebras of compact operators [11]. They showed that a Lie algebra of compact operators with subadditive spectrum has invariant subspaces [11], and it was implicit in their proof that this condition actually implied triangularizability. This gave an extension of Engel's Theorem for Lie algebras of compact operators, which was also proven recently in [10].

On the other hand, results on the existence of invariant subspaces for Jordan algebras of compact operators have only recently been obtained. In particular, there is currently no Jordan analogue of the theory which was developed in [11]. This is one reason why our results have been developed in a different manner.

We begin by analyzing the properties of bounded linear operators which satisfy a certain spectral additivity condition. The results we obtain here, when combined with new "Cartan-like" conditions for the existence of invariant subspaces in Lie and Jordan algebras of compact operators, provides us with a general method of proving the existence of invariant subspaces for these algebras, in particular when their elements have subadditive or submultiplicative spectrum. We will show that this implies a Lie or Jordan algebra of compact operators with subadditive spectrum is triangularizable, obtaining a new proof of Shulman and Turovskii's result for Lie algebras. Finally, we obtain simple conditions on the trace of the finite-rank elements in the algebra which imply the existence of invariant subspaces. As far as we know, some of our results are new even in finite dimensions.

2. Preliminaries

In this paper we confine ourselves to the field $\mathbb{C}$ of complex numbers. For a Banach space $\mathcal{X}$, we let $\mathcal{B}(\mathcal{X})$ and $\mathcal{K}(\mathcal{X})$ denote the algebras of all bounded and compact operators on $\mathcal{X}$ respectively.

For A in $\mathcal{B}(\mathcal{X})$, we let $\sigma(A)$ and $r(A)$ denote the spectrum and spectral radius of A respectively. If A is of finite rank, then the trace of A is well-defined; we denote it by $\text{tr}(A)$.

To clarify our exposition, we will sometimes make use of shorthand notation. For example, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are subsets of $\mathcal{B}(\mathcal{X})$, then we will write $\mathcal{S}_1^n$ for the linear span of $\{A^n: A \in \mathcal{S}_1\}$, and $\mathcal{S}_1\mathcal{S}_2$ for the linear span of $\{AB: A \in \mathcal{S}_1, \ B \in \mathcal{S}_2\}$.

A family $\mathcal{F}$ of operators is called reducible if there is a nontrivial closed subspace invariant under every member of $\mathcal{F}$. We say that $\mathcal{F}$ is triangularizable if its lattice of invariant subspaces contains a maximal subspace chain $\mathcal{C}$. (If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two members of $\mathcal{C}$ with $\mathcal{M}_1 \subseteq \mathcal{M}_2$ and no other member between $\mathcal{M}_1$ and $\mathcal{M}_2$, then $\dim(\mathcal{M}_1 \ominus \mathcal{M}_2) \leqslant 1$.)

To establish the triangularizability of a family of operators, we will require the following lemma from [9, Lemma 7.1.11].

Lemma 2.1 *(The Triangularization Lemma). Let $\mathcal{P}$ be a property of a family of operators such that*

(1) *every family of operators with property $\mathcal{P}$ is reducible, and*
(2) *if $\mathcal{F}$ has property $\mathcal{P}$ and if $\mathcal{M}_1$ and $\mathcal{M}_2$ are invariant subspaces of $\mathcal{F}$ with $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then $\hat{\mathcal{F}}$ has property $\mathcal{P}$, where $\hat{\mathcal{F}}$ is the set of all quotient operators on $\mathcal{M}_1/\mathcal{M}_2$ induced by $\mathcal{F}$.*

Then every family of operators with property $\mathcal{P}$ is triangularizable.

If $\mathcal{P}$ is a property of a family of operators which satisfies hypothesis (2) of the Triangularization Lemma, then we say that $\mathcal{P}$ is inherited by quotients.

The following result from [9, Corollary 8.4.2] will be used in combination with the Triangularization Lemma.

Lemma 2.2. *The property of sublinear spectrum is inherited by quotients.*

Definition 2.3. A pair of bounded operators A and B on a Banach space are said to have *submultiplicative spectrum* if

$$\sigma(AB) \subseteq \sigma(A)\sigma(B),$$

where $\sigma(A)\sigma(B)$ means the set of all $\alpha\beta$, with α and β in $\sigma(A)$ and $\sigma(B)$ respectively. A family $\mathcal{F}$ of bounded operators on a Banach space are said to have submultiplicative spectrum if every pair of elements in $\mathcal{F}$ has submultiplicative spectrum.

The property of submultiplicative spectrum is not strong enough to imply the reducibility of a semigroup of compact operators; indeed, the existence of finite irreducible matrix groups with sublinear spectrum was shown in [6]. We will show however, that Lie and Jordan algebras of compact operators with submultiplicative spectrum are reducible.

3. Operators with stable spectrum

It turns out that the following property, obviously closely related to the property of sublinear spectrum, is of particular importance for obtaining many of our reducibility results.

Definition 3.1. Let T be a bounded operator on a Banach space. We say that a bounded operator A has T-*stable spectrum* if

$$r(A + \lambda T) \leqslant r(A)$$

for every complex number λ. A family of bounded operators on a Banach space is said to have T-stable spectrum if each of its elements has T-stable spectrum.

It will sometimes be useful to reference a family of bounded operators with T-stable spectrum without making explicit mention of T. Therefore, we will say that a family of bounded operators has stable spectrum if it has T-stable spectrum, for some nonzero T.

Example 3.2. Consider the matrices

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Note that A is in Jordan normal form, so $\sigma(A) = \{-1, 0\}$, and $B^3 = 0$, so $\sigma(B) = \{0\}$. The characteristic polynomial of $A + \lambda B$ is $t^3 - t^2$, which implies that $\sigma(A + \lambda B) = \sigma(A)$ for all λ in $\mathbb{C}$. This shows that A is B-stable.

Calculating the inverse of $\mu - A$, we have

$$(\mu - A)^{-1} B = \frac{1}{\mu^2(1-\mu)} \begin{pmatrix} 0 & \mu^2 & \mu^2 \\ -1 & -\mu(1-\mu) & -(1-\mu)^2 \\ \mu(1-\mu) & 0 & \mu(1-\mu) \end{pmatrix},$$

and it is routine to verify that this matrix is nilpotent. Calculating the inverse of $1 - \mu B$, we have

$$(1 - \mu B)^{-1} A = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ 2 & 0 & 1 \end{pmatrix},$$

and this matrix has characteristic polynomial $t^3 - t^2$, showing that it has the same eigenvalues as A.

In this section, we will show that the results in Example 3.2 hold in general, which will be important for our main results. An important tool will be the theory of subharmonic functions, based on the result of Vesentini that if f is an analytic function from a domain of the complex numbers into a Banach algebra, then the functions $\lambda \to r(f(\lambda))$ and $\lambda \to \log(r(f(\lambda)))$ are subharmonic [12]. We will require the following two fundamental results from the theory of subharmonic functions (see for example [1, Theorem A.1.3] and [1, Theorem A.1.29] resp.).

Theorem 3.3 *(Maximum Principle for Subharmonic Functions). Let f be a subharmonic function on a domain D of $\mathbb{C}$. If there exists λ_0 in D such that $f(\lambda) \leqslant f(\lambda_0)$ for all λ in D, then $f(\lambda) = f(\lambda_0)$ for all λ in D.*

We state here only a special case of H. Cartan's Theorem.

Theorem 3.4 *(H. Cartan's Theorem). Let f be a subharmonic function from a domain D of $\mathbb{C}$. If $f(\lambda) = -\infty$ on a nonempty open ball in D, then $f(\lambda) = -\infty$ for all λ in D.*

Remark 3.5. For bounded operators A and T on a Banach space, the function $\lambda \to A + \lambda T$ is analytic, so by Vesentini's results, the functions $\lambda \to r(A + \lambda T)$ and $\lambda \to \log(r(A + \lambda T))$ are subharmonic. If A has T-stable spectrum, the Maximum Principle for subharmonic functions immediately implies that $r(A + \lambda T) = r(A)$ for all complex numbers λ.

If A and T have sublinear spectrum and T is quasinilpotent, then A has T-stable spectrum. The following lemma from [11, Lemma 4.2] shows that the quasinilpotence of T is a necessary condition for A to be T-stable.

Lemma 3.6. *Let A and T be bounded operators on a Banach space. If A has T-stable spectrum, then T is quasinilpotent.*

Proof. By Remark 3.5, $r(A + \lambda T) = r(A)$ for all λ in $\mathbb{C}$, so

$$r(\lambda^{-1} A + T) = |\lambda|^{-1} r(A)$$

for all nonzero λ in $\mathbb{C}$. Thus by the subharmonicity of the function $\mu \to r(\mu A + T)$,

$$r(T) = \limsup_{\mu \to 0} r(\mu A + T) = \limsup_{\mu \to 0} |\mu| r(A) = 0. \qquad \square$$

Lemma 3.7. *Let A and T be bounded operators on a Banach space. Then A has T-stable spectrum if and only if $(\mu - A)^{-1}T$ is quasinilpotent for all $\mu \notin \sigma(A)$.*

Proof. By Remark 3.5, $r(A + \lambda T) = r(A)$ for all λ in $\mathbb{C}$, so for μ in $\mathbb{C}$ with $|\mu| > r(A)$, both $\mu - A$ and $\mu - A - \lambda T$ are invertible. Therefore,

$$\lambda^{-1}(\mu - A)^{-1}(\mu - A - \lambda T) = \lambda^{-1} - (\mu - A)^{-1}T$$

is invertible for all nonzero λ in $\mathbb{C}$. This means that the values of the operator-valued function $\mu \to (\mu - A)^{-1}T$, which is analytic for $\mu \notin \sigma(A)$, are quasinilpotent whenever $|\mu| > r(A)$.

Consider the subharmonic function $\mu \to \log(r((\mu - A)^{-1}T))$ defined for $\mu \notin \sigma(A)$. Since $\log(r((\mu - A)^{-1}T)) = -\infty$ whenever $|\mu| > r(A)$, by H. Cartan's Theorem, $\log(r((\mu - A)^{-1}T)) = -\infty$ for all $\mu \notin \sigma(A)$. In other words, $(\mu - A)^{-1}T$ is quasinilpotent for all $\mu \notin \sigma(A)$. $\square$

Lemma 3.8. *Let A and T be bounded operators on a Banach space. Then A has T-stable spectrum if and only if $\sigma(A + \lambda T) \subseteq \sigma(A)$ for all λ in $\mathbb{C}$.*

Proof. Suppose $\mu \in \sigma(A + \lambda T)$, but that $\mu \notin \sigma(A)$. Then clearly λ is nonzero, and

$$\lambda^{-1}(\mu - A)^{-1}(\mu - A - \lambda T) = \lambda^{-1} - (\mu - A)^{-1}T$$

is not invertible. But by Lemma 3.7, $(\mu - A)^{-1}T$ is quasinilpotent for all $\mu \notin \sigma(A)$, which gives a contradiction. $\square$

The next result also follows from [1, Theorem 3.4.14], but it is interesting to see that it can be proved in the following way.

Lemma 3.9. *Let A and T be bounded operators on a Banach space. If A has T-stable spectrum and $\sigma(A)$ has no interior points, then $\sigma(A + \lambda T) = \sigma(A)$ for all λ in $\mathbb{C}$.*

Proof. This follows immediately from Corollary 3.8 and the Spectral Maximum Principle of [1]. $\square$

Lemma 3.10. *Let A and T be bounded operators on a Banach space. If A has T-stable spectrum and $\sigma(A)$ has no interior points, then $\sigma((1 - \nu T)^{-1}A) = \sigma(A)$ for all ν in $\mathbb{C}$.*

Proof. First suppose λ is nonzero, and that $\lambda \notin \sigma(A)$. By Lemma 3.9, $\sigma(\lambda^{-1}A + \nu T) = \sigma(\lambda^{-1}A)$, and by Lemma 3.6, T is quasinilpotent. These two facts imply that $1 - \nu T$ and $1 - \lambda^{-1}A - \nu T$ are both invertible, and hence that

$$\lambda(1 - \nu T)^{-1}(1 - \lambda^{-1}A - \nu T) = \lambda - (1 - \nu T)^{-1}A$$

is invertible for all ν in $\mathbb{C}$. Therefore, $\lambda \notin \sigma((1 - \nu T)^{-1}A)$ for all ν in $\mathbb{C}$.

Now suppose $0 \notin \sigma(A)$. Then A is invertible, implying $(1 - \nu T)^{-1} A$ is invertible, and hence by the quasinilpotence of T, that $0 \notin \sigma((1 - \nu T)^{-1} A)$ for all ν in $\mathbb{C}$.

We have shown that $\sigma((1 - \nu T)^{-1} A) \subseteq \sigma(A)$ for all ν in $\mathbb{C}$. Since $\sigma(A)$ has no interior points, the result now follows from the Spectral Maximum Principle of [1]. □

Lemma 3.11. *Let f be an entire function from $\mathbb{C}$ into $\mathcal{B}(\mathcal{X})$. Suppose that*

(1) *there exists a complex number λ_0 such that $\sigma(f(\lambda_0)) = \sigma(f(\lambda))$, and*
(2) *there exists N such that $\mathrm{rank}(f(\lambda)) \leqslant N$ for all complex numbers λ.*

Then $\mathrm{tr}(f(\lambda)) = \mathrm{tr}(f(\lambda_0))$ for all λ in $\mathbb{C}$.

Proof. Since f takes finite-rank values and is entire, the function $\lambda \to \mathrm{tr}(f(\lambda))$ is also entire. For each λ, the trace of $f(\lambda)$ is the sum, with multiplicity, of the eigenvalues of $f(\lambda)$, so we may write it as a finite sum

$$\mathrm{tr}\big(f(\lambda)\big) = \sum_{\alpha \in \sigma(f(\lambda_0))} n_\alpha(\lambda)\alpha,$$

where $n_\alpha(\lambda)$ denotes the multiplicity of the eigenvalue α with respect to $f(\lambda)$. But clearly

$$\sum_{\alpha \in \sigma(f(\lambda_0))} n_\alpha(\lambda)|\alpha| \leqslant \mathrm{rank}\big(f(\lambda)\big)\big\|f(\lambda_0)\big\| \leqslant N\big\|f(\lambda_0)\big\|,$$

which implies that the function $\lambda \to \mathrm{tr}(f(\lambda))$ is bounded. By Liouville's Theorem, it now follows that $\mathrm{tr}(f(\lambda)) = \mathrm{tr}(f(\lambda_0))$ for all λ in $\mathbb{C}$. □

The next result extends [11, Lemma 4.2], and gives a symmetric trace condition which will be useful for our results.

Lemma 3.12. *Let A and B be bounded operators on a Banach space. If A is B-stable and one of A or B is of finite rank, then $\mathrm{tr}(A^n B) = \mathrm{tr}(AB^n) = 0$ for all $n \geqslant 1$.*

Proof. First suppose that A is of finite rank. Since B is quasinilpotent by Lemma 3.6, the function $\nu \to (1 - \nu B)^{-1} A$ is entire. Moreover, $\sigma((1 - \nu B)^{-1} A) = \sigma(A)$ for all ν in $\mathbb{C}$ by Lemma 3.10. Then, taking nth powers, the function $\nu \to ((1 - \nu B)^{-1} A)^n$ is also entire, and $\sigma(((1 - \nu B)^{-1} A)^n) = \sigma(A^n)$ for all ν in $\mathbb{C}$. Clearly $\mathrm{rank}(((1 - \nu B)^{-1} A)^n) \leqslant \mathrm{rank}(A)$, so $\mathrm{tr}(((1 - \nu B)^{-1} A)^n) = \mathrm{tr}(A^n)$ for all ν in $\mathbb{C}$ by Lemma 3.11.

For $|\nu| < \|B\|^{-1}$, we may expand $(1 - \nu B)^{-1} A$ as a power series in ν,

$$(1 - \nu B)^{-1} A = \sum_{k \geqslant 0} B^k A \nu^k.$$

Hence

$$\big((1 - \nu B)^{-1} A\big)^n = \bigg(\sum_{k \geqslant 0} B^k A \nu^k\bigg)^n.$$

The coefficient of v^k in the above expansion is $B^k A$, and for $n \geqslant 1$, the coefficient of v is

$$BA^n + ABA^{n-1} + \cdots + A^{n-1}BA.$$

But we may also expand the constant function $\mathrm{tr}(((1-vB)^{-1}A)^n)$ as a power series in v, and the linearity of the trace implies that for $n = 1$, the coefficient of v^k in this expansion is $\mathrm{tr}(B^k A)$, and for $n \geqslant 1$, that the coefficient of v is

$$\mathrm{tr}\big(BA^n + ABA^{n-1} + \cdots + A^{n-1}BA\big) = n \, \mathrm{tr}\big(A^n B\big).$$

Comparing the coefficients on the left- and right-hand sides of the equation $\mathrm{tr}(((1-vB)^{-1}A)^n) = \mathrm{tr}(A^n)$ therefore gives $\mathrm{tr}(A^n B) = 0$ for all $n \geqslant 1$, and $AB^k = 0$ for all $k \geqslant 1$.

Now suppose that B is of finite rank. The function $v \to (1-vA)^{-1}B$ is analytic for $v^{-1} \notin \sigma(A)$, with quasinilpotent values by Lemma 3.7. Taking nth powers, the function $v \to ((1-vA)^{-1}B)^n$ is also analytic with quasinilpotent values for $v^{-1} \notin \sigma(A)$. This means $\mathrm{tr}(((1-vA)^{-1}B)^n) = 0$ for all $v \notin \sigma(A)$.

As above, for $|v| < \|A\|$, we may expand $((1-vA)^{-1}B)^n$ as a power series in v,

$$\big((1-vA)^{-1}B\big)^n = \bigg(\sum_{k \geqslant 0} A^k B v^k\bigg)^n.$$

For $n = 1$, the coefficient of v^k in the above expansion is $A^k B$, and for $n \geqslant 1$, the coefficient of v is

$$AB^n + BAB^{n-1} + \cdots + B^{n-1}A.$$

Proceeding as before, we may also expand the constant function $\mathrm{tr}(((1-vA)^{-1}B)^n)$ as a power series in v, and the linearity of the trace implies that for $n = 1$, the coefficient of v^k in this expansion is $\mathrm{tr}(A^k B)$, and for $n \geqslant 1$, that the coefficient of v is

$$\mathrm{tr}\big(AB^n + BAB^{n-1} + \cdots + B^{n-1}AB\big) = n \, \mathrm{tr}\big(AB^n\big).$$

Comparing the coefficients of the left- and right-hand sides of the equation $\mathrm{tr}(((1-vA)^{-1}B)^n) = 0$ therefore gives $\mathrm{tr}(A^k B) = 0$ for all $k \geqslant 1$, and $\mathrm{tr}(AB^n) = 0$ for all $n \geqslant 1$. $\quad\square$

4. Spectral conditions on a Lie algebra of compact operators

In this section we study Lie algebras of compact operators which satisfy spectral properties like sublinearity and submultiplicativity.

Recall that an operator Lie algebra $\mathcal{L}$ is a subspace of $\mathcal{B}(\mathcal{X})$ which is closed under the Lie commutator product $[A, B] = AB - BA$, for $A, B \in \mathcal{L}$. A Lie ideal $\mathcal{I}$ of $\mathcal{L}$ is a Lie subalgebra of $\mathcal{L}$ such that $[A, B] \in \mathcal{I}$ whenever $A \in \mathcal{L}$ and $B \in \mathcal{I}$. For A in $\mathcal{L}$, we define the bounded linear operator ad_A on $\mathcal{L}$ by $\mathrm{ad}_A(B) = [A, B]$.

Definition 4.1. An operator Lie algebra $\mathcal{L}$ is said to be an *Engel Lie algebra* if ad_A is quasinilpotent for every A in $\mathcal{L}$. An ideal of $\mathcal{L}$ is said to be an *Engel ideal* if it is an Engel Lie algebra.

There are two particularly important classes of Engel Lie algebras:

(1) Commutative operator Lie algebras are Engel Lie algebras, since $\text{ad}_A = 0$ for every $A \in \mathcal{L}$.
(2) Operator Lie algebras in which every element is compact and quasinilpotent are Engel Lie algebras. Indeed, for $A \in \mathcal{L}$, ad_A is the restriction of $L_A - R_A$ to $\mathcal{L}$, where L_A and R_A denote the (commuting) operators of left and right multiplication by A on $\mathcal{B}(\mathcal{X})$ respectively. We always have $\sigma(L_A) \subseteq \sigma(A)$ and $\sigma(R_A) \subseteq \sigma(A)$, so L_A and R_A are quasinilpotent. It follows that $L_A - R_A$ is quasinilpotent, and hence so is its restriction to $\mathcal{L}$.

We require the following two important results of Shulman and Turovskii. The first result from [10, Corollary 11.6] is an extension of Engel's well-known theorem to Lie algebras of compact operators. The second result from [11, Corollary 4.20] establishes the abundance of finite-rank operators in an irreducible Lie algebra of compact operators.

Theorem 4.2. *A Lie algebra of compact operators which contains a nonzero Engel Lie ideal is reducible. In particular, an Engel Lie algebra of compact operators is triangularizable.*

Theorem 4.3. *A uniformly closed Lie algebra of compact operators which does not contain any nonzero finite-rank nilpotent elements is triangularizable.*

For a general Lie algebra of compact operators, it is often more tractable to study its ideal of finite-rank operators due to such niceties as the existence of a trace, so we are especially interested in situations when the reducibility of this ideal implies the reducibility of the entire Lie algebra. Shulman and Turovskii raised the question [11] of whether a Lie algebra of compact operators is reducible whenever its ideal of finite-rank operators is reducible. That such a result holds for an associative algebra of compact operators is a consequence of Lomonosov's Theorem (see for example [9, Theorem 7.4.7]). The next result, a generalization of [5, Lemma 2.3], provides at least one example of a situation in which this type of result is true in the Lie algebra case.

Theorem 4.4. *Let $\mathcal{L}$ be a uniformly closed Lie algebra of compact operators. If $\mathcal{L}$ contains a nonzero element A, with the property that $\text{tr}(AF) = 0$ for every finite-rank operator F in $\mathcal{L}$, then $\mathcal{L}$ is reducible.*

Proof. Let $\mathcal{F}$ denote the ideal of finite-rank operators in $\mathcal{L}$, and let $\mathcal{I} = \{A \in \mathcal{L}: \text{tr}(AF) = 0$ for all $F \in \mathcal{F}\}$. Using the identity

$$\text{tr}\big([A, B]F\big) = \text{tr}\big(A[B, F]\big),$$

it is easy to verify that $\mathcal{I}$ is a Lie ideal of $\mathcal{L}$. Consider the Lie ideal $\mathcal{I}_\mathcal{F} = \mathcal{I} \cap \mathcal{F}$. If $\mathcal{I}_\mathcal{F} = 0$, then $\mathcal{I}$ is triangularizable by Theorem 4.3, and thus $\mathcal{L}$ is reducible by Theorem 4.2. Hence we may suppose that $\mathcal{I}_\mathcal{F} \neq 0$.

We have $\text{tr}(AB) = 0$ for all $A, B \in \mathcal{I}_\mathcal{F}$, which implies by [4, Theorem 4.5] that $[\mathcal{I}_\mathcal{F}, \mathcal{I}_\mathcal{F}]$ consists of nilpotent elements. Note that $[\mathcal{I}_\mathcal{F}, \mathcal{I}_\mathcal{F}]$ is a Lie ideal of $\mathcal{L}$. Indeed, for $A \in \mathcal{L}$ and $F, G \in \mathcal{I}_\mathcal{F}$,

$$\big[A, [F, G]\big] = -\big[F, [G, A]\big] - \big[G, [A, F]\big]$$

by the Jacobi identity. Hence $\mathcal{I}_\mathcal{F}$ is triangularizable by Theorem 4.2. If $[\mathcal{I}_\mathcal{F}, \mathcal{I}_\mathcal{F}] = 0$, then $\mathcal{I}_\mathcal{F}$ is a commutative Lie ideal, and hence is triangularizable by Theorem 4.2. Otherwise, the triangularizable ideal $[\mathcal{I}_\mathcal{F}, \mathcal{I}_\mathcal{F}]$ is nonzero. Either way, $\mathcal{L}$ has a nonzero triangularizable ideal, which implies by Theorem 4.2 that it is reducible.   □

Lemma 4.5. *Let $\mathcal{L}$ be a Lie algebra of compact operators with a nonzero element T. If $\mathcal{L}$ has T-stable spectrum, then $\mathcal{L}$ is reducible.*

Proof. By the continuity of the spectrum of compact operators, we may suppose that $\mathcal{L}$ is uniformly closed. Let $\mathcal{F}$ be the Lie ideal of finite-rank operators in $\mathcal{L}$. By Lemma 3.12, $\mathrm{tr}(FT) = 0$ for all F in $\mathcal{F}$, so the result follows by Theorem 4.4.   □

Theorem 4.6. *A Lie algebra of compact operators with subadditive spectrum is triangularizable.*

Proof. Let $\mathcal{L}$ be a Lie algebra of compact operators with sublinear spectrum. By the Triangularization Lemma and Lemma 2.2, it suffices to show the reducibility of $\mathcal{L}$. As in the proof of Lemma 4.5, we may suppose that $\mathcal{L}$ is uniformly closed. Then by Theorem 4.3, if $\mathcal{L}$ does not contain a nonzero finite-rank nilpotent element, then it is reducible; hence we may suppose that some nonzero T in $\mathcal{L}$ is nilpotent, and of finite rank. Then by the hypothesis of sublinear spectrum, $\mathcal{L}$ has T-stable spectrum, so the result follows by Lemma 4.5.   □

Theorem 4.7. *A Lie algebra of compact operators with submultiplicative spectrum is reducible.*

Proof. Let $\mathcal{L}$ be a Lie algebra of compact operators with submultiplicative spectrum. As in the proof of Lemma 4.5, we may suppose that $\mathcal{L}$ is uniformly closed. Let $\mathcal{F}$ be the ideal of finite-rank elements in $\mathcal{L}$. By Theorem 4.3, if $\mathcal{L}$ does not contain a nonzero finite-rank nilpotent element, then it is reducible; hence we may suppose that some nonzero T in $\mathcal{L}$ is nilpotent, and of finite rank. Then for every F in $\mathcal{F}$, $\sigma(FT) = 0$ by the hypothesis of submultiplicative spectrum, so $\mathrm{tr}(FT) = 0$. Hence $\mathcal{L}$ is reducible by Theorem 4.4.   □

Theorem 4.8. *Let $\mathcal{L}$ be a uniformly closed Lie algebra of compact operators with the property that $\mathrm{tr}(FGH) = \mathrm{tr}(FHG)$ whenever $F, G, H \in \mathcal{L}$ are of finite rank. Then $\mathcal{L}$ is reducible.*

Proof. Let $\mathcal{F}$ be the ideal of finite-rank operators in $\mathcal{L}$. If $\mathcal{F}$ is commutative, then it is an Engel ideal of $\mathcal{L}$, and $\mathcal{L}$ is reducible by Theorem 4.2. Hence we may suppose that $[G, H] \neq 0$ for some $G, H \in \mathcal{L}$. Then for all F in $\mathcal{F}$, $\mathrm{tr}(F[G, H]) = 0$, so $\mathcal{L}$ is reducible by Theorem 4.4.   □

Theorem 4.9. *Let $\mathcal{L}$ be a uniformly closed Lie algebra of compact operators with the property that $\mathrm{tr}(F^2) = 0$ for every finite-rank element F in $\mathcal{L}$. Then $\mathcal{L}$ is reducible.*

Proof. Let $\mathcal{F}$ be the ideal of finite-rank operators in $\mathcal{L}$. Then for any $F, G \in \mathcal{F}$,

$$
\begin{aligned}
0 &= \mathrm{tr}\big((F + G)^2\big) \\
&= \mathrm{tr}\big(F^2\big) + \mathrm{tr}\big(G^2\big) + \mathrm{tr}(FG + GF) \\
&= \mathrm{tr}(FG + GF) \\
&= 2\,\mathrm{tr}(FG),
\end{aligned}
$$

so $\mathrm{tr}(FG) = 0$. If $\mathcal{F} = 0$, then $\mathcal{L}$ is reducible by Theorem 4.3. Otherwise, $\mathcal{L}$ is reducible by Theorem 4.4. $\quad\square$

5. Spectral conditions on a Jordan algebra of compact operators

In this section we turn our attention to Jordan algebras, and show that most of the results we obtained for Lie algebras are also true in this setting.

An operator Jordan algebra $\mathcal{J}$ is a subspace of $\mathcal{B}(\mathcal{X})$ which is closed under the Jordan anti-commutator product $\{A, B\} = AB + BA$, for $A, B \in \mathcal{J}$. It is easy to verify that this is equivalent to $\mathcal{J}$ being closed under taking positive integer powers. A Jordan ideal $\mathcal{I}$ of $\mathcal{J}$ is a Jordan subalgebra of $\mathcal{J}$ such that $\{A, B\} \in \mathcal{I}$ whenever $A \in \mathcal{J}$ and $B \in \mathcal{I}$.

The methods of this section will differ from those used in the section on Lie algebras. This is mainly because for obtaining reducibility results, the closest Jordan analogue of an Engel Lie ideal will be a Jordan ideal in which every element is quasinilpotent. This is due to the following result, the Jordan analogue of Theorem 4.2, which was recently obtained in [5, Theorem 11.3].

Theorem 5.1. *A Jordan algebra of compact operators with a nonzero ideal of quasinilpotent operators is reducible. In particular, a Jordan algebra of compact quasinilpotent operators is triangularizable.*

Let $\mathcal{J}$ be a uniformly closed Jordan algebra of compact operators, and let A be a non-quasinilpotent element of $\mathcal{J}$. For nonzero λ in $\sigma(A)$, it is well known that the Riesz projection P_λ of A corresponding to λ is of finite rank, and moreover that it may be written as a uniform limit of polynomials in A with zero constant coefficient. It follows that $P_\lambda \in \mathcal{J}$. This fact, combined with Theorem 5.1, implies the following result.

Theorem 5.2. *A uniformly closed Jordan algebra of compact operators which does not contain any nonzero finite-rank operators is reducible.*

For Jordan algebras, we are also interested in situations when the reducibility of this ideal implies the reducibility of the entire Jordan algebra. The next result establishes the Jordan analogue of Theorem 4.4.

Theorem 5.3. *Let $\mathcal{J}$ be a uniformly closed Jordan algebra of compact operators. If $\mathcal{J}$ contains a nonzero element A with the property that $\mathrm{tr}(AF) = 0$ for every finite-rank operator F in $\mathcal{J}$, then $\mathcal{J}$ is reducible.*

Proof. Let $\mathcal{F}$ denote the Jordan ideal of finite-rank elements in $\mathcal{J}$, and let $\mathcal{I} = \{A \in \mathcal{J}: \mathrm{tr}(AF) = 0$ for all $F \in \mathcal{F}\}$. Using the identity

$$\mathrm{tr}\big(\{A, B\}F\big) = \mathrm{tr}\big(A\{B, F\}\big) = 0,$$

it is straightforward to verify that $\mathcal{I}$ is a Jordan ideal of $\mathcal{J}$. We claim that $\mathcal{I}$ consists of quasinilpotent elements. Indeed, suppose otherwise that for some $A \in \mathcal{I}$, $\lambda \in \sigma(A)$ is nonzero, and let P_λ be the Riesz projection of A corresponding to λ. Then P_λ belongs to $\mathcal{J}$, and $\mathrm{tr}(P_\lambda A P_\lambda) = n\lambda$, where n is the spectral multiplicity of λ. But

$$\operatorname{tr}(P_\lambda A P_\lambda) = \operatorname{tr}\left(A P_\lambda^2\right) = \operatorname{tr}(A P_\lambda) = 0$$

by hypothesis, which gives a contradiction. The result now follows by Theorem 5.1. □

Using the structure which is presented in a general Jordan algebra of compact operators, and applying what we know about operators with stable spectrum, we are able to prove the following result.

Lemma 5.4. *Let $\mathcal{J}$ be a Jordan algebra of compact operators with T in $\mathcal{J}$. If $\mathcal{J}$ has T-stable spectrum, then AT is quasinilpotent for all $A \in \mathcal{J}$.*

Proof. For $A \in \mathcal{J}$, $\nu A + \nu^2 A^2 + \cdots + \nu^n A^n + \lambda T \in \mathcal{J}$, for all $n \geqslant 1$ and $\lambda, \nu \in \mathbb{C}$. Also,

$$\sigma\left(\nu A + \nu^2 A^2 + \cdots + \nu^n A^n + \lambda T\right) = \sigma\left(\nu A + \nu^2 A^2 + \cdots + \nu^n A^n\right),$$

so

$$\sigma\left(1 + \nu A + \nu^2 A^2 + \cdots + \nu^n A^n + \lambda T\right) = \sigma\left(1 + \nu A + \nu^2 A^2 + \cdots + \nu^n A^n\right).$$

For sufficiently small ν, taking $n \to \infty$, and using the continuity of the spectrum of compact operators gives

$$\sigma\left((1 - \nu A)^{-1} + \lambda T\right) = \sigma\left((1 - \nu A)^{-1}\right).$$

Hence $(1 - \nu A)^{-1}$ has T-stable spectrum, so by Lemma 3.7, $(1 - \nu A)T = T - \nu AT$ is quasinilpotent for sufficiently small ν. For such ν, the subharmonic function $\nu \to \log(r(T - \nu AT))$ satisfies $\log(r(T - \nu AT)) = -\infty$, so by H. Cartan's Theorem, $\log(r(T - \nu AT)) = -\infty$, i.e. $T - \nu AT$ is quasinilpotent for all $\nu \in \mathbb{C}$. But this means T has AT-stable spectrum, so AT is quasinilpotent by Lemma 3.6. □

Lemma 5.5. *Let $\mathcal{J}$ be a Jordan algebra of compact operators, and let T be a nonzero element of $\mathcal{J}$. If $\mathcal{J}$ has T-stable spectrum, then $\mathcal{J}$ is reducible.*

Proof. By the continuity of the spectrum of compact operators, we may suppose that $\mathcal{J}$ is uniformly closed. Consider the Jordan ideal $\mathcal{F}$ of finite-rank operators in $\mathcal{J}$. By Theorem 5.2, if $\mathcal{F}$ is zero then $\mathcal{J}$ is reducible; hence we may assume that $\mathcal{F}$ is nonzero.

For A in $\mathcal{J}$ and F in $\mathcal{F}$, $\{A, F\}$ belongs to $\mathcal{F}$, so by Lemma 3.12,

$$\operatorname{tr}\left(\{A, F\}T\right) = \operatorname{tr}\left(\{F, T\}A\right) = 0.$$

If $\{F, T\} \neq 0$ for some F in $\mathcal{F}$, then $\mathcal{J}$ is reducible by Lemma 5.3. Otherwise, if $\{F, T\} = 0$ for all F in $\mathcal{F}$, then

$$\operatorname{tr}(TF) = \operatorname{tr}\left(\frac{1}{2}\{F, T\}\right) = 0$$

for all F in $\mathcal{F}$, which by Lemma 5.3 again implies the reducibility of $\mathcal{J}$. □

Theorem 5.1 roughly says that a Jordan algebra of compact operators with too many quasinilpotent operators is reducible. The next result says that a Jordan algebra of compact operators with too few quasinilpotent operators is also reducible. For its proof, we will require two classical theorems. The Motzkin–Taussky Theorem states that a linear space of finite-rank diagonalizable operators is commutative (see for example [3, Theorem 2.7]). The Kleinecke–Shirokov Theorem states that if A and B are bounded operators on a Banach space, and if $[A, [A, B]] = 0$, then $[A, B]$ is quasinilpotent (see for example [1, Theorem 5.1.3]).

Theorem 5.6. *Let $\mathcal{J}$ be a uniformly closed Jordan algebra of compact operators. If $\mathcal{J}$ does not contain any nonzero finite-rank nilpotent elements, then $\mathcal{J}$ is reducible.*

Proof. Suppose $\mathcal{J}$ does not contain any nonzero finite-rank nilpotent elements, and let $\mathcal{F}$ be the Jordan ideal of finite-rank elements in $\mathcal{J}$. For $F \in \mathcal{F}$, since $\mathcal{J}$ contains every polynomial in F with zero constant coefficient, in particular it contains the polynomial

$$p(F) = F \prod_\alpha (\alpha - F),$$

where the product is taken over all nonzero α in $\sigma(F)$. Since some power of $p(F)$ is zero, $p(F)$ is nilpotent, and hence $p(F) = 0$ by hypothesis, which implies that F is diagonalizable. Hence every element in $\mathcal{F}$ is diagonalizable, so by the Motzkin–Taussky Theorem, the elements of $\mathcal{F}$ pairwise commute.

Fix some $F \in \mathcal{F}$. For A in $\mathcal{J}$, $[F, [F, A]]$ is of finite rank and belongs to $\mathcal{J}$, since

$$\big[F, [F, A]\big] = \big\{F, \{F, A\}\big\} - \big\{A, \{F, F\}\big\}.$$

From above, we therefore have that F and $[F, [F, A]]$ commute, i.e. that $[F, [F, [F, A]]] = 0$, so the Kleinecke–Shirokov Theorem implies that $[F, [F, A]]$ is nilpotent. Since, by hypothesis, $\mathcal{J}$ does not contain any nonzero finite-rank nilpotent elements, $[F, [F, A]] = 0$, and applying the Kleinecke–Shirokov Theorem again implies that $[F, A]$ is nilpotent. But then $[F, A]^2$ is also finite-rank and nilpotent, and moreover, it belongs to $\mathcal{J}$ since

$$[F, A]^2 = \{A, F\}^2 + 2\big\{A^2, F^2\big\} - \big\{A, \{A, F^2\}\big\} - \big\{F, \{F, A^2\}\big\}.$$

Hence $[F, A]^2 = 0$, where we again use the hypothesis that $\mathcal{J}$ does not contain any nonzero finite-rank nilpotent elements.

Let $B \in \mathcal{J}$ with nonzero $\beta \in \sigma(B)$, and let P be the Riesz projection of B corresponding to β. Then P is of finite rank, so from above, for $A \in \mathcal{J}$,

$$0 = [P, A]^2 = (PA)^2 - PA^2P - APA + (AP)^2.$$

Now consider $(1 - P)AP + PA(1 - P) = \{A, P\} - 2PAP$, which belongs to $\mathcal{J}$. We have

$$\big((1 - P)AP + PA(1 - P)\big)^2 = (1 - P)APA(1 - P) + PA(1 - P)AP$$
$$= -\big((PA)^2 - PA^2P - APA + (AP)^2\big)$$
$$= -[P, A]^2$$
$$= 0,$$

which implies that the finite-rank element $(1 - P)AP + PA(1 - P)$ is nilpotent. Since by hypothesis, $\mathcal{J}$ does not contain any nonzero finite-rank nilpotent elements, it follows that $(1 - P)AP + PA(1 - P) = 0$ for all $A \in \mathcal{J}$.

We have

$$0 = (1 - P)AP + PA(1 - P) = AP + PA - 2PAP,$$

so $AP = 2PAP - PA$. Multiplying on the left by P then gives $PAP = 2PAP - PA = PA$, so it follows that the range of P is invariant under A. Since P was chosen to be nontrivial, this shows that $\mathcal{J}$ is reducible. $\quad\square$

Theorem 5.7. *A Jordan algebra of compact operators with subadditive spectrum is triangularizable.*

Proof. Let $\mathcal{J}$ be a Jordan algebra of compact operators with sublinear spectrum. By the continuity of the spectrum of compact operators, we may suppose that $\mathcal{J}$ is uniformly closed. As in Theorem 4.6, it suffices to show the reducibility of $\mathcal{J}$. By Theorem 5.6, if $\mathcal{J}$ does not contain any nonzero quasinilpotent elements, then $\mathcal{J}$ is reducible; hence we may suppose that $\mathcal{J}$ contains a nonzero quasinilpotent element T. By the hypothesis of sublinear spectrum, $\mathcal{J}$ has T-stable spectrum, so the result now follows from Lemma 5.5. $\quad\square$

Theorem 5.8. *A Jordan algebra of compact operators with submultiplicative spectrum is reducible.*

Proof. Let $\mathcal{J}$ be a Jordan algebra of compact operators with submultiplicative spectrum. As in the proof of Theorem 5.7, we may suppose that $\mathcal{J}$ is uniformly closed. Let $\mathcal{F}$ be the ideal of finite-rank elements in $\mathcal{J}$. By Theorem 5.6, if $\mathcal{F}$ does not contain a nonzero nilpotent element, then $\mathcal{J}$ is reducible; hence we may suppose that some nonzero T in $\mathcal{F}$ is nilpotent. Then for every F in $\mathcal{F}$, $\sigma(FT) = 0$ by the hypothesis of submultiplicative spectrum, so $\mathrm{tr}(FT) = 0$. Hence $\mathcal{J}$ is reducible by Theorem 4.4. $\quad\square$

Corollary 5.9. *Let $\mathcal{J}$ be a uniformly closed Jordan algebra of compact operators with the property that $\mathrm{tr}(F^2) = 0$ for every finite-rank element F in $\mathcal{J}$. Then $\mathcal{J}$ is reducible.*

Proof. Let $\mathcal{F}$ be the ideal of finite-rank elements in $\mathcal{J}$. The hypothesis implies that every $F \in \mathcal{F}$ is nilpotent; indeed, otherwise some $F \in \mathcal{F}$ would have nonzero $\alpha \in \sigma(F)$, and the Riesz projection of F corresponding to α would have $\mathrm{tr}(P^2) \neq 0$, since $\sigma(P) = \{0, 1\}$. Hence the result follows by Theorem 5.1. $\quad\square$

Acknowledgments

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