Negativity compensation in the nonnegative inverse eigenvalue problem

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Abstract

If a set \( \mathcal{A} \) of complex numbers can be partitioned as \( \mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_s \) in such a way that each \( \mathcal{A}_i \) is realized as the spectrum of a nonnegative matrix, say \( \mathcal{A}_i \), then \( \mathcal{A} \) is trivially realized as the spectrum of the nonnegative matrix \( \mathcal{A} = \bigoplus \mathcal{A}_i \). In [Linear Algebra Appl. 369 (2003) 169] it was shown that, in some cases, a real set \( \mathcal{A} \) can be realized even if some of the \( \mathcal{A}_i \) are not realizable themselves. Here we systematize and extend these results, in particular allowing the sets to be complex. The leading idea is that one can associate to any nonrealizable set \( \mathcal{A} \) a certain negativity \( N(\mathcal{A}) \), and to any realizable set \( \mathcal{A} \) a certain positivity \( M(\mathcal{A}) \). Then, under appropriate conditions, if \( M(\mathcal{A}) \geq N(\mathcal{A}) \) we can conclude that \( \mathcal{A} \cup \mathcal{A} \) is the spectrum of a nonnegative matrix. Additionally, we prove a complex generalization of Suleimanova’s theorem.

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1. Introduction

A set \( \mathcal{A} \) of complex numbers is said to be realizable if \( \mathcal{A} \) is the spectrum of an entrywise nonnegative matrix. It is clear that if a set \( \mathcal{A} \) of complex numbers can be

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partitioned as \( A = A_1 \cup \cdots \cup A_s \) in such a way that each \( A_i \) is realizable, then \( A \) is realizable: if \( A_i \) is a nonnegative matrix with spectrum \( \lambda_i \) for \( i = 1, 2, \ldots, s \) then the block diagonal matrix \( A = \bigoplus A_i \) is nonnegative and has spectrum \( \lambda \). The purpose of the present paper is to give conditions under which \( A \) is realizable even if some of the \( A_i \) are not realizable, provided there are other subsets \( A_j \) which are realizable and, in a certain way, compensate the nonrealizability of the former ones.

To do this, our main tool will be a result, due to Brauer [7] (Theorem 3.1), which shows how to modify one single eigenvalue of a matrix via a rank-one perturbation, without changing any of the remaining eigenvalues. This, together with the properties of real matrices with constant row sums, are the basic ingredients of our technique. This approach was first adopted by Soto [35] in connection with the nonnegative inverse eigenvalue problem (hereafter NIEP), i.e. the problem of characterizing all possible spectra of (entrywise) nonnegative matrices. In the references we provide a wide bibliography on the NIEP [1–40], and, in particular, on conditions which are sufficient for realizability of spectra.

Soto obtained conditions which are sufficient for realizability of partitioned real spectra, with the partition allowing some of its pieces to be nonrealizable. One remarkable feature of the results in [35] is that, unlike most of the previous conditions which are sufficient for realizability of spectra, the proofs are constructive in the sense that one can explicitly construct nonnegative matrices realizing the prescribed spectra. This is a fundamental difference of our results with previous related results in the literature. Some of these focus on manipulating the characteristic polynomial [22,32], eventually employing encoded versions [4,5]. Also, Wuwen [39] contains a result closely related to our basic Lemma 4.5 (see Theorem 4.1 in Section 4), but it does not offer the possibility of actually constructing any specific matrix realizing the given spectrum.

In the present paper we intend to further exploit the advantages provided by Brauer’s theorem to obtain conditions which are sufficient for realizability of sets of complex numbers. The paper is organized as follows:

We begin by introducing the basic concepts and notation used throughout the paper in Section 2. After briefly recalling both Brauer’s theorem and Suleimanova’s sufficient conditions [37], Section 3 contains the proof of a complex analogue of Suleimanova’s result with the negative real semi-axis replaced by the sector \( \{ z \in \mathbb{C} : \Re z \leq 0, |\Re z| \geq |\Im z| \} \) of the complex plane.

In Section 4 we define and analyze what we call the Brauer negativity and the Brauer realizability margin. The Brauer negativity of a self-conjugate set \( A = \{\lambda_0; \lambda_1, \ldots, \lambda_n\} \in \mathbb{C} \) with \( \lambda_0 \in \mathbb{R} \) and \( \lambda_0 \geq \lambda_i \) for any \( \lambda_i \in \mathbb{R} \) measures how far the set \( A \) is from being realizable, namely, how much \( \lambda_0 \) must be increased to obtain a realizable set. In particular, \( A \) is realizable if and only if its Brauer negativity is zero. On the other hand, the Brauer realizability margin of a realizable set measures how much its Perron root can be diminished while remaining the Perron root of the resulting set. The central idea of Section 4 is that if we have a nonrealizable set \( I \) with Brauer negativity \( \delta \) and a realizable set \( A \) with Brauer realizability margin \( \epsilon \) and \( \epsilon \geq \delta \) then
under some conditions, that are rigorously quantified in Lemma 4.5, we can conclude that \( \Gamma \cup \Lambda \) is a realizable set. All the proofs in Section 5 heavily rely on Lemma 4.5.

Section 5 presents our main result, Theorem 5.1, a realizability criterion for sets of complex numbers which can be partitioned in such a way that the negativity of the nonrealizable pieces can be compensated by the positivity of the realizable ones. We must stress that, although the proof is constructive to a certain extent, it does not allow in general, as in [35], to explicitly construct a nonnegative matrix with the given spectrum. Also, since computing the Brauer negativity and the Brauer realizability margin is not always simple, Theorem 5.1 may be sometimes hard to use in practice. However, one can use the fact that any realizability criterion, present or future, gives both an upper bound for the Brauer negativity of a nonrealizable set, and a lower bound for the Brauer realizability margin of a realizable set. This is the idea underlying Corollary 5.1, a weaker but more applicable version of Theorem 5.1, which allows to eventually employ different realizability criteria on each piece of the partition in order to estimate their negativity or their positivity. This flexibility allows to view Corollary 5.1 as a procedure to obtain new realizability criteria starting from previous ones.

Finally, we present in Section 6 two specific examples, one real and one complex, to illustrate our results.

2. Preliminaries and notation

A set \( A = \{\lambda_0, \lambda_1, \ldots, \lambda_n\} \) of complex numbers is said to be realizable if there exists an entrywise nonnegative \( n+1 \) by \( n+1 \) matrix with spectrum \( A \). The set of all realizable sets is denoted by \( \mathcal{R} \).

If a complex set \( A = \{\lambda_0, \lambda_1, \ldots, \lambda_n\} \) is realizable, then the nonreal elements of \( A \) come in conjugate pairs. Hence, the conjugate set \( \bar{A} = \{\bar{\lambda}_0, \bar{\lambda}_1, \ldots, \bar{\lambda}_n\} \) coincides with \( A \). Moreover, the Perron–Frobenius theorem (see [26]) implies that if \( A = \{\lambda_0, \lambda_1, \ldots, \lambda_n\} \) is realizable then one of its elements, say \( \lambda_0 \), is real and such that \( \lambda_0 \geq |\lambda_i| \) for \( i = 1, \ldots, n \). Therefore, the set

\[
\mathcal{A} \equiv \{ A = \{\lambda_0, \lambda_1, \ldots, \lambda_n\} \subset \mathbb{C} : \bar{A} = A, \lambda_0 \in \mathbb{R}, \lambda_0 \geq \lambda_i \text{ for any } \lambda_i \in \mathbb{R} \}
\]

must include all possible spectra of nonnegative matrices, appropriately reordered to single out the Perron root \( \lambda_0 \) (notice the semicolon separating \( \lambda_0 \)). Unless otherwise stated, we assume from now on that any set under examination is in \( \mathcal{A} \). We denote

\[
\mathcal{A}_R = \{ A \in \mathcal{A} : A \text{ is realizable} \}.
\]

A real matrix \( A = (a_{ij})_{i,j=1}^n \) is said to have constant row sums if all its rows sum up to a same constant, say \( \alpha \), i.e.

\[
\sum_{j=1}^{n} a_{ij} = \alpha, \quad i = 1, \ldots, n.
\]
The set of all real matrices with constant row sums equal to \( \alpha \) is denoted by \( \mathcal{S}_\alpha \). We will make frequent use of the fact:

**Lemma 2.1** (Johnson [15]). Any realizable set is realized in particular by a nonnegative matrix with constant row sums equal to its Perron root.

Also, we will use that any matrix in \( \mathcal{S}_\alpha \) has eigenvector \( e = (1, \ldots, 1)^T \) corresponding to the eigenvalue \( \alpha \). For simplicity, we denote in what follows by \( e \) any vector of the appropriate dimension with all its entries equal to one. Likewise, we denote by \( e_1 = (1, 0, \ldots, 0)^T \) the first column of the identity matrix of any appropriate dimension.

3. Brauer’s theorem and a complex Suleimanova-type theorem

As pointed out in the introduction, our main motivation is to exploit the advantages provided by the following result, due to Brauer [7] in the study of the NIEP.

**Theorem 3.1** (Brauer [7]). Let \( A \) be an \( n \times n \) arbitrary matrix with eigenvalues \( \lambda_1, \ldots, \lambda_n \). Let \( v = (v_1, \ldots, v_n)^T \) be an eigenvector of \( A \) associated with the eigenvalue \( \lambda_k \) and let \( q \) be any \( n \)-dimensional vector. Then the matrix \( A + vq^T \) has eigenvalues \( \lambda_1, \ldots, \lambda_k - 1, \lambda_k + v^T q, \lambda_{k+1}, \ldots, \lambda_n \).

An immediate consequence of Brauer’s theorem is the following useful and well-known result:

**Lemma 3.1.** If \( \lambda_0 \), \( \lambda_1, \ldots, \lambda_n \) \( \in \mathcal{A} \) and \( \alpha > 0 \), then \( \lambda_0 + \alpha \), \( \lambda_1, \ldots, \lambda_n \) \( \in \mathcal{A} \).

**Proof.** From Lemma 2.1 we know that there exists a nonnegative matrix \( A \in \mathcal{S}_{\lambda_0} \) with spectrum \( \lambda_0 \), \( \lambda_1, \ldots, \lambda_n \). The matrix \( A + \alpha e e^T \in \mathcal{S}_{\lambda_0 + \alpha} \) is nonnegative and, by Theorem 3.1, has spectrum \( \lambda_0 + \lambda_1, \ldots, \lambda_n \). \( \square \)

The next theorem is due to Suleimanova, and is usually considered one of the most important results in the real NIEP:

**Theorem 3.2** (Suleimanova [37]). Let \( \lambda_0, \lambda_1, \ldots, \lambda_n \) \( \in \mathbb{R} \) with \( \lambda_i \leq 0 \) for \( i = 1, \ldots, n \). Then \( \lambda_0 \) is realizable if and only if \( \sum_{i=0}^{n} \lambda_i \geq 0 \).

An illustration of the interest of Brauer’s theorem is that it allows us to prove the following complex generalization of Suleimanova’s theorem:

**Theorem 3.3.** Let \( \lambda_0, \lambda_1, \ldots, \lambda_n \) \( \in \mathcal{A} \) with

\[
A' = [\lambda_1, \ldots, \lambda_n] \subset \{ z \in \mathbb{C} : \text{Re} z \leq 0, |\text{Re} z| \geq |\text{Im} z| \},
\]

Then \( \lambda_0 \) is realizable if and only if \( \sum_{i=0}^{n} \lambda_i \geq 0 \).
Proof. Suppose that the elements of $A'$ are ordered in such a way that $\lambda_{2p+1}, \ldots, \lambda_n \in \mathbb{R}$ and $\lambda_1, \ldots, \lambda_{2p} \in \mathbb{C} \setminus \mathbb{R}$ with

$$x_k = \text{Re} \lambda_{2k-1} = \text{Re} \lambda_{2k} \quad \text{and} \quad y_k = \text{Im} \lambda_{2k-1} = -\text{Im} \lambda_{2k}$$

for $k = 1, \ldots, p$. Consider now the matrix

$$B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_1 + y_1 & x_1 & -y_1 & 0 & 0 & 0 & 0 \\
-x_1 - y_1 & y_1 & x_1 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-x_p + y_p & 0 & 0 & x_p & -y_p & 0 & 0 \\
-x_p - y_p & 0 & 0 & y_p & x_p & 0 & 0 \\
-\lambda_{2p+1} & 0 & 0 & 0 & 0 & \lambda_{2p+1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\lambda_n & 0 & 0 & 0 & 0 & 0 & \lambda_n 
\end{bmatrix}.$$ 

It is not difficult to check that $B \in \mathcal{C}\mathcal{J}_0$ with spectrum $\{0, \lambda_1, \ldots, \lambda_n\}$, and all the elements on the first column of $B$ are nonnegative.

Define $q = (q_0, q_1, \ldots, q_n)^T$ with $q_0 = 0,

$$q_k = -\text{Re} \lambda_k \quad \text{for} \quad k = 1, \ldots, 2p \quad \text{and} \quad q_k = -\lambda_k \quad \text{for} \quad k = 2p + 1, \ldots, n.$$ 

From Brauer's Theorem 3.1 we deduce that the matrix $B + eq^T$, which is nonnegative, has spectrum $\{-\sum_{i=1}^n \lambda_i, \lambda_1, \ldots, \lambda_n\}$. Since $\lambda_0 \geq -\sum_{i=1}^n \lambda_i$, we conclude from Lemma 3.1 that $A = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ is the spectrum of a nonnegative matrix as well. □

4. Brauer negativity and its compensation

The basic idea of this section, which will be applied to the NIEP in Section 5, is that one can associate to each set $A \in \mathcal{A}$ its so-called Brauer realizability margin, a quantity reflecting in a certain particular way how far the set is from being realized as the spectrum of a nonnegative matrix. This negativity can be diminished by joining the set with a realizable set, at best until the negativity is fully compensated and the joint set becomes realizable.

In order to define this negativity we need to introduce some further notation:

**Definition 4.1.** Given a set $A = \{\lambda_0; \lambda_1, \ldots, \lambda_n\} \in \mathcal{A}$, we define the Brauer realizability margin of $A$ as

$$\mathcal{M}(A) \equiv \max \{\epsilon \geq 0 : \{\lambda_0 - \epsilon; \lambda_1, \ldots, \lambda_n\} \in \mathcal{A}\},$$

and denote

$$\mathcal{M}_\epsilon = \{A \in \mathcal{A} : \mathcal{M}(A) = \epsilon\}.$$
Definition 4.2. Given a set $\mathcal{A} = \{\lambda_0, \lambda_1, \ldots, \lambda_n\} \in \mathcal{A}$, we define the Brauer negativity of $A$ as

$$\mathcal{N}(A) \equiv \min \{\delta \geq 0 : \{\lambda_0 + \delta; \lambda_1, \ldots, \lambda_n\} \in \mathcal{A}\},$$

and denote

$$\mathcal{N}_\delta = \{A \in \mathcal{A} : \mathcal{N}(A) = \delta\}.$$

Note that a set $\mathcal{A} \in \mathcal{A}$ is realizable if and only if $\mathcal{N}(\mathcal{A}) = 0$. That is, $\mathcal{N}_0 = \mathcal{A}$. An interesting outcome of Theorem 3.1 concerns the zero–nonzero pattern of nonnegative matrices with constant row sums realizing the extremal spectra in Definitions 4.1 and 4.2.

Lemma 4.1. Let $\mathcal{A} = \{\lambda_0; \lambda_1, \ldots, \lambda_n\} \in \mathcal{A}$ with $\mathcal{M}(\mathcal{A}) = \epsilon$. Let $A_{-\epsilon} \in \mathcal{G}_{\lambda_0 - \epsilon}$ be nonnegative and with spectrum $\{\lambda_0 - \epsilon; \lambda_1, \ldots, \lambda_n\} \in \mathcal{A}$. Then every column of $A_{-\epsilon}$ contains at least one zero entry.

Proof. Suppose there is a column of $A_{-\epsilon}$, say the first one, such that all its entries are positive, and let $\alpha > 0$ be the smallest entry in that column. It follows from Theorem 3.1 that the matrix $A = A_{-\epsilon} - \alpha ee^T \in \mathcal{G}_{\lambda_0 - (\epsilon + \alpha)}$, which is nonnegative with Perron root $\lambda_0 - (\epsilon + \alpha)$, has spectrum $\{\lambda_0 - (\epsilon + \alpha); \lambda_1, \ldots, \lambda_n\} \in \mathcal{A}$. This contradicts the maximality of $\mathcal{M}(\mathcal{A}) = \epsilon$ in Definition 4.1. □

Lemma 4.2. Let $\mathcal{A} = \{\lambda_0; \lambda_1, \ldots, \lambda_n\} \in \mathcal{A}$ with $\mathcal{N}(\mathcal{A}) = \delta$. Let $A_\delta \in \mathcal{G}_{\lambda_0 + \delta}$ be nonnegative and with spectrum $\{\lambda_0 + \delta; \lambda_1, \ldots, \lambda_n\} \in \mathcal{A}$. Then every column of $A_\delta$ contains at least one zero entry.

Proof. Suppose there is a column of $A_\delta$, say the first one, such that all its entries are positive, and let $\alpha > 0$ be the smallest entry in that column. It follows from Theorem 3.1 that the matrix $A = A_\delta - \alpha ee^T \in \mathcal{G}_{\lambda_0 + (\delta - \alpha)}$, which is nonnegative with Perron root $\lambda_0 + (\delta - \alpha)$, has spectrum $\{\lambda_0 + (\delta - \alpha); \lambda_1, \ldots, \lambda_n\} \in \mathcal{A}$. This contradicts the minimality of $\mathcal{N}(A) = \delta$ in Definition 4.2. □

The two concepts introduced in Definitions 4.1 and 4.2 measure how far is the set $\mathcal{A} \in \mathcal{A}$ from changing its realizability properties under a transformation which shifts its dominant real element, leaving the remaining elements untouched. This is precisely what Brauer’s Theorem 3.1 allows us to do, hence the naming of both quantities.

Definition 4.3. Let $A = (a_{ij})_{i,j=1}^n$ be a real $n$ by $n$ matrix. We define the negativity index of the $j$th column $A_j$ of $A$ as

$$\mathcal{N}(A_j) \equiv \max\{0, -a_{1j}, \ldots, -a_{nj}\}.$$
We define the negativity index of \( A \) as

\[
\mathcal{N}(A) \equiv \sum_{j=1}^{n} N(A_j).
\]

Notice that, with this definition, a real matrix \( A \) is nonnegative if and only if \( \mathcal{N}(A) = 0 \). We now use Brauer’s Theorem 3.1 to reduce the negativity index of a given real matrix with constant row sums, changing its spectrum by increasing the eigenvalue associated with \( e \).

**Lemma 4.3.** Let \( A \in \mathcal{C}(\lambda_0) \) with spectrum \( \{\lambda_0, \lambda_1, \ldots, \lambda_n\} \subset \mathbb{C} \). If \( 0 \leq \delta \leq \mathcal{N}(A) \) then there exists \( B \in \mathcal{C}(\lambda_0 + \delta) \) with spectrum \( \{\lambda_0 + \delta, \lambda_1, \ldots, \lambda_n\} \) and \( \mathcal{N}(B) = \mathcal{N}(A) - \delta \).

**Proof.** Let \( v = (v_0, \ldots, v_n)^T \) be any real vector such that \( \sum_{j=0}^{n} v_j = \delta \) and \( 0 \leq v_j \leq N(A_j) \) for each \( j = 0, \ldots, n \). Since \( e = (1, \ldots, 1)^T \) is the eigenvector with eigenvalue \( \lambda_0 \) then, by Theorem 3.1, we conclude that the matrix \( B = A + \delta ee^T \) satisfies the conditions of the statement. \( \Box \)

The connection between the Brauer negativity of \( \{\lambda_0; \lambda_1, \ldots, \lambda_n\} \in \mathcal{A} \) and the negativity indices of the matrices with constant row sums equal to \( \lambda_0 \) and spectrum \( \{\lambda_0; \lambda_1, \ldots, \lambda_n\} \) is established in the next result:

**Lemma 4.4.** Let \( \{\lambda_0; \lambda_1, \ldots, \lambda_n\} \in \mathcal{A} \), then

\[
\mathcal{N}(\mathcal{A}) = \min \left\{ \mathcal{N}(A) : A \in \mathcal{C}(\lambda_0) \text{ and } A \text{ has spectrum } \mathcal{A} \right\}.
\]  

**Proof.** Denote by \( \mathcal{N}(\mathcal{A}) \) the quantity on the right-hand side of (1).

First, let \( \mathcal{N}(\mathcal{A}) = \delta \). Then the set \( A_\delta = \{\lambda_0 + \delta; \lambda_1, \ldots, \lambda_n\} \) is realizable. In particular, by Lemma 2.1, the set \( A_\delta \) is realizable by a nonnegative matrix \( A_\delta \in \mathcal{C}(\lambda_0 + \delta) \). Hence the matrix \( A = A_{\delta} - \delta ee^T \) is in \( \mathcal{C}(\lambda_0) \) and, by Theorem 3.1, has spectrum \( A \). The last \( n \) columns of \( A \) are nonnegative, since they are columns of \( A_{\delta} \). And the negativity index of the first column of \( A \) is exactly \( \delta \), since by Lemma 4.2 the first column of \( A_{\delta} \) contains at least one zero entry. Consequently \( \mathcal{N}(A) = \delta \) and \( \mathcal{N}(A) \leq \mathcal{N}(\mathcal{A}) \).

Now, let \( \mathcal{N}(\mathcal{A}) = \gamma \). To prove that \( \mathcal{N}(\mathcal{A}) \leq \mathcal{N}(\mathcal{A}) \) it suffices to show that \( A_\gamma = \{\lambda_0 + \gamma; \lambda_1, \ldots, \lambda_n\} \) is realizable. Take a matrix \( B \in \mathcal{C}(\lambda_0) \) with spectrum \( A \) and \( \mathcal{N}(B) = \gamma \). By Lemma 4.3, there exists a matrix \( B_\gamma \) with spectrum \( A_\gamma \) and negativity index \( \mathcal{N}(B) - \gamma = 0 \). Hence, \( B_\gamma \) is a nonnegative matrix that realizes \( A_\gamma \). This concludes the proof. \( \Box \)

A simple consequence of Lemma 4.4 is the subadditivity of \( \mathcal{N} \): let

\[
A_1 = \{\lambda_0^{(1)}, \lambda_1^{(1)}, \ldots, \lambda_n^{(1)}\} \quad \text{and} \quad A_2 = \{\lambda_0^{(2)}, \lambda_1^{(2)}, \ldots, \lambda_n^{(2)}\}
\]
be two sets in $\mathcal{S}$. For $i = 1, 2$ let $A_i \in \mathcal{S}_{\lambda_i(0)}$ with spectrum $A_i$ and such that $\mathcal{N}(A_i) = \mathcal{N}(A_i)$. With no loss of generality we may assume that $\lambda_i(1) \geq \lambda_i(2)$. Then, adding a column of nonnegative entries $\lambda_i(1) - \lambda_i(2)$ to the lower left block of the direct sum $A_1 \oplus A_2$, one obtains a matrix $B \in \mathcal{S}_{\lambda_i(0)}$ with spectrum $\lambda_i(0)$ and negativity $\mathcal{N}(B) = \mathcal{N}(A_1) + \mathcal{N}(A_2)$. Hence, using the equivalence obtained in (1),

$$\mathcal{N}(A_1 \cup A_2) \leq \mathcal{N}(A_1) + \mathcal{N}(A_2).$$

Now, we are in the position to show that the amount of negativity of a nonrealizable set can be diminished by merging it with a realizable spectrum:

**Lemma 4.5.** Let $\lambda(\lambda_0, \lambda_1, \ldots, \lambda_k) \in \mathcal{S}$ and $\delta > 0$ such that $\lambda(\lambda_0 + \delta; \lambda_1, \ldots, \lambda_k) \in \mathcal{S}$. Let $A = [\lambda_0; \lambda_1, \ldots, \lambda_k] \in \mathcal{S}$ with $\lambda_0 > \gamma_0$ and $\epsilon > 0$ such that $A_{-\epsilon} = [\lambda_0 - \epsilon; \lambda_1, \ldots, \lambda_k] \in \mathcal{S}$. Set

$$\rho = \max(\lambda_0 - \epsilon, \gamma_0).$$

Then there exists $M \in \mathcal{S}_{\lambda_0}$ with spectrum $\lambda_0 \cup \lambda$ and negativity index

$$\mathcal{N}(M) \leq \max(0, \delta - (\lambda_0 - \rho)).$$

In particular, if $\lambda_0 - \rho \geq \delta$, then $\lambda_0 \cup \lambda$ is realizable.

**Proof.** Since $\Gamma_0$ is realizable, by Lemma 2.1 there exists a nonnegative matrix $A_0 \in \mathcal{S}_{\gamma_0}$ of order $k + 1$ with spectrum $\Gamma_0$. The matrix $A = A_0 - \delta \epsilon \epsilon^T \in \mathcal{S}$ has spectrum $\Gamma$ with $\mathcal{N}(A) \leq \delta$, and the last $k$ columns of $A$ are nonnegative.

Since $[\lambda_0 - \epsilon; \lambda_1, \ldots, \lambda_k]$ is realizable and $\rho \geq \lambda_0 - \epsilon$, then we conclude from Lemmas 2.1 and 3.1 that there exists a nonnegative matrix $C \in \mathcal{S}_\rho$ of order $j + 1$ with spectrum $[\rho; \lambda_1, \ldots, \lambda_k]$.

Now, consider the matrix

$$M = \begin{bmatrix} A & (\rho - \gamma_0) \epsilon \epsilon^T \\ 0 & C \end{bmatrix}$$

which is in $\mathcal{S}_\rho$ and has spectrum $\lambda_0 \cup \{\rho; \lambda_1, \ldots, \lambda_k\}$. Hence, the matrix

$$M = M_\rho + (\lambda_0 - \rho) \epsilon \epsilon^T$$

satisfies the conditions in the statement. $\square$

The wider the realizability margin $\epsilon$ the more the negativity $\delta$ can be reduced, provided $\lambda_0 > \gamma_0$. If $\lambda_0 \leq \gamma_0$, one can easily check that no improvement of the negativity $\delta$ is produced using an argument similar to the proof of Lemma 4.5. Note that Lemma 4.5 is not sharp, since it is written for convenience in terms of arbitrary $\epsilon$ and $\delta$ (this is how it will be employed in Section 5). Of course, the sharpest result
can be obtained simply replacing $\epsilon$ by the Brauer realizability margin and $\delta$ by the Brauer negativity.

Corollary 4.1. Consider the sets $\Gamma = \{\gamma_0; \gamma_1, \ldots, \gamma_k\} \in \mathcal{A} - \mathcal{AR}$ and $A = \{\lambda_0; \lambda_1, \ldots, \lambda_j\} \in \mathcal{A}$ with $\lambda_0 > \gamma_0$. Let
\[ \rho = \max(\lambda_0 - \mathcal{M}(A), \gamma_0). \]

Then there exists $M \in \mathcal{CF}_{\lambda_0}$ with spectrum $\Gamma \cup A$ and negativity index
\[ N(M) = \max\{0, N(\Gamma) - (\lambda_0 - \rho)\}. \]

In particular, if $\lambda_0 - \rho \geq N(\Gamma)$, then $\Gamma \cup A$ is realizable.

The proof of Corollary 4.1 is identical to that of Lemma 4.5 replacing $\epsilon$ by $\mathcal{M}(A)$ and $\delta$ by $N(\Gamma)$, except for one detail: here the matrix $A_\delta$ right at the beginning of the proof has at least one zero entry in each column, according to Lemma 4.2. Therefore, the matrix $A = A_\delta - \delta ee^T_1$ has negativity index equal to $\delta$, hence the equality in the conclusion of the corollary.

Using a different approach, Wuwen proves in [39] a result very close to both Lemma 4.5 and Corollary 4.1, namely

Theorem 4.1 (Wuwen [39]). Let $\{\lambda_0; \lambda_1, \ldots, \lambda_n\} \in \mathcal{A}$ with $\lambda_1 \in \mathbb{R}$. Then, for any $\delta \geq 0$ we have that
\[ \{\lambda_0 + \delta; \lambda_1 \pm \delta, \ldots, \lambda_n\} \in \mathcal{A}. \]

One can easily check that Theorem 4.1 works much in the same way as Lemma 4.5 in the case when $\lambda_0 - \rho \geq \delta$, so the joint spectrum $\Gamma \cup A$ is realizable. Actually, in that case the assumptions of Theorem 4.1 are less demanding than those of Lemma 4.5. The reason for this is that Theorem 4.1 is not concerned with constructing the realizing nonnegative matrix. However, Lemma 4.5 applies to more general situations than Theorem 4.1, since the latter does not give any information at all in the case when $\lambda_0 - \rho < \delta$.

5. Realizability of spectra via partition

We are now in the position to prove our main result, a criterion for the realizability of sets of complex numbers which can be partitioned in such a way that the negativity of one of its pieces is compensated by the realizability margin of the remaining ones.

Theorem 5.1. Let $A$ be a set of complex numbers which can be partitioned as
\[ A = \Gamma \cup A_1 \cup A_2 \cup \cdots \cup A_p \]
in such a way that
\[ \Gamma = \{ \gamma_0; \gamma_1, \ldots, \gamma_q \} \in \mathcal{N} \delta, \] where \( \delta > 0 \),
and for each \( i = 1, \ldots, p \)
\[ A_i = \{ \lambda^{(i)}_0, \lambda^{(i)}_1, \ldots, \lambda^{(i)}_{m_i} \} \in \mathcal{M}_{\epsilon_i}, \] where \( \epsilon_i \geq 0 \) and \( \lambda^{(i)}_0 > \gamma_0 \).
For each \( i = 1, \ldots, p \) let
\[ \rho_i = \max \{ \lambda^{(i)}_0 - \epsilon_i, \gamma_0 \}. \]
If
\[ \delta \leq \mu \left( \bigcup_{i=1}^{p} [\rho_i, \lambda^{(i)}_0] \right) \] (2)
for the usual measure \( \mu \) in \( \mathbb{R} \), then \( \Delta \) is realizable.

**Proof.** We begin by doing some simplifying assumptions:

(i) We may assume without loss of generality that the \( A_i \) are ordered so that
\[ \lambda^{(1)}_0 - \epsilon_1 \leq \lambda^{(2)}_0 - \epsilon_2 \leq \cdots \leq \lambda^{(p)}_0 - \epsilon_p. \]
(ii) We may additionally assume that
\[ \lambda^{(i-1)}_0 < \lambda^{(i)}_0, \quad i = 2, \ldots, p. \]
Otherwise, \([\rho_i, \lambda^{(i)}_0] \subseteq [\rho_{i-1}, \lambda^{(i-1)}_0]\) and the interval \([\rho_i, \lambda^{(i)}_0]\) does not contribute to the measure in (2). Since \( A_i \) is realizable, it can be removed from \( \Delta \) without affecting the realizability of \( \Delta \).

In order to directly apply Lemma 4.5 it is convenient to rewrite formula (2). To do so, set \( \lambda^{(0)}_0 = \gamma_0 \) and for each \( i = 1, \ldots, p \) define
\[ \rho'_i = \max \{ \lambda^{(i)}_0 - \epsilon_i, \lambda^{(i-1)}_0 \}. \]
We claim that
\[ \mu \left( \bigcup_{i=1}^{k} [\rho_i, \lambda^{(i)}_0] \right) = \sum_{i=1}^{k} (\lambda^{(i)}_0 - \rho_i). \] (3)
for each \( k \in \{1, \ldots, p\} \). We proceed to prove (3) recursively.

Eq. (3) is trivially true for \( k = 1 \), since \( \rho_1 = \rho'_1 \). Now, suppose (3) is true for \( k = j \). We will show that then it is also true for \( k = j + 1 \). Notice that the two simplifying assumptions imply that when we add the new index \( j + 1 \), the only change
in the measure (2) can come from above \( \lambda_{(j)}^{(j)} \). The situation depends mainly on the overlap of the intervals \([\rho_{j}, \lambda_{(j)}^{(j)}] \) and \([\rho_{j+1}, \lambda_{(j+1)}^{(j+1)}] \):

- If they do not overlap, then the new interval contributes with its whole length to the measure of the union, i.e. it produces an increase of \( \epsilon_{j+1} \) on the left-hand side of (3). On the other hand, in this case \( \rho_{j}^{'} = \lambda_{(j+1)}^{(j+1)} - \epsilon_{j+1} \), so the right-hand side also increases exactly by an amount of \( \epsilon_{j+1} \).
- If the two intervals overlap, then \( \rho_{j}^{'} = \lambda_{(j)}^{(j)} \), so the right-hand side of (3) increases by \( \lambda_{(j+1)}^{(j+1)} - \lambda_{(j)}^{(j)} \). Since the two intervals overlap, with \( \lambda_{(j+1)}^{(j+1)} > \lambda_{(j)}^{(j)} \), the new interval contributes to the measure only with the part which is above \( \lambda_{(j)}^{(j)} \), i.e. the measure also increases by an amount \( \lambda_{(j+1)}^{(j+1)} - \lambda_{(j)}^{(j)} \).

This concludes the proof of (3). Now, by Lemma 4.5 applied to \( \Gamma \cup A_{1} \), there exists \( M_{1} \in \mathcal{E}_{\lambda_{(i)}}^{(i)} \) with spectrum \( \Gamma \cup A_{1} \) and negativity index

\[
\mathcal{N}(M_{1}) \leq \max \{0, \delta - (\lambda_{(i)}^{(i)} - \rho_{1}^{i})\}.
\]

In particular, if \( \lambda_{(1)}^{(1)} - \rho_{1}^{1} \geq \delta \), then \( \Gamma \cup A_{1} \) is realizable and the result follows from (3). Otherwise, apply Lemma 4.5 to \( (\Gamma \cup A_{1}) \cup A_{2} \) there exists \( M_{2} \in \mathcal{E}_{\lambda_{(i)}}^{(i)} \) with spectrum \( \Gamma \cup A_{1} \cup A_{2} \) and negativity index

\[
\mathcal{N}(M_{2}) \leq \max \{0, \delta - (\lambda_{(1)}^{(1)} - \rho_{1}^{1}) - (\lambda_{(2)}^{(2)} - \rho_{2}^{2})\}.
\]

In particular, if \( \lambda_{(1)}^{(1)} - \rho_{1}^{1} \leq \delta \), then \( \Gamma \cup A_{1} \cup A_{2} \) is realizable and the result follows using (3). This argument can be repeated for each \( i = 1, \ldots, p \), concluding the proof. \( \square \)

Theorem 5.1 is the sharpest result we can get from this approach, since both the negativity and the realizability margin are defined as optimal quantities. However, the hypotheses of Theorem 5.1 are not easy to check in specific examples, since no systematic procedure is available to compute neither of the two optimal quantities. Hence, it may be useful to have a weaker, more applicable version.

As a set of conditions is said to be a realizability criterion if any set \( A = \{\lambda_{0}, \lambda_{1}, \ldots, \lambda_{n}\} \subset \mathbb{C} \) satisfying the conditions \( K \) is realizable. For instance, the two conditions

\[
\lambda_{i} \leq 0, \quad i = 1, \ldots, n,
\]

\[
\sum_{i=0}^{n} \lambda_{i} \geq 0
\]

constitute Suleimanova’s realizability criterion (see Theorem 3.2). To each realizability criterion \( K \) we associate the set
and further define, by analogy with Definitions 4.1 and 4.2,

\[ M_K(A) \equiv \max \left\{ \epsilon \geq 0 : \{\lambda_0 - \epsilon; \lambda_1, \ldots, \lambda_n\} \in \mathcal{A} \mathcal{R}_K \right\} , \]

\[ N_K(A) \equiv \min \left\{ \delta \geq 0 : \{\lambda_0 + \delta; \lambda_1, \ldots, \lambda_n\} \in \mathcal{A} \mathcal{R}_K \right\} . \]

It is clear that

\[ M_K(A) \leq M(A), \quad N_K(A) \geq N(A) \]

for any realizability criterion \( K \). Therefore, if we denote

\[ M_{K,\epsilon} = \{A \in \mathcal{A} \mathcal{R}_K : M_K(A) = \epsilon\} , \]

and

\[ N_{K,\delta} = \{A \in \mathcal{A} : N_K(A) = \delta\} , \]

we trivially obtain the following Corollary of Theorem 5.1.

**Corollary 5.1.** Let \( \mathcal{A} \) be a set of complex numbers which can be partitioned as

\[ \mathcal{A} = \Gamma \cup A_1 \cup A_2 \cup \cdots \cup A_p \]

in such a way that there exists a realizability criterion \( K \) such that

\[ \Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_q\} \in N_{K,\delta}, \quad \text{where } \delta > 0, \]

and that for each \( i = 1, \ldots, p \) there exists a realizability criterion \( C_i \) such that

\[ A_i = \{\lambda^{(i)}_0, \lambda^{(i)}_1, \ldots, \lambda^{(i)}_n\} \in M_{C_i,\epsilon_i}, \quad \text{where } \epsilon_i \geq 0 \text{ and } \lambda^{(i)}_0 > \gamma_0 \]

For each \( i = 1, \ldots, p \) let

\[ \rho_i = \max \left\{ \lambda^{(i)}_0 - \epsilon_i, \gamma_0 \right\} . \]

If

\[ \delta \leq \mu \left( \bigcup_{i=k}^p \left[\rho_i, \lambda^{(i)}_0\right] \right) \]

for the usual measure \( \mu \) in \( \mathbb{R} \), then \( \mathcal{A} \) is realizable.

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**6. Examples**

We will provide two examples, one real and the other one complex.

**Real example.** Consider the set

\[ \mathcal{A} = \{6.5, 5, 1, -4, -4, -6\} . \]
Note that $\mathcal{A}$ is not a realizable set since the sum of its components is equal to $-0.5$, and therefore $\mathcal{N}^+(\mathcal{A}) \geq 0.5$. We will show that indeed $\mathcal{N}^+(\mathcal{A}) = 0.5$. Let $\mathcal{A} = \Gamma \cup \mathcal{A}$ where

$$\Gamma = \{5; 1, 1, -4, -4\} \quad \text{and} \quad \mathcal{A} = \{6.5; -6\}.$$ Let $\mathcal{R}$ be the Reams realizability criterion [32], and let $\mathcal{S}$ be the Suleimanova realizability criterion given by Theorem 3.2. It can be verified that for $\delta = 1$, $\epsilon = 0.5$ we have

$$\Gamma \in \mathcal{N}_{R, \delta} \quad \text{and} \quad \mathcal{A} \in \mathcal{N}_{S, \epsilon}.$$ Applying Corollary 4.1 we conclude that there exists a matrix $\mathcal{M} \in \mathcal{F}_{6.5}$ with spectrum $\mathcal{A}$ and $\mathcal{N}(\mathcal{M}) = 0.5$.

Now we will construct such a matrix according to the proof of Lemma 4.5. First, following Reams we obtain the nonnegative matrix

$$E = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
14 & 0 & 1 & 0 & 0 \\
18 & \frac{7}{5} & 0 & 1 & 0 \\
0 & \frac{9}{5} & 0 & 0 & 1 \\
516 & \frac{585}{4} & \frac{15}{2} & \frac{35}{2} & 0
\end{bmatrix}$$

with spectrum $\Gamma_1 = \{6; 1, 1, -4, -4\}$. The Perron vector of $E$ associated with the Perron root 6 is $v = (1, 6, 22, 93, 531)^T$. Consider the positive diagonal matrix $D$ obtained by placing the entries of $v$ in order, down the diagonal. Following Johnson [15], the nonnegative matrix

$$D^{-1}ED \in \mathcal{F}_{6}$$

has spectrum $\Gamma_1$. At this point we follow the steps of the proof of Lemma 4.5 with $\delta = 1$, $\epsilon = 0.5$, $A_3 = D^{-1}ED$ and $C = \begin{bmatrix} 0 & 6 \\ 0 & 0 \end{bmatrix}$ to construct the matrix

$$M = \begin{bmatrix}
-1/2 & 6 & 0 & 0 & 0 & 0 & 1 & 0 \\
11/5 & 0 & 11/3 & 0 & 0 & 1 & 0 \\
7/2 & 21/4 & 0 & 93/22 & 0 & 1 & 0 \\
-1/2 & 9/37 & 0 & 0 & 177/37 & 1 & 0 \\
167/384 & 195/118 & 55/177 & 1085/384 & 0 & 1 & 0 \\
1/4 & 0 & 0 & 0 & 0 & 0 & 6 \\
1/2 & 0 & 0 & 0 & 0 & 0 & 6 & 0
\end{bmatrix} \in \mathcal{F}_{6.5}$$

which has spectrum $\mathcal{A}$ and $\mathcal{N}(\mathcal{M}) = 0.5$.

\textbf{Note.} Consider now the set

$$\mathcal{A}' = \{7, 5, 1, 1, -4, -4, -6\}.$$
If we proceed as in example above we will construct the nonnegative matrix

\[
M' = \begin{bmatrix}
0 & 6 & 0 & 0 & 0 & 1 & 0 \\
7 & 0 & 11 & 0 & 0 & 1 & 0 \\
9 & 21 & 22 & 0 & 22 & 0 & 1 \\
0 & 9 & 0 & 0 & 177 & 3 & 0 \\
172 & 195 & 55 & 1085 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 6 \\
1 & 0 & 0 & 0 & 0 & 6 & 0
\end{bmatrix} \in \mathcal{C}_7
\]

which has spectrum \( \lambda_{\text{aff}} \) and \( N(M) = 0 \).

The realizability of \( \lambda' \) can be deduced easily from the realizability of \( \lambda = \{6; 1, 1, -4, -4\} \) and the realizability of \( \{6; -6\} \) by applying Theorem 4.1, but Wuwen’s result does not provide a method to construct a specific nonnegative matrix with spectrum \( \lambda' \).

**Complex example.** Let

\[
\lambda = \{13, 11, -12, -1 + i, -1 - i, -2 + 2i, -2 - 2i, -3 + i, -3 - i\}
\]

with \( \lambda = \lambda' \cup A \), where

\[
\lambda' = \{11; -1 + i, -1 - i, -2 + 2i, -2 - 2i, -3 + i, -3 - i\}
\]

and

\[
\lambda = \{13; -12\}.
\]

Let \( C \) be the realizability criterion given in Theorem 3.3, and let \( S \) be the Suleimanova realizability criterion given by Theorem 3.2. It can be verified that for \( \delta = \epsilon = 1 \) we have

\[
\lambda' \in N_{C, \delta} \quad \text{and} \quad \lambda \in M_{S, \epsilon}.
\]

Applying Corollary 5.1 we conclude that \( \lambda \) is realizable.

Now we will construct a nonnegative matrix with spectrum \( \lambda' \). We will do this by simply following the steps in the proof of Lemma 4.5. First, following Theorem 3.3 we construct the matrix

\[
A_{\delta} = \begin{bmatrix}
0 & 1 & 1 & 2 & 2 & 3 & 3 \\
2 & 0 & 0 & 2 & 2 & 3 & 3 \\
0 & 2 & 0 & 2 & 2 & 3 & 3 \\
4 & 1 & 1 & 0 & 0 & 3 & 3 \\
0 & 1 & 1 & 4 & 0 & 3 & 3 \\
4 & 1 & 1 & 2 & 2 & 0 & 2 \\
2 & 1 & 1 & 2 & 2 & 4 & 0
\end{bmatrix} \in \mathcal{C}_{12}
\]

with spectrum

\[
\lambda' = \{12; -1 + i, -1 - i, -2 + 2i, -2 - 2i, -3 + i, -3 - i\}.
\]

Then \( A = A_{\delta} - ee^T \) is a matrix with constant row sums equal to 11 and spectrum \( \lambda' \).

Now, since
\[ B = \begin{pmatrix} 0 & 12 \\ 12 & 0 \end{pmatrix} \]

has spectrum \{12, -12\} we have the matrix
\[ M_\delta = \begin{pmatrix} A & e e^T \\ 0 & B \end{pmatrix} \]

with constant row sums equal to 12 and spectrum \( \Gamma \cup \{12, -12\} \). Thus, \( M = M_\delta + e e^T \) is nonnegative with spectrum \( \Lambda \) and constant row sums 13.

Following Theorem 4.1 we may also use the partition \( \Gamma_\delta \cup \{12, -12\} \) to directly conclude that \( \Lambda \) is realizable. However, as in the previous real example, Wuwen’s result does not provide a method to construct a specific nonnegative matrix with spectrum \( \Lambda \).

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References

[18] R.B. Kellogg, Matrices similar to a positive or essentially positive matrix, Linear Algebra Appl. 4 (1971) 191–204.