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Frequent Oscillatory Criteria for Partial Difference Equations with Several Delays

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Abstract—This paper is concerned with partial difference equations with several delays of the form

$$aA_{m+1,n} + bA_{m,n+1} - dA_{m,n} + \sum_{i=1}^{r} p_i(m,n)\, A_{m-\sigma_i,n-\tau_i} = 0, \qquad m,n = 0,1,2,\dots,$$

where a, b, and d are three positive real constants, σ_i, τ_i, and r are positive integers, and $\{p_i(m,n)\}$ are real double sequences, $i = 1,2,\dots,r$. Some new frequent oscillation criteria for this equation are derived. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Recently, there are many papers that have been devoted to the development of qualitative theory of difference equations [1–6]. In this paper, we shall consider the difference equation of the form

$$aA_{m+1,n} + bA_{m,n+1} - dA_{m,n} + \sum_{i=1}^{r} p_i(m,n)\, A_{m-\sigma_i,n-\tau_i} = 0, \qquad m,n = 0,1,2,\dots, \tag{1}$$

where a, b, and d are positive real constants, σ_i, τ_i, and r are positive integers, and $\{p_i(m,n)\}_{m,n=0}^{\infty}$ are real double sequences, $i = 1,2,\dots,r$.

By a solution of (1), we mean a nontrivial double sequence, $\{A_{m,n}\}$, which is defined for $m \ge -u$ and $n \ge -v$, and satisfies (1) for $m \ge 0$ and $n \ge 0$, where

$$u = \max\{\sigma_1,\sigma_2,\dots,\sigma_r\} \qquad \text{and} \qquad v = \max\{\tau_1,\tau_2,\dots,\tau_r\}.$$

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The usual concept of oscillation of a sequence $\{A_{m,n}\}$ is the following: the sequence $\{A_{m,n}\}$ is said to be eventually positive (or negative), if $A_{m,n} > 0$ (or $A_{m,n} < 0$), for all large m and n. It is said to be oscillatory, if it is neither eventually positive nor eventually negative.

If $r = 1$ and $a = b = d = 1$, then, equation (1) leads to

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n}A_{m-\sigma,n-\tau} = 0, \qquad m,n = 0,1,2,\dots, \tag{2}$$

which has received much attention in literature. In particular, Zhang, *et al.* [1] proved that every solution of equation (2) oscillates, if $p_{m,n} \geq 0$ eventually and

$$\liminf_{m,n\to\infty} p_{m,n} > \frac{(\sigma+\tau)^{\sigma+\tau}}{(\sigma+\tau+1)^{\sigma+\tau+1}}. \tag{3}$$

Since the above usual concept of oscillation does not catch all the fine details of an oscillatory sequence, a strengthened oscillation called frequent oscillation has been posed by Tian *et al.* in [2], by introducing the concept of frequency measure. Since frequent oscillation implies usual oscillation, the obtained frequent oscillation criteria in [2,3] are also oscillation ones of (1). For example, Tian and Zhang [3] proved that every solution of equation (2) is frequently oscillatory (and, hence, oscillatory), if $p_{m,n} \geq 0$ eventually and

$$\liminf_{m,n\to\infty} p_{m,n} > \frac{1}{(\sigma+\tau+1)\sqrt{C^{2\tau}_{2\sigma+2\tau}}}, \qquad C^k_n = \frac{n!}{k!(n-k)!}, \tag{4}$$

where $n! = n(n-1)\dots 2\cdot 1$, for any integer $n > 0$. Obviously, (4) improves (3).

In this paper, we shall be interested in establishing some new frequent oscillation criteria of all solutions for equation (1). For the sake of completeness, the definitions of frequency measure will be briefly sketched as follows.

Let $Z = \{\dots,-1,0,1,\dots\}$, $N_k = \{k,k+1,k+2,\dots\}$, for any $k \in Z$, and

$$Z^2 = \{(m,n) \mid m,n \in Z\} \qquad \text{and} \qquad N_k^2 = \{(m,n) \mid m,n \in N_k\}.$$

An element of Z^2 is called a lattice point. The union, intersection, and difference of two sets A and B of lattice points will be denoted by $A+B$ (or $A\cup B$), $A\cdot B$ (or $A\cap B$) and $A-B$ (or $A\backslash B$), respectively. Let Ω be a set of lattice points. The size of Ω is denoted by $|\Omega|$, i.e., $|\Omega|$ denotes the number of all elements in the set Ω. Given integers m and n, the translation operators X^m and Y^n are defined by

$$X^m\Omega = \{(i+m,j) \in Z^2 \mid (i,j) \in \Omega\} \qquad \text{and} \qquad Y^n\Omega = \{(i,j+n) \in Z^2 \mid (i,j) \in \Omega\},$$

respectively, and $\Omega^{(m,n)} = \{(i,j) | (i,j) \in \Omega,\ i \leq m,\ j \leq n\}$. Let α, β and θ, δ be integers, such that $\alpha \leq \beta$ and $\theta \leq \delta$. The union $\sum_{i=\alpha}^{\beta}\sum_{j=\theta}^{\delta} X^iY^j\Omega$ is called a derived set of Ω. Hence, [2]

$$(i,j) \in Z^2 \setminus \sum_{i=\alpha}^{\beta}\sum_{j=\theta}^{\delta} X^iY^j(\Omega) \Longleftrightarrow (i-k,\ j-l) \in Z^2 \setminus \Omega, \tag{5}$$

for $\alpha \leq k \leq \beta$ and $\theta \leq l \leq \delta$.

DEFINITION 1.1. *Let Ω be a set of integers. If $\limsup_{m,n\to\infty} |\Omega^{(m,n)}|/mn$ exists, then, the limit, denoted by $\mu^*(\Omega)$, will be called the upper frequency measure of Ω. If $\liminf_{m,n\to\infty} |\Omega^{(m,n)}|/mn$ exists, then, the limit, denoted by $\mu_*(\Omega)$, will be called the lower frequency measure of Ω. If $\mu^*(\Omega) = \mu_*(\Omega)$, then, the common limit, denoted by $\mu(\Omega)$, will be called the frequency measure of Ω.*

DEFINITION 1.2. *Let $A = \{A_{m,n} | m \geq -u,\ n \geq -v\}$ be real double sequence and $\lambda \in [0,1]$ be a constant. If $\mu^*(A \leq 0) \leq \lambda$, then, A is said to be frequently positive of upper degree λ, and if $\mu^*(A \geq 0) \leq \lambda$, then, A is said to be frequently negative of upper degree λ. The sequence A is said to be frequently oscillatory of upper degree λ, if it is neither frequently positive nor frequently negative of the same upper degree λ. The concept of frequently positive of lower degree, etc., is similarly defined by means of μ_*. If a sequence A is frequently oscillatory of upper degree 0, it is said to be frequently oscillatory.*

Obviously, if a double sequence is eventually positive (or eventually negative), then, it is frequently positive (or frequently negative). Thus, if the sequence is frequently oscillatory, then, it is oscillatory.

We will adopt the usual notation for level sets of a double sequence, that is, let $A : \Omega \to R$ be a double sequence, then, the set $\{(m,n) \in \Omega | A_{m,n} \leq c\}$ will be denoted by $(A \leq c)$ or $(A_{m,n} \leq c)$, where c is a real constant. The notations $(A \geq c)$, $(A_{m,n} < c)$, etc., will have similar meanings.

We first recall three results from [2] needed in the following.

LEMMA 1.1. *Let Ω and Γ be subsets of N_k^2, where $k \in Z$. Then,*

$$\mu^*(\Omega + \Gamma) \leq \mu^*(\Omega) + \mu^*(\Gamma).$$

Furthermore, if Ω and Γ are disjoint, then,

$$\mu_*(\Omega) + \mu_*(\Gamma) \leq \mu_*(\Omega + \Gamma) \leq \mu_*(\Omega) + \mu^*(\Gamma) \leq \mu^*(\Omega + \Gamma) \leq \mu^*(\Omega) + \mu^*(\Gamma),$$

so that, $\mu_*(\Omega) + \mu^*(N_k^2 - \Omega) = 1$.

LEMMA 1.2. *Let Ω and Γ be subsets of N_k^2. If $\mu_*(\Omega) + \mu^*(\Gamma) > 1$, then, $\Omega \cap \Gamma$ is an infinite set.*

LEMMA 1.3. *Let $\Omega \subset N_k^2$, α, β, θ and δ be integers, such that $\alpha \leq \beta$ and $\theta \leq \delta$. Then,*

$$\mu^*\left(\sum_{i=\alpha}^{\beta}\sum_{j=\theta}^{\delta} X^i Y^j(\Omega)\right) \leq (\beta - \alpha + 1)(\delta - \theta + 1)\mu^*(\Omega),$$

$$\mu_*\left(\sum_{i=\alpha}^{\beta}\sum_{j=\theta}^{\delta} X^i Y^j(\Omega)\right) \leq (\beta - \alpha + 1)(\delta - \theta + 1)\mu_*(\Omega).$$

2. PREPARATORY LEMMAS

To obtain our main results, we need the following technical lemmas.

LEMMA 2.1. *Let k, m, and n be three positive integers, and let $\{A_{i,j}\}$ be a sequence, such that $A_{i,j} > 0$ for $i \in \{m, m+1, \dots, m+k\}$ and $j \in \{n, n+1, \dots, n+k\}$. If $dA_{i,j} \geq aA_{i+1,j} + bA_{i,j+1}$, for $i \in \{m, m+1, \dots, m+k\}$ and $j \in \{n, n+1, \dots, n+k\}$, then,*

$$d^k A_{m,n} \geq \sum_{i=0}^{k} a^{k-i} b^i C_k^i A_{m+k-i,n+i}. \tag{6}$$

PROOF. Obviously, (6) holds for $k = 1$. Assume that (6) holds for an integer $s \in \{1, 2, \dots, k-1\}$.

Then, in view of the following inequality

$$\begin{aligned}
&\sum_{i=0}^{s} a^{s-i}b^i C_s^i \left(aA_{m+s+1-i,n+i} + bA_{m+s-i,n+i+1}\right) \\
&\quad \geq a^{s+1}A_{m+s+1,n} + \sum_{i=1}^{s} a^{s+1-i}b^i C_s^i A_{m+s+1-i,n+i} \\
&\qquad + \sum_{i=0}^{s-1} a^{s-i}b^{1+i} C_s^i A_{m+s-i,n+i+1} + b^{s+1}A_{m,n+s+1} \\
&\quad \geq a^{s+1}A_{m+s+1,n} + \sum_{i=1}^{s} a^{s+1-i}b^i \left(C_s^i + C_s^{i-1}\right) A_{m+s+1-i,n+i} + b^{s+1}A_{m,n+s+1} \\
&\quad = \sum_{i=0}^{s+1} a^{s+1-i}b^i C_{s+1}^i A_{m+s+1-i,n+i},
\end{aligned} \tag{7}$$

(6) holds for $s+1$. By induction, (6) holds, and this completes the proof. ∎

LEMMA 2.2. *Let k, m, and n be three positive integers, such that $m \geq 2u$ and $n \geq 2v$. Assume that equation (1) has a solution $\{A_{i,j}\}$, such that $A_{i,j} > 0$, for $i \in \{m-2u, \dots, m+k\}$ and $j \in \{n-2v, \dots, n+k\}$, $p_s(i,j) \geq q_s \geq 0$, for $i \in \{m-u, \dots, m+k\}$ and $j \in \{n-v, \dots, n+k\}$, where q_s are real constants, $s = 1, 2, \dots, r$. Then,*

$$\begin{aligned}
d^{k+1}A_{m,n} \geq \sum_{i=0}^{k+1} a^{k+1-i}b^i C_{k+1}^i A_{m+k+1-i,n+i} + (k+1)\, q \sum_{i=0}^{k} a^{k-i}b^i C_k^i A_{m+k-i-\alpha,n+i-\beta} \\
+ q^2 \sum_{i=1}^{k} i d^{k-i} \sum_{j=0}^{i-1} a^{i-1-j} b^j C_{i-1}^j A_{m+i-1-j-2\alpha,n+j-2\beta},
\end{aligned} \tag{8}$$

where $\alpha = \min\{\sigma_1, \sigma_2, \dots, \sigma_r\}$ and $\beta = \min\{\tau_1, \tau_2, \dots, \tau_r\}$, and

$$q = \sum_{s=1}^{r} \frac{q_s a^{\sigma_s-\alpha} b^{\tau_s-\beta} C_{\sigma_s-\alpha+\tau_s-\beta}^{\tau_s-\beta}}{d^{\sigma_s-\alpha+\tau_s-\beta}}. \tag{9}$$

PROOF. In view of (1), for any $i \in \{m-u, \dots, m+k\}$ and $j \in \{n-v, \dots, n+k\}$, we have

$$dA_{i,j} = aA_{i+1,j} + bA_{i,j+1} + \sum_{s=1}^{r} p_s(i,j)\, A_{i-\sigma_s,j-\tau_s} \geq aA_{i+1,j} + bA_{i,j+1}.$$

Then, from Lemma 2.1, for any $i \in \{m, \dots, m+k\}$ and $j \in \{n, \dots, n+k\}$, we get

$$d^{\sigma_s+\tau_s-\alpha-\beta} A_{i-\sigma_s,j-\tau_s} \geq a^{\sigma_s-\alpha} b^{\tau_s-\beta} C_{\sigma_s+\tau_s-\alpha-\beta}^{\tau_s-\beta} A_{i-\alpha,j-\beta}, \qquad s = 1, 2, \dots, r,$$

and so that,

$$\begin{aligned}
dA_{i,j} &\geq aA_{i+1,j} + bA_{i,j+1} + \left(\sum_{s=1}^{r} p_s(i,j)\, C_{\sigma_s+\tau_s-\alpha-\beta}^{\tau_s-\beta} \frac{a^{\sigma_s-\alpha} b^{\tau_s-\beta}}{d^{\sigma_s+\tau_s-\alpha-\beta}} \right) A_{i-\alpha,j-\beta} \\
&= aA_{i+1,j} + bA_{i,j+1} + q_{i,j} A_{i-\alpha,j-\beta},
\end{aligned} \tag{10}$$

where

$$q_{i,j} = \sum_{s=1}^{r} \frac{p_s(i,j)\, a^{\sigma_s-\alpha} b^{\tau_s-\beta} C_{\sigma_s-\alpha+\tau_s-\beta}^{\tau_s-\beta}}{d^{\sigma_s+\tau_s-\alpha-\beta}}. \tag{11}$$

Obviously, $q_{i,j} \geq q$, for $i \in \{m-u, \dots, m+k\}$ and $j \in \{n-v, \dots, n+k\}$. Hence, from (10), we obtain

$$dA_{m,n} \geq aA_{m+1,n} + bA_{m,n+1} + q_{m,n}A_{m-\alpha,n-\beta},$$

and

$$\begin{aligned} dA_{m+1,n} &\geq aA_{m+2,n} + bA_{m+1,n+1} + q_{m+1,n}A_{m+1-\alpha,n-\beta}, \\ dA_{m,n+1} &\geq aA_{m+1,n+1} + bA_{m,n+2} + q_{m,n+1}A_{m-\alpha,n+1-\beta}, \\ dA_{m-\alpha,n-\beta} &\geq aA_{m+1-\alpha,n-\beta} + bA_{m-\alpha,n+1-\beta} + q_{m-\alpha,n-\beta}A_{m-2\alpha,n-2\beta}. \end{aligned}$$

Thus, from the above inequalities, we have

$$\begin{aligned} d^2A_{m,n} \geq\ & a^2A_{m+2,n} + 2abA_{m+1,n+1} + b^2A_{m,n+2} + a\left(q_{m,n} + q_{m+1,n}\right)A_{m+1-\alpha,n-\beta} \\ & + b\left(q_{m,n} + q_{m,n+1}\right)A_{m-\alpha,n+1-\beta} + \left(q_{m,n}q_{m-\alpha,n-\beta}\right)A_{m-2\alpha,n-2\beta}. \end{aligned} \tag{12}$$

Hence,

$$\begin{aligned} d^2A_{m,n} \geq \sum_{i=0}^{2} a^{2-i}b^iC_2^iA_{m+2-i,n+i} + 2q\sum_{i=0}^{1} a^{1-i}b^iC_1^iA_{m+1-i-\alpha,n+i-\beta} \\ + q^2\sum_{i=1}^{1} id^{1-i}\sum_{j=0}^{i-1} a^{i-1-j}b^jC_{i-1}^jA_{m+i-1-j-2\alpha,n+j-2\beta}. \end{aligned}$$

Assume that (8) holds for a positive integer $s \in \{1, 2, \dots, k\}$. Then, from (7), (10), and the assumption,

$$\begin{aligned} & d^{k+1}A_{m,n} \\ & \geq \sum_{i=0}^{k} a^{k-i}b^iC_k^i\left(aA_{m+k+1-i,n+i} + bA_{m+k-i,n+1+i} + qA_{m+k-i-\alpha,n+i-\beta}\right) \\ & \quad + kq\sum_{i=0}^{k-1} a^{k-1-i}b^iC_{k-1}^i\left(aA_{m+k-i-\alpha,n+i-\beta} + bA_{m+k-1-i-\alpha,n+1+i-\beta} + qA_{m+k-1-i-2\alpha,n+i-2\beta}\right) \\ & \quad + q^2\sum_{i=1}^{k-1} id^{k-i}\sum_{j=0}^{i-1} a^{i-1-j}b^jC_{i-1}^jA_{m+i-1-j-2\alpha,n+j-2\beta} \\ & \geq \sum_{i=0}^{k+1} a^{k+1-i}b^iC_{k+1}^iA_{m+k+1-i,n+i} + q\sum_{i=0}^{k} a^{k-i}b^iC_k^iA_{m+k-i-\alpha,n+i-\beta} \\ & \quad + kq\sum_{i=0}^{k} a^{k-i}b^iC_k^iA_{m+k-i-\alpha,n+i-\beta} + kq^2\sum_{i=0}^{k-1} a^{k-1-i}b^iC_{k-1}^iA_{m+k-1-i-2\alpha,n+i-2\beta} \\ & \quad + q^2\sum_{i=1}^{k-1} id^{k-i}\sum_{j=0}^{i-1} a^{i-1-j}b^jC_{i-1}^jA_{m+i-1-j-2\alpha,n+j-2\beta} \\ & = \sum_{i=0}^{k+1} a^{k+1-i}b^iC_{k+1}^iA_{m+k+1-i,n+i} + (k+1)q\sum_{i=0}^{k} a^{k-i}b^iC_k^iA_{m+k-i-\alpha,n+i-\beta} \\ & \quad + q^2\sum_{i=1}^{k} id^{k-i}\sum_{j=0}^{i-1} a^{i-1-j}b^jC_{i-1}^jA_{m+i-1-j-2\alpha,n+j-2\beta}. \end{aligned}$$

Hence, (8) holds, and this completes the proof. ∎

From Lemma 2.2, we can obtain the following corollaries.

COROLLARY 2.1. *Assume that $\alpha > 0$ and $\beta > 0$, and for integers $m \geq 3u$ and $n \geq 3v$, equation (1) has a solution $\{A_{i,j}\}$, such that $A_{i,j} > 0$, for $i \in \{m-3u,\dots,m+v\}$ and $j \in \{n-3v,\dots,n+u\}$, $p_s(i,j) \geq q_s \geq 0$ for $i \in \{m-2u,\dots,m+v\}$ and $j \in \{n-2v,\dots,n+u\}$, where q is defined in (9). Then,*

$$\left(d^{\alpha+\beta} - q\,(\alpha+\beta)\,C^{\beta}_{\alpha+\beta}\left(\frac{a^{\alpha}b^{\beta}}{d}\right)\right)A_{m-\alpha,n-\beta} \geq a^{\alpha}b^{\beta}C^{\beta}_{\alpha+\beta}A_{m,n}. \tag{13}$$

PROOF. From (1), for $i \in \{m-2u,\dots,m+v\}$ and $j \in \{n-2v,\dots,n+u\}$, we have

$$dA_{i-1,j} \geq aA_{i,j} \qquad \text{and} \qquad dA_{i,j-1} \geq bA_{i,j}.$$

In view of Lemma 2.2 and the inequality, $C^i_k + C^{i-1}_k = C^i_{k+1}$, we have

$$\begin{aligned} d^{\alpha+\beta}A_{m-\alpha,n-\beta} &\geq \sum_{i=0}^{\alpha+\beta} a^{\alpha+\beta-i}b^iC^i_{\alpha+\beta}A_{m+\beta-i,n-\beta+i} \\ &\quad + q(\alpha+\beta)\sum_{i=0}^{\alpha+\beta-1} a^{\alpha+\beta-1-i}b^iC^i_{\alpha+\beta-1}A_{m+\beta-1-i-\alpha,n+i-2\beta} \\ &\geq a^{\alpha}b^{\beta}C^{\beta}_{\alpha+\beta}A_{m,n} + (\alpha+\beta)\,qa^{\alpha-1}b^{\beta}C^{\beta}_{\alpha+\beta-1}A_{m-\alpha-1,n-\beta} \\ &\quad + (\alpha+\beta)\,qa^{\alpha}b^{\beta-1}C^{\beta-1}_{\alpha+\beta-1}A_{m-\alpha,n-\beta-1} \\ &\geq a^{\alpha}b^{\beta}C^{\beta}_{\alpha+\beta}A_{m,n} + (\alpha+\beta)\,q\left(\frac{a^{\alpha}b^{\beta}}{d}\right)C^{\beta}_{\alpha+\beta}A_{m-\alpha,n-\beta}. \end{aligned}$$

Hence, (13) holds, and this completes the proof. ∎

COROLLARY 2.2. *Assume that $\alpha > 0$ and $\beta > 0$, and for integers $m \geq 2u$ and $n \geq 2v$, equation (1) has a solution $\{A_{i,j}\}$, such that $A_{i,j} > 0$, for $i \in \{m-2u,\dots,m+u+v+1\}$ and $j \in \{n-2v,\dots,n+u+v+1\}$, and $p_s(i,j) \geq q_s \geq 0$, for $i \in \{m-u,\dots,m+u+v\}$ and $j \in \{n-v,\dots,n+u+v\}$. Let q be defined in (9). Then,*

$$\left(d^{\alpha+\beta} - qd^{-1}a^{\alpha}b^{\beta}(1+\beta)C^{\beta}_{\alpha+\beta}\right)A_{m+1,n} \geq (\alpha+\beta)\,qa^{\alpha-1}b^{\beta}C^{\beta}_{\alpha+\beta-1}A_{m,n}, \tag{14}$$

and

$$\left(d^{\alpha+\beta} - qd^{-1}a^{\alpha}b^{\beta}(1+\alpha)C^{\beta}_{\alpha+\beta}\right)A_{m,n+1} \geq (\alpha+\beta)\,qa^{\alpha}b^{\beta-1}C^{\beta-1}_{\alpha+\beta-1}A_{m,n}. \tag{15}$$

PROOF. From (1), we have $dA_{m+1,n-1} \geq bA_{m+1,n}$. From (10), for any $i \in \{m,\dots,m+u+v\}$ and $j \in \{n,\dots,n+u+v\}$, we obtain

$$dA_{i,j} \geq qA_{i-\alpha,j-\beta}.$$

In view of Lemma 2.2, we get

$$\begin{aligned} d^{\alpha+\beta}A_{m+1,n} &\geq \sum_{i=0}^{\alpha+\beta} a^{\alpha+\beta-i}b^iC^i_{\alpha+\beta}A_{m+1+\alpha+\beta-i,n+i} \\ &\quad + q\,(\alpha+\beta)\sum_{i=0}^{\alpha+\beta-1} a^{\alpha+\beta-1-i}b^iC^i_{\alpha+\beta-1}A_{m+\alpha+\beta-i-\alpha,n+i-\beta} \\ &\geq a^{\alpha}b^{\beta}C^{\beta}_{\alpha+\beta}A_{m+1+\alpha,n+\beta} + (\alpha+\beta)\,qa^{\alpha-1}b^{\beta}C^{\beta}_{\alpha+\beta-1}A_{m,n} \\ &\quad + (\alpha+\beta)\,qa^{\alpha}b^{\beta-1}C^{\beta-1}_{\alpha+\beta-1}A_{m+1,n-1} \\ &\geq qd^{-1}a^{\alpha}b^{\beta}C^{\beta}_{\alpha+\beta}A_{m+1,n} + (\alpha+\beta)\,qa^{\alpha-1}b^{\beta}C^{\beta}_{\alpha+\beta-1}A_{m,n} \\ &\quad + qd^{-1}\,(\alpha+\beta)\,a^{\alpha}b^{\beta}C^{\beta-1}_{\alpha+\beta-1}A_{m+1,n} \\ &= qd^{-1}a^{\alpha}b^{\beta}(1+\beta)C^{\beta}_{\alpha+\beta}A_{m+1,n} + (\alpha+\beta)\,qa^{\alpha-1}b^{\beta}C^{\beta}_{\alpha+\beta-1}A_{m,n}. \end{aligned}$$

Hence, (14) holds. Similarly, (15) holds, and this completes the proof. ∎

COROLLARY 2.3. *Assume that $\alpha > 0$ and $\beta > 0$, and for integers $m \geq 2u + v$ and $n \geq 2v + u$, equation (1) has a solution $\{A_{i,j}\}$, such that $A_{i,j} > 0$, for $i \in \{m - 2u - v, \dots, m + 2u + v + 1\}$ and $j \in \{n - 2v - u, \dots, n + u + 2v + 1\}$, and $p_s(i,j) \geq q_s \geq 0$, for $i \in \{m - u - v, \dots, m + 2u + v\}$ and $j \in \{n - v - u, \dots, n + u + 2v\}$. Then,*

$$d^{\alpha+\beta+1} A_{m-h,n+h} \geq (\alpha + \beta + 1)\, q a^{\alpha+h} b^{\beta-h} C_{\alpha+\beta}^{\beta-h} A_{m,n}, \tag{16}$$

where q is defined in (9) and $-\alpha \leq h \leq \beta$.

PROOF. For any $-\alpha \leq h \leq \beta$, from Lemma 2.2, we get

$$\begin{aligned} d^{\alpha+\beta+1} A_{m-h,n+h} &\geq (\alpha + \beta + 1)\, q \sum_{i=0}^{\alpha+\beta} a^{\alpha+\beta-i} b^i C_{\alpha+\beta}^i A_{m-h+\alpha+\beta-i-\alpha, n+h+i-\beta} \\ &\geq (\alpha + \beta + 1)\, q a^{\alpha+h} b^{\beta-h} C_{\alpha+\beta}^{\beta-h} A_{m,n} \end{aligned}$$

Hence, (16) holds, and this completes the proof. ∎

3. MAIN RESULTS

THEOREM 3.1. *Assume that $\alpha > 0$ and $\beta > 0$, and there exist nonnegative constants $q_i \geq 0$, $\theta_i \geq 0$, and $\omega \in [0,1]$, such that $\mu^* \{p_i(m,n) < q_i\} = \theta_i \geq 0$, $i = 1, 2, \dots, r$, and*

$$\begin{aligned} d^{\alpha+\beta+1} &\leq q^2 \{ D d^{-1} a^{\alpha+1} b^\beta C_{\alpha+\beta+1}^{\beta} + \overline{D} d^{-1} a^\alpha b^{\beta+1} C_{\alpha+\beta+1}^{\beta+1} \\ &+ (\alpha + \beta + 1)^2 E a^{2\alpha} b^{2\beta} C_{2\alpha+2\beta}^{2\beta} + d^{-1} (\alpha + \beta) B a^\alpha b^\beta C_{\alpha+\beta}^{\beta} \}, \end{aligned} \tag{17}$$

and

$$(4u + 2v + 1)(2u + 4v + 1)(\theta_1 + \dots + \theta_r) + (5u + 2v + 2)(2u + 5v + 2)\, \omega < 1,$$

where q is defined in (9) and

$$B = \frac{a^\alpha b^\beta C_{\alpha+\beta}^{\beta}}{\left(d^{\alpha+\beta} - q(\alpha+\beta)(a^\alpha b^\beta / d) C_{\alpha+\beta}^{\beta} \right)}, \qquad E = 1/d^{\alpha+\beta+1},$$

$$D = \frac{(\alpha+\beta) a^{\alpha-1} b^\beta C_{\alpha+\beta-1}^{\beta}}{\left\{ d^{\alpha+\beta} - q d^{-1} a^\alpha b^\beta (1+\beta) C_{\alpha+\beta}^{\beta} \right\}},$$

$$\bar{D} = \frac{(\alpha+\beta) a^\alpha b^{\beta-1} C_{\alpha+\beta-1}^{\beta-1}}{\left\{ d^{\alpha+\beta} - q d^{-1} a^\alpha b^\beta (1+\alpha) C_{\alpha+\beta}^{\beta} \right\}}.$$

Then, every solution of equation (1) is frequently oscillatory of lower-degree ω.

PROOF. Suppose to the contrary, let $A = \{A_{m,n}\}$ be a frequently positive solution of equation (1), such that $\mu_* \{A \leq 0\} \leq \omega$. In view of Lemma 1.1 and 1.3, we have

$$\begin{aligned} &\mu_* \left\{ N_0^2 \setminus \sum_{s=1}^{r} \sum_{i=-2u-v}^{2u+v} \sum_{j=-u-2v}^{u+2v} X^i Y^j (p_s(m,n) < q_s) \right\} \\ &\quad + \mu^* \left\{ N_0^2 \setminus \sum_{i=-2u-v-1}^{3u+v} \sum_{j=-u-2v-1}^{u+3v} X^i Y^j (A \leq 0) \right\} \\ &= 2 - \mu^* \left\{ \sum_{s=1}^{r} \sum_{i=-2u-v}^{2u+v} \sum_{j=-u-2v}^{u+2v} X^i Y^j (p_s(m,n) < q_s) \right\} \\ &\quad - \mu_* \left\{ \sum_{i=-2u-v-1}^{3u+v} \sum_{j=-u-2v-1}^{u+3v} X^i Y^j (A \leq 0) \right\} \\ &\geq 2 - (4u + 2v + 1)(2u + 4v + 1)(\theta_1 + \dots + \theta_r) \\ &\quad - (5u + 2v + 2)(2u + 5v + 2)\, \omega > 1. \end{aligned}$$

Hence, by Lemma 1.2, the intersection

$$\left\{N_0^2 \setminus \sum_{s=1}^{r} \sum_{i=-2u-v}^{2u+v} \sum_{j=-u-2v}^{u+2v} X^i Y^j \left(p_s(m,n) < q_s\right)\right\} \cap \left\{N_0^2 \setminus \sum_{i=-2u-v-1}^{3u+v} \sum_{j=-u-2v-1}^{u+3v} X^i Y^j (A \le 0)\right\}$$

is an infinite subset of N_0^2, which together with (5) implies that there exists a lattice point (m,n), such that $A_{i,j} > 0$, for $i \in \{m-3u-v, \ldots, m+2u+v+1\}$ and $j \in \{n-3v-u, \ldots, n+u+2v+1\}$, and $p_s(i,j) \ge q_s$, for $i \in \{m-2u-v, \ldots, m+2u+v\}$ and $j \in \{n-2v-u, \ldots, n+u+2v\}$, $s = 1, 2, \ldots, r$.

If $\alpha \ge \beta$, then, from (10) and Corollary 2.1–2.3, we get

$$A_{m-h,n+h} \ge E(\alpha+\beta+1) q a^{\alpha+h} b^{\beta-h} C_{\alpha+\beta}^{\beta-h} A_{m,n}, \qquad \text{for } -\alpha \le h \le \beta,$$
$$dA_{m+\alpha+1,n+\beta} \ge qA_{m+1,n}, \qquad dA_{m+\alpha,n+\beta+1} \ge qA_{m,n+1},$$
$$A_{m+1,n} \ge DqA_{m,n}, \qquad dA_{m,n+1} \ge \bar{D}qA_{m,n},$$
$$A_{m-\alpha-1,n-\beta} \ge BA_{m-1,n}, \qquad dA_{m-1,n} \ge aA_{m,n}, \qquad dA_{m,n-1} \ge bA_{m,n}.$$

Hence, from Lemma 2.2,

$$\begin{aligned}
d^{\alpha+\beta+1} A_{m,n} \ge{}& \sum_{i=0}^{\alpha+\beta+1} a^{\alpha+\beta+1-i} b^i C_{\alpha+\beta+1}^i A_{m+\alpha+\beta+1-i,n+i} \\
&+ (\alpha+\beta+1)\, q \sum_{i=0}^{\alpha+\beta} a^{\alpha+\beta-i} b^i C_{\alpha+\beta}^i A_{m+\alpha+\beta-i-\alpha,n+i-\beta} \\
&+ q^2 \sum_{i=1}^{\alpha+\beta} i d^{\alpha+\beta-i} \sum_{j=0}^{i-1} a^{i-1-j} b^j C_{i-1}^j A_{m+i-1-j-2\alpha,n+j-2\beta} \\
>{}& a^{\alpha+1} b^\beta C_{\alpha+\beta+1}^\beta A_{m+\alpha+1,n+\beta} + a^\alpha b^{\beta+1} C_{\alpha+\beta+1}^{\beta+1} A_{m+\alpha,n+\beta+1} \\
&+ q\,(\alpha+\beta+1) \sum_{i=0}^{2\beta} a^{\alpha+\beta-i} b^i C_{\alpha+\beta}^i A_{m+\beta-i,n+i-\beta} \\
&+ q^2\,(\alpha+\beta)\, a^{\alpha-1} b^\beta C_{\alpha+\beta-1}^\beta A_{m-\alpha-1,n-\beta} \\
&+ q^2\,(\alpha+\beta)\, a^\alpha b^{\beta-1} C_{\alpha+\beta-1}^{\beta-1} A_{m-\alpha,n-\beta-1} \\
\ge{}& \left(q^2 D d^{-1} a^{\alpha+1} b^\beta C_{\alpha+\beta+1}^\beta + q^2 \bar{D} d^{-1} a^\alpha b^{\beta+1} C_{\alpha+\beta+1}^{\beta+1}\right) A_{m,n} \\
&+ (\alpha+\beta+1)^2 q^2 E a^{2\alpha} b^{2\beta} \left(\sum_{i=0}^{2\beta} C_{\alpha+\beta}^i C_{\alpha+\beta}^{2\beta-i}\right) A_{m,n} \\
&+ q^2 d^{-1} B\,(\alpha+\beta)\, a^\alpha b^\beta C_{\alpha+\beta}^\beta A_{m,n}.
\end{aligned}$$

In view of the equality, $\sum_{i=0}^{2\beta} C_{\alpha+\beta}^i C_{\alpha+\beta}^{2\beta-i} = C_{2\alpha+2\beta}^{2\beta}$, we have

$$\begin{aligned}
d^{\alpha+\beta+1} > q^2 \Big\{& D d^{-1} a^{\alpha+1} b^\beta C_{\alpha+\beta+1}^\beta + \bar{D} d^{-1} a^\alpha b^{\beta+1} C_{\alpha+\beta+1}^{\beta+1} \\
&+ (\alpha+\beta+1)^2 E a^{2\alpha} b^{2\beta} C_{2\alpha+2\beta}^{2\beta} + d^{-1} B(\alpha+\beta) a^\alpha b^\beta C_{\alpha+\beta}^\beta \Big\},
\end{aligned}$$

which is contrary to (17).

If $\alpha < \beta$, then, similar to the above proof,

$$d^{\alpha+\beta+1}A_{m,n} > \left(q^2Dd^{-1}a^{\alpha+1}b^{\beta}C^{\beta}_{\alpha+\beta+1} + q^2\bar{D}d^{-1}a^{\alpha}b^{\beta+1}C^{\beta+1}_{\alpha+\beta+1}\right)A_{m,n}$$
$$+ (\alpha+\beta+1)^2q^2Ea^{2\alpha}b^{2\beta}\left(\sum_{i=\beta-\alpha}^{\alpha+\beta} C^i_{\alpha+\beta}C^{2\beta-i}_{\alpha+\beta}\right)A_{m,n}$$
$$+ q^2d^{-1}B(\alpha+\beta)a^{\alpha}b^{\beta}C^{\beta}_{\alpha+\beta}A_{m,n}.$$

In view of the equality,

$$\sum_{i=\beta-\alpha}^{\alpha+\beta} C^i_{\alpha+\beta}C^{2\beta-i}_{\alpha+\beta} = \sum_{i=\beta-\alpha}^{\alpha+\beta} C^{\alpha+\beta-i}_{\alpha+\beta}C^{2\alpha-(\alpha+\beta-i)}_{\alpha+\beta} = \sum_{i=0}^{2\alpha} C^i_{\alpha+\beta}C^{2\alpha-i}_{\alpha+\beta} = C^{2\alpha}_{2\alpha+2\beta} = C^{2\beta}_{2\alpha+2\beta},$$

similar to the above proof, we also obtain a contradiction to (17), and this completes the proof. ∎

COROLLARY 3.1. *Assume that $\alpha > 0$ and $\beta > 0$, and there exist positive constants $q_i \ge 0$, such that $\mu\{p_i(m,n) < q_i\} = 0$, $i = 1, 2, \dots, r$, and*

$$q \ge \frac{d^{\alpha+\beta+1}}{(\alpha+\beta+1)\,a^{\alpha}b^{\beta}\sqrt{C^{2\beta}_{2\alpha+2\beta}}}. \tag{18}$$

Then, every solution of equation (1) is frequently oscillatory of lower-degree ω (and hence, oscillatory), where $\omega \in [0, 1/(5u+2v+2)(2u+5v+2))$.

In fact, in view of $d^{\alpha+\beta+1} = Ed^{2(\alpha+\beta+1)}$, from (18), we have

$$d^{\alpha+\beta+1} \le q^2\left(\alpha+\beta+1\right)^2 Ea^{2\alpha}b^{2\beta}C^{2\beta}_{2\alpha+2\beta}.$$

Hence, (17) holds. From Theorem 3.1, Corollary 3.1 holds.

Similarly, from Theorem 3.1, it is easy to obtain the following corollaries.

COROLLARY 3.2. *Assume that $\sigma > 0$ and $\tau > 0$, and*

$$\mu\left\{p_{m,n} > \frac{1}{\sqrt{(\sigma+\tau)\left\{C^{\tau}_{\sigma+\tau-1}C^{\tau}_{\sigma+\tau+1} + C^{\tau-1}_{\sigma+\tau-1}C^{\tau+1}_{\sigma+\tau+1} + \left(C^{\tau}_{\sigma+\tau}\right)^2\right\} + (\sigma+\tau+1)^2 C^{2\tau}_{2\sigma+2\tau}}}\right\} = 1. \tag{19}$$

Then, every solution of equation (2) is frequently oscillatory of lower degree ω (and hence, oscillatory), where $\omega \in [0, 1/(5u+2v+2)(2u+5v+2))$.

COROLLARY 3.3. *Assume that $\sigma > 0$ and $\tau > 0$, and*

$$\liminf_{m,n\to\infty} p_{m,n} > \frac{1}{\sqrt{(\sigma+\tau)\left\{C^{\tau}_{\sigma+\tau-1}C^{\tau}_{\sigma+\tau+1} + C^{\tau-1}_{\sigma+\tau-1}C^{\tau+1}_{\sigma+\tau+1} + \left(C^{\tau}_{\sigma+\tau}\right)^2\right\} + (\sigma+\tau+1)^2 C^{2\tau}_{2\sigma+2\tau}}}. \tag{20}$$

Then, every solution of equation (2) is frequently oscillatory (and hence, oscillatory).

Obviously, (20) improves (3) and (4).

4. EXAMPLES

In this section, we give two examples to illustrate the above results.

EXAMPLE 4.1. Consider the partial difference equation with two delays of the form

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n}A_{m-1,n-2} + q_{m,n}A_{m-2,n-1} = 0, \tag{21}$$

where

$$p_{m,n} = -1 \quad \text{and} \quad q_{m,n} = -1, \qquad \text{for } (m,n) \in S = \{(i,j)\,|i = 2^s,\ j = 2^t,\ s,t = 0,1,2,\dots\},$$

and $p_{m,n} = 0.05$ and $q_{m,n} = 0.07$, for any $(m,n) \notin S$.

Let $a = b = d = 1$, $r = 2$, $\sigma_1 = 1$, $\tau_1 = 2$, $\sigma_2 = 2$, and $\tau_2 = 1$, then, $\alpha = 1$ and $\beta = 1$. It is obvious that $\mu\{p_{m,n} \geq 0.05 = q_1\} = 1$ and $\mu\{q_{m,n} \geq 0.07 = q_2\} = 1$, $E = 1$, and

$$q = \sum_{s=1}^{r} \frac{q_s a^{\sigma_s-\alpha} b^{\tau_s-\beta} C_{\sigma_s-\alpha+\tau_s-\beta}^{\tau_s-\beta}}{d^{\sigma_s-\alpha+\tau_s-\beta}} = q_1 + q_2 = 0.12,$$

$$B = \frac{a^\alpha b^\beta C_{\alpha+\beta}^{\beta}}{\left(d^{\alpha+\beta} - q(\alpha+\beta)\, d^{-1} a^\alpha b^\beta C_{\alpha+\beta}^{\beta}\right)} > 2,$$

$$D = \frac{(\alpha+\beta)\, a^{\alpha-1} b^\beta C_{\alpha+\beta-1}^{\beta}}{\left(d^{\alpha+\beta} - q d^{-1} a^\alpha b^\beta (1+\beta)\, C_{\alpha+\beta}^{\beta}\right)} > 2,$$

$$\bar{D} = \frac{(\alpha+\beta)\, a^{\alpha} b^{\beta-1} C_{\alpha+\beta-1}^{\beta-1}}{\left(d^{\alpha+\beta} - q d^{-1} a^\alpha b^\beta (1+\alpha)\, C_{\alpha+\beta}^{\beta}\right)} > 2,$$

Obviously,

$$\frac{1}{\sqrt{D \times C_3^1 + \bar{D} \times C_3^2 + 9 \times C_4^2 + 2 \times B \times C_2^1}} < \frac{1}{\sqrt{6+6+54+8}} = \frac{1}{\sqrt{74}} < q.$$

Hence, (17) holds. By Theorem 3.1, every solution of (21) is frequently oscillatory of lower-degree $\omega \in [0, 1/256)$ and hence, oscillatory.

EXAMPLE 4.2. Consider the partial difference equation of the form

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n} A_{m-1,n-2} = 0, \tag{22}$$

where $p_{m,n} = 1/16$, for any $m, n = 0, 1, 2, \dots$.

Let $\sigma = 1$ and $\tau = 2$. It is easy to see that $p_{m,n} = 1/16 = 0.0625$,

$$\frac{(\sigma+\tau)^{\sigma+\tau}}{(\sigma+\tau+1)^{\sigma+\tau+1}} = \frac{3^3}{4^4} \approx 0.1055 \qquad \text{and} \qquad \frac{1}{(\sigma+\tau+1)\sqrt{C_{2\sigma+2\tau}^{2\tau}}} = \frac{1}{4\sqrt{15}} \approx 0.0645,$$

and

$$\frac{1}{\sqrt{(\sigma+\tau)\left\{C_{\sigma+\tau-1}^{\tau} C_{\sigma+\tau+1}^{\tau} + C_{\sigma+\tau-1}^{\tau-1} C_{\sigma+\tau+1}^{\tau+1} + \left(C_{\sigma+\tau}^{\tau}\right)^2\right\} + (\sigma+\tau+1)^2 C_{2\sigma+2\tau}^{2\tau}}} = \frac{1}{\sqrt{309}} \approx 0.0569.$$

Hence, from Corollary 3.3, every solution of (22) is frequently oscillatory, and hence, oscillatory. But it is difficult to obtain the same conclusion from the corresponding results in [1,3].

5. SUMMARY

In this paper, we discuss the strengthen oscillation (frequent oscillation) of a class of partial difference equations and obtain some new oscillatory criteria for the equations which improve the existing ones in the literature. It is necessary for us to continue studying frequent oscillation of another partial difference equations for further research.

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