



Coding true arithmetic in the Medvedev degrees of Π_1^0 classes

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ABSTRACT

Let \mathcal{E}_s denote the lattice of Medvedev degrees of non-empty Π_1^0 subsets of 2^ω , and let \mathcal{E}_w denote the lattice of Muchnik degrees of non-empty Π_1^0 subsets of 2^ω . We prove that the first-order theory of \mathcal{E}_s as a partial order is recursively isomorphic to the first-order theory of true arithmetic. Our coding of arithmetic in \mathcal{E}_s also shows that the Σ_3^0 -theory of \mathcal{E}_s as a lattice and the Σ_4^0 -theory of \mathcal{E}_s as a partial order are undecidable. Moreover, we show that the degree of \mathcal{E}_s as a lattice is $\mathbf{0}'''$ in the sense that $\mathbf{0}'''$ computes a presentation of \mathcal{E}_s and that every presentation of \mathcal{E}_s computes $\mathbf{0}'''$. Finally, we show that the Σ_3^0 -theory of \mathcal{E}_w as a lattice and the Σ_4^0 -theory of \mathcal{E}_w as a partial order are undecidable.

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1. Introduction

1.1. Mass problems and reducibilities

A *mass problem* is a set $X \subseteq \omega^\omega$ thought of as an abstract mathematical problem, namely the problem of finding a member of X . Medvedev introduced his notion of reducibility among the mass problems as a formalization of Kolmogorov's idea of a "calculus of problems" [24]. For sets $X, Y \subseteq \omega^\omega$, $X \leq_s Y$ (read X *Medvedev reduces* or *strongly reduces* to Y) if and only if there is a Turing functional Φ such that $(\forall g \in Y)(\Phi(g) \in X)$. Under the interpretation of subsets of ω^ω as mathematical problems, $X \leq_s Y$ means that problem Y is at least as hard as problem X in a strongly intuitionistic sense: solutions to Y can be converted to solutions to X by a uniform effective procedure.

Medvedev reducibility induces a degree structure on $\mathcal{P}(\omega^\omega)$ in the same way that Turing reducibility induces a degree structure on ω^ω . For sets $X, Y \subseteq \omega^\omega$, $X \equiv_s Y$ (read X is *Medvedev equivalent* or *strongly equivalent* to Y) if and only if $X \leq_s Y$ and $Y \leq_s X$. \mathcal{D}_s denotes the *Medvedev degrees*, that is, the set of all \equiv_s -equivalence classes $\text{deg}_s(X)$ for $X \subseteq \omega^\omega$. The preordering \leq_s of $\mathcal{P}(\omega^\omega)$ induces a partial ordering of \mathcal{D}_s , also named \leq_s . Muchnik introduced a non-uniform variant of Medvedev reducibility [26]. For sets $X, Y \subseteq \omega^\omega$, $X \leq_w Y$ (read X *Muchnik reduces* or *weakly reduces* to Y) if and only if $(\forall g \in Y)(\exists f \in X)(f \leq_T g)$. *Muchnik equivalence* (or *weak equivalence*) \equiv_w and the *Muchnik degrees* \mathcal{D}_w are defined analogously to \equiv_s and \mathcal{D}_s but with \leq_w in place of \leq_s .

\mathcal{D}_s and \mathcal{D}_w extend the Turing degrees \mathcal{D}_T . The natural maps $\text{deg}_T(f) \mapsto \text{deg}_s(\{f\})$ and $\text{deg}_T(f) \mapsto \text{deg}_w(\{f\})$ are upper-semilattice embeddings of \mathcal{D}_T into \mathcal{D}_s and \mathcal{D}_w respectively. Moreover, the range of each of these embeddings is definable

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in the corresponding structure. This fact is due to Dymet for \mathcal{D}_s ([12] Corollary 2.1), and the proof for \mathcal{D}_w is simpler (see also [41] Theorem 2.2). \mathcal{D}_s and \mathcal{D}_w enjoy a much richer algebraic structure than \mathcal{D}_T does. Most importantly, \mathcal{D}_s and \mathcal{D}_w are both distributive lattices. In fact, \mathcal{D}_s is a Brouwer algebra, and \mathcal{D}_w is both a Brouwer algebra and a Heyting algebra. Heyting and Brouwer algebras provide semantics for propositional logic, and the interpretation of \mathcal{D}_s and \mathcal{D}_w as semantics for propositional logic was an original motivation for their study. This interpretation continues to drive much of the research in this area. Sorbi's survey [41] is a good introduction to \mathcal{D}_s and \mathcal{D}_w .

A classic problem in computability theory is to determine the complexity of the first-order theory of a given degree structure, such as \mathcal{D}_T , \mathcal{D}_s , or \mathcal{D}_w . The benchmarks are theories of arithmetic, the comparisons are made via recursive isomorphisms, and the results typically express that the first-order theories of the degree structures are as complicated as possible. The original result of this sort, due to Simpson, is that the first-order theory of \mathcal{D}_T is recursively isomorphic to the second-order theory of true arithmetic [35]. Lewis, Nies, and Sorbi and independently the author have determined that the first-order theories of \mathcal{D}_s and \mathcal{D}_w are both recursively isomorphic to the third-order theory of true arithmetic [21,32].

1.2. Substructures of \mathcal{D}_s and \mathcal{D}_w

Various substructures arise in the study of degree structures, and the complexities of their first-order theories naturally come into question. In the Turing degrees, two popular substructures are $\mathcal{D}_T(\leq_T \mathbf{0}')$, the Turing degrees below $\mathbf{0}'$, and \mathcal{R} , the Turing degrees of r.e. sets. Both $\mathcal{D}_T(\leq_T \mathbf{0}')$ and \mathcal{R} have first-order theories that are recursively isomorphic to the first-order theory of true arithmetic. The $\mathcal{D}_T(\leq_T \mathbf{0}')$ case is due to Shore [33]. The \mathcal{R} case is due to unpublished work of Harrington and Slaman (see also [27]). For degree structures on the mass problems, substructures naturally arise by restricting the family of mass problems under consideration to natural topological classes. For instance, restricting to the degrees of closed subsets of ω^ω yields $\mathcal{D}_{s,cl}$, the closed Medvedev degrees, and $\mathcal{D}_{w,cl}$, the closed Muchnik degrees. Restricting to the degrees of compact subsets of ω^ω (or equivalently restricting to closed subsets of 2^ω) yields $\mathcal{D}_{s,cl}^{01}$, the compact Medvedev degrees, and $\mathcal{D}_{w,cl}^{01}$, the compact Muchnik degrees. $\mathcal{D}_{s,cl}$ and $\mathcal{D}_{s,cl}^{01}$ are sublattices of \mathcal{D}_s , and $\mathcal{D}_{w,cl}$ and $\mathcal{D}_{w,cl}^{01}$ are sublattices of \mathcal{D}_w . These sublattices have received attention in for example [3,22,31,32,41]. The author has determined that the four structures $\mathcal{D}_{s,cl}$, $\mathcal{D}_{s,cl}^{01}$, $\mathcal{D}_{w,cl}$, and $\mathcal{D}_{w,cl}^{01}$ all have first-order theories that are recursively isomorphic to the second-order theory of true arithmetic [32].

In this paper, we consider \mathcal{E}_s , the sublattice of \mathcal{D}_s consisting of the Medvedev degrees of non-empty Π_1^0 subsets of 2^ω . We also consider \mathcal{E}_w , the sublattice of \mathcal{D}_w consisting of the Muchnik degrees of non-empty Π_1^0 subsets of 2^ω , though to a much lesser extent. \mathcal{E}_s and its sister-structure \mathcal{E}_w are the effective counterparts of $\mathcal{D}_{s,cl}^{01}$ and $\mathcal{D}_{w,cl}^{01}$. They have enjoyed considerable attention from many authors, beginning with Simpson's suggestion to the Foundations of Mathematics discussion group that \mathcal{E}_w is analogous to \mathcal{R} but with more natural examples [36]. This analogy with \mathcal{R} drives much of the research on \mathcal{E}_s and \mathcal{E}_w . For example, every non-minimum member of \mathcal{E}_s and \mathcal{E}_w is join-reducible [4], reflecting Sacks's splitting theorem for \mathcal{R} [28], and \mathcal{E}_s is dense [7], reflecting Sacks's density theorem for \mathcal{R} [30]. The question of whether \mathcal{E}_w is dense remains open. See the recent surveys by Simpson [39] and Hinman [14] for an overview of \mathcal{E}_s and \mathcal{E}_w .

1.3. Undecidability in \mathcal{E}_s and \mathcal{E}_w

Our main result is that the first-order theory of \mathcal{E}_s is recursively isomorphic to the first-order theory of true arithmetic. This result holds in both the language of lattices and in the language of partial orders because, in any lattice, the lattice operations are arithmetically definable from the partial order. In light of the analogies between \mathcal{E}_s and \mathcal{R} , our main result can be seen as a companion to the result that the first-order theory of \mathcal{R} is recursively isomorphic to the first-order theory of true arithmetic. We are able to prove that the first-order theory of \mathcal{E}_w is undecidable, but, beyond that result, the question of the exact complexity of the first-order theory of \mathcal{E}_w remains wide open. Cole and Simpson conjecture that the first-order theory of \mathcal{E}_w is recursively isomorphic to $\mathcal{O}^{(\omega)}$ (the ω th Turing jump of Kleene's \mathcal{O}), the obvious upper bound [11].

We also consider the decidability of fragments of the first-order theories of \mathcal{E}_s and \mathcal{E}_w . Here we need to be careful to specify whether we are working in the language of lattices or in the language of partial orders. Binns has shown that the Σ_1^0 -theories of \mathcal{E}_s and \mathcal{E}_w as lattices are identical and decidable [4]. Cole and Kihara have shown that the Σ_2^0 -theory of \mathcal{E}_s as a partial order is decidable [10]. The corresponding result for \mathcal{E}_w is not known. The decidability of the Σ_2^0 -theories of \mathcal{E}_s and \mathcal{E}_w as lattices and the Σ_3^0 -theories of \mathcal{E}_s and \mathcal{E}_w as partial orders are not known. Our method of coding arithmetic in distributive lattices proves that the Σ_3^0 -theories of \mathcal{E}_s and \mathcal{E}_w as lattices and the Σ_4^0 -theories of \mathcal{E}_s and \mathcal{E}_w as a partial orders are all undecidable.

There has been a huge amount of difficult work on the decidability of various fragments of the first-order theories of \mathcal{D}_T and \mathcal{R} . We summarize the results for \mathcal{R} for comparison (see [34] for a survey of this area). The Σ_1^0 -theory of \mathcal{R} as an upper-semilattice is decidable [29]. The decidability of the Σ_2^0 -theory of \mathcal{R} as either a partial order or an upper semi-lattice is unknown. However, the Σ_3^0 -theory of \mathcal{R} as a partial order is undecidable [19]. Moreover, if one extends the partial infimum function on \mathcal{R} (as an upper-semilattice) to any total function, then the Σ_2^0 -theory of the resulting structure is undecidable [25]. These two undecidability results for \mathcal{R} suggest by analogy that the Σ_2^0 -theories of \mathcal{E}_s and \mathcal{E}_w as lattices and the Σ_3^0 -theories of \mathcal{E}_s and \mathcal{E}_w as partial orders may all be undecidable. Table 1 summarizes the current state of knowledge concerning the decidability of various fragments of the first-order theories of \mathcal{R} , \mathcal{E}_s , and \mathcal{E}_w .

Table 1
Decidability of various fragments of the first-order theories of \mathcal{R} , \mathcal{E}_s , and \mathcal{E}_w .

	Σ_1^0	Σ_2^0	Σ_3^0	Σ_4^0
\mathcal{R} as a partial order	decidable	?	undecidable	undecidable
\mathcal{R} as an upper-semilattice	decidable	?	undecidable	undecidable
\mathcal{E}_s as a partial order	decidable	decidable	?	undecidable
\mathcal{E}_s as a lattice	decidable	?	undecidable	undecidable
\mathcal{E}_w as a partial order	decidable	?	?	undecidable
\mathcal{E}_w as a lattice	decidable	?	undecidable	undecidable

We also prove that \mathcal{E}_s is as complicated as possible in terms of degree of presentation. Specifically, we prove that the degree of \mathcal{E}_s as a lattice is $\mathbf{0}'''$. This means that $\mathbf{0}'''$ computes a presentation of \mathcal{E}_s as a lattice and that $\mathbf{0}'''$ is computable in every presentation of \mathcal{E}_s as a lattice. A corollary is that \mathcal{E}_s has no recursive presentation as a partial order. The natural presentation of \mathcal{E}_w has Turing degree \mathcal{O} [11], so it is reasonable to expect that \mathcal{E}_w has degree \mathcal{O} , though this question remains open. For comparison, it follows from the results of [27] (though it is not stated explicitly) that the degree of \mathcal{R} as an upper-semilattice is $\mathbf{0}^{(4)}$.

This paper is organized as follows. Section 2 provides the necessary background material. Section 3 presents our scheme for coding arithmetic in distributive lattices. Section 4 presents the theory of meet-irreducibles in \mathcal{E}_s necessary to implement our coding in \mathcal{E}_s . Section 5 implements our coding in \mathcal{E}_s , thereby proving our results concerning the complexity of the first-order theory of \mathcal{E}_s . Section 6 proves that the degree of \mathcal{E}_s as a lattice is $\mathbf{0}'''$. Section 7 proves our undecidability results concerning the first-order theory of \mathcal{E}_w .

2. Background

Here we present the relevant background concerning classical computability theory, distributive lattices, Π_1^0 classes and their Medvedev and Muchnik degrees, and arithmetic. Much of the notation should be familiar from the standard sources, such as [20] and [40]. Unfortunately, notation for the Medvedev degrees is far from standardized. We follow [39] in the hope that its notation will become standard.

2.1. Computability theory

Let $n \in \omega$, $\sigma, \tau \in \omega^{<\omega}$, $f, g \in \omega^\omega$, and $X, Y \subseteq \omega^\omega$. Then $f \upharpoonright n$ is the initial segment of f of length n ; $|\sigma|$ is the length of σ ; $\sigma \subseteq \tau$ means that σ is an initial segment of τ ; $\sigma \subset f$ means that σ is an initial segment of f ; $\sigma \frown f$ is the concatenation of σ and f , defined by

$$(\sigma \frown f)(n) = \begin{cases} \sigma(n) & \text{if } n < |\sigma| \\ f(n - |\sigma|) & \text{if } n \geq |\sigma|, \end{cases}$$

with $n \frown f$ abbreviating $(n) \frown f$ for sequences (n) of length 1; $f \oplus g$ is the function defined by

$$(f \oplus g)(n) = \begin{cases} f(m) & \text{if } n = 2m \\ g(m) & \text{if } n = 2m + 1; \end{cases}$$

$\sigma \frown X = \{\sigma \frown f \mid f \in X\}$; and $X \otimes Y = \{f \oplus g \mid f \in X \wedge g \in Y\}$.

The function $\langle \cdot, \cdot \rangle : \omega \times \omega \rightarrow \omega$ is a fixed recursive bijection. Φ_e denotes the e th Turing functional. Φ always denotes a Turing functional, and if $f \in \omega^\omega$, then $\Phi(f)$ is the partial function computed when Φ uses f as its oracle. For $\sigma \in \omega^{<\omega}$, $\Phi(\sigma)$ is the partial function that, on input $n \in \omega$, is computed by running Φ on input n for at most $|\sigma|$ steps and using σ to answer oracle queries. The restriction on the running time of $\Phi(\sigma)$ ensures that oracle queries are only made of numbers $< |\sigma|$. Consequently, if $\Phi(\sigma)(n) \downarrow$, then $\Phi(f)(n) = \Phi(\sigma)(n)$ for all $f \supset \sigma$.

Let $A, B \subseteq \omega$. $A \leq_1 B$ if and only if there is a one-to-one recursive function f such that $\forall n(n \in A \leftrightarrow f(n) \in B)$. A and B are *recursively isomorphic* if and only if there is such an f that is a bijection. The Myhill isomorphism theorem states that A and B are recursively isomorphic if and only if $A \equiv_1 B$, that is, if and only if $A \leq_1 B$ and $B \leq_1 A$ (see [40] Section I.5).

2.2. Distributive lattices

The usual options for lattice notation conflict with either the logical notation (\vee and \wedge) or the arithmetic notation ($+$ and \times). To avoid this conflict, we follow [39] and write sup for join and inf for meet.

A lattice \mathcal{L} is *distributive* if and only if sup and inf distribute over each other:

- $(\forall x, y, z \in \mathcal{L})(\text{sup}(x, \text{inf}(y, z)) = \text{inf}(\text{sup}(x, y), \text{sup}(x, z)))$, and
- $(\forall x, y, z \in \mathcal{L})(\text{inf}(x, \text{sup}(y, z)) = \text{sup}(\text{inf}(x, y), \text{inf}(x, z)))$.

An element x of a lattice \mathcal{L} is *meet-reducible* if and only if $(\exists y, z \in \mathcal{L})(y > x) \wedge (z > x) \wedge (x = \inf(y, z))$. Otherwise x is *meet-irreducible*. We frequently use the following well-known characterization without mention:

Lemma 2.1 (See [2] Section III.2). *If \mathcal{L} is a distributive lattice, then $x \in \mathcal{L}$ is meet-irreducible if and only if $(\forall y, z \in \mathcal{L})(x \geq \inf(y, z) \rightarrow x \geq y \vee x \geq z)$.*

Proof. Suppose x is meet-irreducible and $x \geq \inf(y, z)$. Then

$$x = \sup(x, \inf(y, z)) = \inf(\sup(x, y), \sup(x, z)).$$

Thus $x = \sup(x, y)$ or $x = \sup(x, z)$, which means $x \geq y$ or $x \geq z$. Conversely, if x is meet-reducible, then by definition there are $y, z > x$ with $x = \inf(y, z)$. \square

Dualizing gives the definitions of join-reducible and join-irreducible, and it gives a characterization of join-irreducible in distributive lattices.

Sometimes we want to ignore the lattice operations of a lattice \mathcal{L} and consider \mathcal{L} as a partial order. When we do, we write $(\mathcal{L}; \leq_{\mathcal{L}})$ to indicate that we are considering only the partial order structure on \mathcal{L} . In particular, $\text{Th}(\mathcal{L})$ denotes the first-order theory of \mathcal{L} as a lattice, and $\text{Th}(\mathcal{L}; \leq_{\mathcal{L}})$ denotes the first-order theory of \mathcal{L} as a partial order.

2.3. Π_1^0 classes and their Medvedev and Muchnik degrees

The Π_1^0 classes are the Π_1^0 subsets of ω^ω , where a set $X \subseteq \omega^\omega$ is Π_1^0 if and only if it is of the form $X = \{f \in \omega^\omega \mid \forall n \varphi(f, n)\}$ for some recursive predicate φ . The Π_1^0 classes have been persistent objects of study throughout computability theory, due in no small part to their applications to recursive mathematics and reverse mathematics. The surveys by Cenzer [6] and by Cenzer and Remmel [8] provide an extensive overview of the theory of the Π_1^0 classes, as does the forthcoming book by Cenzer and Remmel [9].

A useful characterization of the Π_1^0 classes is as the sets of paths through recursive trees. A *tree* is a set $T \subseteq \omega^{<\omega}$ closed under initial segments: $(\forall \sigma, \tau \in \omega^{<\omega})(\sigma \in T \wedge \tau \subseteq \sigma \rightarrow \tau \in T)$. A function $f \in \omega^\omega$ is a *path* through T if and only if $(\forall n \in \omega)(f \upharpoonright n \in T)$. If T is a tree, then $[T]$ denotes the set of all paths through T . A set $X \subseteq \omega^\omega$ is a Π_1^0 class if and only if $X = [T]$ for some recursive tree T (see [8] Lemma 2.2). Recall the usual product topology on ω^ω . Basic open sets are of the form $I(\sigma) = \{f \in \omega^\omega \mid \sigma \subset f\}$ for $\sigma \in \omega^{<\omega}$. A set $X \subseteq \omega^\omega$ is closed in this topology if and only if $X = [T]$ for some (not necessarily recursive) tree T . For this reason, the Π_1^0 classes are sometimes called the effectively closed sets.

For sets $X, Y \subseteq \omega^\omega$, $X \leq_s Y$ if and only if there is a Turing functional Φ such that $(\forall g \in Y)(\Phi(g) \in X)$, a condition which we abbreviate by $\Phi(Y) \subseteq X$. Similarly, $X \leq_w Y$ if and only if $(\forall g \in Y)(\exists f \in X)(f \leq_T g)$. We consider \leq_s and \leq_w restricted to non-empty Π_1^0 subsets of 2^ω . Henceforth the term “ Π_1^0 class” refers exclusively to non-empty Π_1^0 subsets of 2^ω , and all trees are subsets of $2^{<\omega}$. Every Π_1^0 class is a closed subset of the compact space 2^ω and is therefore compact. The compactness of the Π_1^0 classes is crucial to many of our arguments. As a first example, compactness allows us to express \leq_s arithmetically.

Lemma 2.2. $[T_0] \leq_s [T_1]$ is Σ_3^0 relative to the trees T_0 and T_1 .

Proof. For a given Turing functional Φ , we show that

$$\Phi([T_1]) \subseteq [T_0] \text{ if and only if } (\forall n \in \omega)(\exists s \in \omega)(\forall \sigma \in 2^s)(\sigma \in T_1 \rightarrow \Phi(\sigma) \upharpoonright n \in T_0),$$

where $\Phi(\sigma) \upharpoonright n \in T_0$ includes the provision that $(\forall i < n)(\Phi(\sigma)(i) \downarrow)$. It then follows that

$$[T_0] \leq_s [T_1] \text{ if and only if } (\exists e \in \omega)(\forall n \in \omega)(\exists s \in \omega)(\forall \sigma \in 2^s)(\sigma \in T_1 \rightarrow \Phi_e(\sigma) \upharpoonright n \in T_0),$$

which gives our Σ_3^0 definition of \leq_s .

For the forward direction, let $n \in \omega$ be given. Let $\Sigma = \{\sigma \in 2^{<\omega} \mid \Phi(\sigma) \upharpoonright n \in T_0\}$. The condition $\Phi([T_1]) \subseteq [T_0]$ implies that $[T_1] \subseteq \bigcup_{\sigma \in \Sigma} I(\sigma)$. By compactness, there is a finite $\Sigma_0 \subseteq \Sigma$ such that $[T_1] \subseteq \bigcup_{\sigma \in \Sigma_0} I(\sigma)$ and an $s \in \omega$ such that $(\forall \sigma \in 2^s)(\sigma \in T_1 \rightarrow (\exists \sigma_0 \in \Sigma_0)(\sigma_0 \subseteq \sigma))$. Then $(\forall \sigma \in 2^s)(\sigma \in T_1 \rightarrow \Phi(\sigma) \upharpoonright n \in T_0)$.

For the reverse direction, consider $f \in [T_1]$. Given any $n \in \omega$, let $s \in \omega$ be such that $(\forall \sigma \in 2^s)(\sigma \in T_1 \rightarrow \Phi(\sigma) \upharpoonright n \in T_0)$. Then $\Phi(f \upharpoonright s) \upharpoonright n \in T_0$, so $\Phi(f) \upharpoonright n \in T_0$. Thus $\forall n(\Phi(f) \upharpoonright n \in T_0)$. Hence $\Phi(f) \in [T_0]$, and therefore $\Phi([T_1]) \subseteq [T_0]$. \square

Let $\mathcal{E}_s = \{\text{deg}_s(X) \mid X \text{ is a } \Pi_1^0 \text{ class}\}$. \mathcal{E}_s is partially ordered by \leq_s . If X and Y are both Π_1^0 classes, then so are $X \otimes Y$ and $0^\wedge X \cup 1^\wedge Y$. Given trees T_0 and T_1 , let $T_0 \otimes T_1 = \{\sigma \oplus \tau \mid \sigma \in T_0 \wedge \tau \in T_1 \wedge |\tau| \leq |\sigma| \leq |\tau| + 1\}$. Then $T_0 \otimes T_1$ and $0^\wedge T_0 \cup 1^\wedge T_1$ are both trees, $[T_0] \otimes [T_1] = [T_0 \otimes T_1]$, and $0^\wedge [T_0] \cup 1^\wedge [T_1] = [0^\wedge T_0 \cup 1^\wedge T_1]$. Define $\sup(\text{deg}_s(X), \text{deg}_s(Y)) = \text{deg}_s(X \otimes Y)$ and $\inf(\text{deg}_s(X), \text{deg}_s(Y)) = \text{deg}_s(0^\wedge X \cup 1^\wedge Y)$. One readily checks that $\sup(\text{deg}_s(X), \text{deg}_s(Y))$ and $\inf(\text{deg}_s(X), \text{deg}_s(Y))$ are, respectively, the \leq_s -least upper bound and \leq_s -greatest lower bound of $\text{deg}_s(X)$ and $\text{deg}_s(Y)$ and that sup and inf distribute over each other. Thus \mathcal{E}_s is a distributive lattice. \mathcal{E}_s has a least element $\mathbf{0}_s = \text{deg}_s(2^\omega)$, and a Π_1^0 class has least degree if and only if it has a recursive member. \mathcal{E}_s also has a greatest element $\mathbf{1}_s$ (see [37] Lemma 3.20). Two examples of Π_1^0 classes with greatest degree are $\text{DNR}_2 = \{f \in 2^\omega \mid \forall e(f(e) \neq \Phi_e(e))\}$ (DNR stands for *diagonally non-recursive*) and the class of all (appropriately Gödel numbered) complete consistent extensions of Peano arithmetic.

Let $\mathcal{E}_w = \{\text{deg}_w(X) \mid X \text{ is a } \Pi_1^0 \text{ class}\}$. \mathcal{E}_w is a distributive lattice with order \leq_w and with joins and meets computed as in \mathcal{E}_s . Notice, however, that $0 \wedge X \cup 1 \wedge Y \equiv_w X \cup Y$, which is not in general true with \equiv_s in place of \equiv_w . \mathcal{E}_w has a least element $\mathbf{0}_w$, a greatest element $\mathbf{1}_w$, and the above examples of Π_1^0 classes with least or greatest Medvedev degree also have least or greatest Muchnik degree. However, it is not the case that every Π_1^0 class with greatest Muchnik degree also has greatest Medvedev degree.

A sequence of trees $\{T_n\}_{n \in \omega}$ is *uniformly recursive* if and only if the set $\{(n, \sigma) \mid \sigma \in T_n\}$ is recursive. A *recursive sequence of Π_1^0 classes* is a sequence of Π_1^0 classes $\{X_n\}_{n \in \omega}$ for which there is a uniformly recursive sequence of trees $\{T_n\}_{n \in \omega}$ such that $X_n = [T_n]$ for each $n \in \omega$. For convenience, we also allow indexing over recursive sets A and consider recursive sequences of Π_1^0 classes of the form $\{X_n\}_{n \in A}$. Though not strictly necessary for our results, a convenient fact is that there is a recursive sequence all Π_1^0 classes (with many repetitions).

Lemma 2.3 (See [9] Chapter XV and [8] Section 2.7). *There is a uniformly recursive sequence of infinite trees $\{T_e\}_{e \in \omega}$ such that for every Π_1^0 class X there is an $e \in \omega$ such that $X = [T_e]$.*

Proof. In fact, [8] Lemma 2.2 proves that every Π_1^0 class is of the form $[T]$ for a primitive recursive tree. Let $\{P_e\}_{e \in \omega}$ be a recursive sequence of all primitive recursive functions. Then define T'_e to be the tree $T'_e = \{\sigma \in 2^{<\omega} \mid (\forall \tau \subseteq \sigma)(P_e(\tau) = 1)\}$. If P_e is the characteristic function of a tree, then T'_e is that tree. Thus if X is a Π_1^0 class, then $X = [T'_e]$ for some $e \in \omega$. We just need to make a final adjustment to ensure that every tree in the sequence is infinite. To this end, let

$$T_e = \{\sigma \in 2^{<\omega} \mid \sigma \in T'_e \vee (\forall m \leq |\sigma|)(\sigma \upharpoonright m \notin T'_e \rightarrow (\forall \tau \in 2^m)(\tau \notin T'_e))\}.$$

If T'_e is infinite, then $T_e = T'_e$. Otherwise, T_e consists of T'_e along with all strings that extend a string in T'_e of maximum length. \square

2.4. Arithmetic

In Section 3, we code structures that model PA^- (Peano arithmetic without induction) in distributive lattices. The language of arithmetic is $\{<, +, \times, 0, 1\}$, where $<$ is a 2-ary relation symbol, $+$ and \times are 2-ary function symbols, and 0 and 1 are constant symbols. The axioms of PA^- are those of a discretely ordered commutative semiring with a unit. For reference, our official axioms of PA^- are those that appear in [18] Section 2.1.

The official PA^- axioms are all Π_1^0 , except for the axiom $\forall x, y(x < y \rightarrow \exists z(x + z = y))$. To reduce the quantifier complexity of this axiom for when we analyze the fragments of $\text{Th}(\mathcal{E}_s)$, we introduce the *monus* symbol “ $\dot{-}$ ” and Skolemize. We call the resulting theory $\text{PA}^{\dot{-}}$.

Definition 2.4. $\text{PA}^{\dot{-}}$ is the theory whose axioms are the same as PA^- but with the axiom $\forall x, y(x < y \rightarrow \exists z(x + z = y))$ replaced by the axiom $\forall x, y(x < y \rightarrow x + (y \dot{-} x) = y)$.

The standard relational model of arithmetic is the structure $\mathcal{N} = (\omega; <, +, \times, 0, 1)$, where $<$ is a 2-ary relation on ω , $+$ and \times are 3-ary relations on ω , and 0 and 1 are constants in ω interpreted as the usual less-than, plus, times, zero, and one respectively. $\text{Th}(\mathcal{N})$ denotes the first-order theory of \mathcal{N} . We use the relational versions of $+$ and \times instead of the usual functional versions because our coding techniques most naturally code relations. Any formula in which $+$ and \times are relation symbols can be trivially translated into an equivalent formula in which $+$ and \times are function symbols. Translations in the other direction require *unnesting*. In general, a formula is said to be *unnested* if and only if every atomic subformula is of the form $x = y$, $c = y$, $f(x_1, \dots, x_n) = y$, or $R(x_1, \dots, x_n)$, where x, y , and the x_i for $i \leq n$ are variables, c is a constant symbol, f is a function symbol, and R is a relation symbol. Every formula can be recursively translated into an equivalent unnested formula (see [15] section 2.6). When unnesting is applied to a first-order formula in the functional language of arithmetic, we get an equivalent formula in which every atomic subformula is of the form $x = y$, $0 = y$, $1 = y$, $x < y$, $x + y = z$, or $x \times y = z$. That is, we get an equivalent formula in the relational language of arithmetic. Therefore the relational and functional versions of $\text{Th}(\mathcal{N})$ are recursively isomorphic.

We also make use of the structure $\mathcal{N}^{\dot{-}} = (\omega; <, +, \times, \dot{-}, 0, 1)$, where $<, +, \times, 0$, and 1 are as for \mathcal{N} , and $\dot{-}$ is the 3-ary relation on ω corresponding to the function

$$x \dot{-} y = \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{if } x < y. \end{cases}$$

Clearly, $\mathcal{N} \models \text{PA}^-$, $\mathcal{N}^{\dot{-}} \models \text{PA}^{\dot{-}}$, and $\text{PA}^{\dot{-}} \vdash \text{PA}^-$.

Let $\mathcal{M} \models \text{PA}^-$. An *initial segment* of \mathcal{M} is a $<$ -downward-closed substructure \mathcal{M}' of \mathcal{M} : $(\forall x \in \mathcal{M}')(\forall y \in \mathcal{M})(\mathcal{M} \models y < x \rightarrow (y \in \mathcal{M}'))$. An *initial interval* of \mathcal{M} is a subset of \mathcal{M} of the form $\{y \in \mathcal{M} \mid \mathcal{M} \models y < x \vee y = x\}$ for some $x \in \mathcal{M}$. The following fact ensures that our coding in the next section correctly codes structures isomorphic to \mathcal{N} .

Lemma 2.5 (See [18] Theorem 2.2). *If $\mathcal{M} \models \text{PA}^-$, then there is an initial segment of \mathcal{M} that is isomorphic to \mathcal{N} . In particular, \mathcal{N} is the unique model of PA^- , up to isomorphism, in which every initial interval is finite.*

For the undecidability of $\Sigma_3^0\text{-Th}(\mathcal{E}_s)$, we also need the following fact.

Lemma 2.6 (See [18] Corollary 2.9). *If φ is a Σ_1^0 sentence and $\mathcal{N} \models \varphi$, then $\text{PA}^- \vdash \varphi$.*

3. Coding arithmetic in distributive lattices

We present our scheme for coding arithmetic in distributive lattices. Although the definitions below make sense in any lattice, they were designed with the particular goal of coding \mathcal{N} into \mathcal{D}_s , \mathcal{D}_w , and their sublattices in mind. For example, meet-irreducible elements play a major role in the coding. One may dualize the coding to replace meet-irreducible by join-irreducible, but this would not suffice for our purposes because all non-zero elements of \mathcal{E}_s are join-reducible [4]. The coding presented here has been slightly modified from the original version developed in [32] in order to reduce the quantifier complexity of coded relations.

3.1. Coding relations

Definition 3.1. For elements s and w of a lattice, s meets to w if and only if $\exists y (y > w \wedge \inf(s, y) = w)$.

Definition 3.2. For a lattice \mathcal{L} and a $w \in \mathcal{L}$,

$$E(w) = \{s \in \mathcal{L} \mid s \text{ is meet-irreducible} \wedge s \text{ meets to } w\}.$$

The next two lemmas prove important properties of E in distributive lattices.

Lemma 3.3. If \mathcal{L} is a distributive lattice, then $E(w)$ is an antichain for every $w \in \mathcal{L}$.

Proof. Suppose for a contradiction that there are $s, s' \in E(w)$ with $s > s'$. Let $y > w$ be such that $\inf(s, y) = w$. Then $s' \geq y$ because s' is meet-irreducible, $s' \geq \inf(s, y)$, and $s' \not\geq s$. Therefore $s > s' \geq y > w$, giving the contradiction $\inf(s, y) = y > w$. \square

Lemma 3.4. If \mathcal{L} is a distributive lattice and $\{s_i\}_{i < n} \subseteq \mathcal{L}$ is a finite antichain of meet-irreducible elements, then $E(\inf_{i < n} s_i) = \{s_i\}_{i < n}$.

Proof. Let $w = \inf_{i < n} s_i$. First we show that $s_i \in E(w)$ for each $i < n$. Fix $i < n$ and let $t_i = \inf\{s_j \mid j < n \wedge j \neq i\}$. Clearly $t_i \geq w$ and $\inf(s_i, t_i) = w$. Moreover, $s_i \not\geq t_i$ because otherwise the meet-irreducibility of s_i implies that $s_i \geq s_j$ for some $j \neq i$, contradicting that $\{s_i\}_{i < n}$ is an antichain. Thus in fact $t_i > w$, so t_i witnesses that s_i meets to w . Hence $s_i \in E(w)$. Conversely, if $x \in E(w)$, then x is meet-irreducible and $x \geq w$. Thus $x \geq s_i$ for some $i < n$, so $x = s_i$ because $E(w)$ is an antichain by Lemma 3.3. Thus $E(w) = \{s_i\}_{i < n}$. \square

Given an element w of a lattice, we think of w as code for the set $E(w)$. The symbol “ E ” stands for “elements,” as in the elements of the set coded by w .¹

Now we code 2-ary and 3-ary relations on $E(w_0)$ for an element w_0 of a lattice \mathcal{L} . The same scheme can code n -ary relations for any $n \in \omega$, but we only need to code 2-ary and 3-ary relations to code \mathcal{N} . The intuition behind the following definition is that if $s_0, u_0 \in E(w_0)$, then $\sup(s_0, u_0)$ should code the pair (s_0, u_0) . However, this coding makes the pairs (s_0, u_0) and (u_0, s_0) indistinguishable because $\sup(s_0, u_0) = \sup(u_0, s_0)$. To solve this problem, we fix additional parameters $w_1, w_2, m \in \mathcal{L}$. Once $w_0, w_1, w_2, m \in \mathcal{L}$ are fixed, any $c \in \mathcal{L}$ can be interpreted as coding a 2-ary relation R_c^2 on $E(w_0)$ and a 3-ary relation R_c^3 on $E(w_0)$.

Definition 3.5. Let \mathcal{L} be a lattice and fix elements $w_0, w_1, w_2, m \in \mathcal{L}$. Then any $c \in \mathcal{L}$ defines a 2-ary relation R_c^2 on $E(w_0)$ and a 3-ary relation R_c^3 on $E(w_0)$ by

$$\begin{aligned} R_c^2(s_0, u_0) & \text{ if and only if } s_0 \in E(w_0) \wedge u_0 \in E(w_0) \\ & \wedge \exists u_1 (u_1 \text{ meets to } w_1 \wedge \sup(u_0, u_1) \geq m \wedge \sup(s_0, u_1) \geq c) \\ R_c^3(s_0, u_0, v_0) & \text{ if and only if } s_0 \in E(w_0) \wedge u_0 \in E(w_0) \wedge v_0 \in E(w_0) \\ & \wedge \exists u_1 \exists v_2 (u_1 \text{ meets to } w_1 \wedge v_2 \text{ meets to } w_2 \\ & \wedge \sup(u_0, u_1) \geq m \wedge \sup(v_0, v_2) \geq m \wedge \sup(s_0, u_1, v_2) \geq c). \end{aligned}$$

3.2. Coding arithmetic

With Definition 3.5 in hand, we can define codes for models of various theories. For PA^- we have the following definitions.

Definition 3.6. In a lattice \mathcal{L} , a code (for a structure in the language of arithmetic) is a sequence of elements

$$\vec{w} = (w_0, w_1, w_2, m, \ell, p, t, z, o)$$

from \mathcal{L} interpreted as coding the structure

$$\mathcal{M}_{\vec{w}} = (E(w_0); R_\ell^2, R_p^3, R_t^3, z, o)$$

where R_ℓ^2 , R_p^3 , and R_t^3 are the relations on $E(w_0)$ defined from ℓ , p , and t , respectively, as in Definition 3.5.

¹ In [32], $E(w)$ was called $\tilde{E}(w)$ (see [32] Definition 4.4) and its definition required that the $s \in \tilde{E}(w)$ also be minimal with respect to being meet-irreducible and meeting to w . The minimality requirement is unnecessary by Lemma 3.3.

In Definition 3.6, w is for “ ω ,” m is for “match,” ℓ is for “less,” p is for “plus,” t is for “times,” z is for “zero,” and o is for “one.”

If \vec{w} is a code in a lattice \mathcal{L} , then sentences in the language of arithmetic are interpreted in $\mathcal{M}_{\vec{w}}$ in the obvious way.

Definition 3.7. Let φ be a sentence in the language of arithmetic. The *translation* of φ is the formula $\varphi'(w_0, w_1, w_2, m, \ell, p, t, z, o)$ (with the displayed variables free) in the language of lattices obtained from φ by making the following replacements.

- Replace $<$ by the formula defining R_ℓ^2 ,
- replace $+$ by the formula defining R_p^3 ,
- replace \times by the formula defining R_t^3 ,
- replace 0 by z ,
- replace 1 by o ,
- replace $\exists x$ by the formula expressing $\exists x \in E(w_0)$, and
- replace $\forall x$ by the formula expressing $\forall x \in E(w_0)$.

If \vec{w} is a code in a lattice \mathcal{L} , then $\mathcal{M}_{\vec{w}} \models \varphi$ means that $\mathcal{L} \models \varphi'(\vec{w})$.

Compare the notation in Definitions 3.5 and 3.6 to the notation in Definition 3.7. In Definitions 3.5 and 3.6, the symbols $w_0, w_1, w_2, m, \ell, p, t, z$, and o denote elements of a lattice \mathcal{L} , whereas in Definition 3.7 these symbols denote the free variables of the formula φ' . We conflate the notation in this manner throughout in order to describe easily what variables interpret what lattice elements.

Definition 3.8. In a lattice \mathcal{L} , a *code for a model of PA^-* is a code \vec{w} such that $\mathcal{M}_{\vec{w}} \models \text{PA}^-$.

If φ is a first-order sentence in the language of arithmetic, then its translation φ' is a first-order formula in the language of lattices. Thus for such a sentence φ , the property “ \vec{w} is a code such that $\mathcal{M}_{\vec{w}} \models \varphi$ ” is first-order. The property “ \vec{w} is a code for a model of PA^- ” is therefore expressible by a first-order formula in the language of lattices.

To code \mathcal{N} , we add extra conditions to Definition 3.8 ensuring that the coded structure is isomorphic to \mathcal{N} . Ultimately, these extra conditions express that every initial interval of the coded structure is finite, which suffices by Lemma 2.5. The following definitions allows us to compare the cardinalities of initial intervals of coded models of PA^- .

Definition 3.9. Let \mathcal{L} be a lattice and let \vec{w} be a code for a model of PA^- . An $a \in \mathcal{L}$ *codes an initial interval of $\mathcal{M}_{\vec{w}}$* if and only if $(\exists s \in E(w_0))(\forall b \in \mathcal{L})(b \in E(a) \leftrightarrow R_\ell^2(b, s) \vee b = s)$.

Definition 3.10. For a lattice \mathcal{L} and elements $r, q \in \mathcal{L}$, $E(r)$ *matches* $E(q)$ if and only if there is a $z \in \mathcal{L}$ such that

- (i) $(\forall x \in E(q))(\exists! y \in E(r))(\text{sup}(x, y) \in E(z))$, and
- (ii) $(\forall x \in E(r))(\exists! y \in E(q))(\text{sup}(x, y) \in E(z))$.

Clearly if $E(r)$ matches $E(q)$, then $|E(r)| = |E(q)|$. The next definition enforces a weak converse of this fact.

Definition 3.11. A lattice \mathcal{L} has the *finite matching property* if and only if whenever $q, q' \in \mathcal{L}$ are such that $|E(q)| = |E(q')| = n$ for some $n \in \omega$ then there is an $r \in \mathcal{L}$ such that $E(r)$ matches both $E(q)$ and $E(q')$.

We can now define a code for \mathcal{N} in a lattice \mathcal{L} and prove that codes for \mathcal{N} always code structures isomorphic to \mathcal{N} provided that \mathcal{L} is distributive, that \mathcal{L} has the finite matching property, and that some code in \mathcal{L} codes a structure isomorphic to \mathcal{N} . It follows that $\text{Th}(\mathcal{N}) \leq_1 \text{Th}(\mathcal{L})$.

Definition 3.12. In a lattice \mathcal{L} , a *code for \mathcal{N}* is a code \vec{w} such that

- (i) \vec{w} is a code for a model of PA^- ,
- (ii) $(\forall s \in E(w_0))(\exists a \in \mathcal{L})(\forall b \in \mathcal{L})(b \in E(a) \leftrightarrow R_\ell^2(b, s) \vee b = s)$ (that is, every initial interval of $\mathcal{M}_{\vec{w}}$ is coded by some $a \in \mathcal{L}$), and
- (iii) For every $a \in \mathcal{L}$ that codes an initial interval of $\mathcal{M}_{\vec{w}}$ and every code \vec{w}' that satisfies items (i) and (ii) above, there is an $a' \in \mathcal{L}$ that codes an initial interval of $\mathcal{M}_{\vec{w}'}$ and an $r \in \mathcal{L}$ such that $E(r)$ matches both $E(a)$ and $E(a')$.

Again, the property “ \vec{w} is a code for \mathcal{N} ” can be expressed by a first-order formula in the language of lattices.

Lemma 3.13. Let \mathcal{L} be a distributive lattice with the finite matching property, and let \vec{w} be a code such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$. Then \vec{w} is a code for \mathcal{N} and, moreover, $\mathcal{M}_{\vec{w}'} \cong \mathcal{N}$ for every \vec{w}' that is a code for \mathcal{N} in \mathcal{L} .

Proof. First let \vec{w} be as in the statement of the lemma and show that \vec{w} satisfies Definition 3.12. Item (i) is satisfied by the assumption $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$. For item (ii), let $s \in E(w_0)$ and notice that $\{b \mid R_\ell^2(b, s) \vee b = s\}$ is finite because it is an initial interval of a structure isomorphic to \mathcal{N} and that it is an antichain because it is a subset of $E(w_0)$ which is an antichain by Lemma 3.3. Thus $a = \text{inf}\{b \mid R_\ell^2(b, s) \vee b = s\}$ witnesses item (ii) for s because $E(a) = \{b \mid R_\ell^2(b, s) \vee b = s\}$ by Lemma 3.4. For item (iii), let $a \in \mathcal{L}$ code an initial interval of $\mathcal{M}_{\vec{w}}$ and let \vec{w}' be a code satisfying items (i) and (ii) of Definition 3.12. $|E(a)| = n$ for some $n \in \omega$ because $E(a)$ is an initial interval of a structure isomorphic to \mathcal{N} . $\mathcal{M}_{\vec{w}'} \models \text{PA}^-$, so by Lemma 2.5 there is an initial

interval of $\mathcal{M}_{\vec{w}'}$ of cardinality n and, by item (ii), there is an $a' \in \mathcal{L}$ coding this initial interval. Thus $|E(a)| = |E(a')| = n$, so by the finite matching property there is an $r \in \mathcal{L}$ such that $E(r)$ matches both $E(a)$ and $E(a')$. Thus \vec{w} is indeed a code for \mathcal{N} .

Now suppose that \vec{w}' is a code for \mathcal{N} in \mathcal{L} . We show that $\mathcal{M}_{\vec{w}'} \cong \mathcal{N}$. By Definition 3.12 item (i), $\mathcal{M}_{\vec{w}'} \models \text{PA}^-$. So by Lemma 2.5, it suffices to show that every initial interval $\mathcal{M}_{\vec{w}'}$ is finite. Thus let $s' \in E(w'_0)$, let $\{b' \mid R_{\ell'}^2(b', s') \vee b' = s'\}$ be the corresponding initial interval, and, by Definition 3.12 item (ii), let $a' \in \mathcal{L}$ code this initial interval. By Definition 3.12 item (iii) there is an $a \in \mathcal{L}$ coding an initial interval of $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$ such that $|E(a)| = |E(a')|$. $E(a)$ is finite, hence the initial interval $\{b' \mid R_{\ell'}^2(b', s') \vee b' = s'\}$ is finite. \square

Lemma 3.14. *Let \mathcal{L} be a distributive lattice with the finite matching property in which there exists a code \vec{w} such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$. Then $\text{Th}(\mathcal{N}) \leq_1 \text{Th}(\mathcal{L}; \leq_{\mathcal{L}})$.*

Proof. Let φ be a sentence in the language of arithmetic. Let θ be the sentence

$$\theta = \exists \vec{w} (\vec{w} \text{ is a code for } \mathcal{N} \wedge \mathcal{M}_{\vec{w}} \models \varphi)$$

in the language of lattices. By Lemma 3.13, there are codes for \mathcal{N} in \mathcal{L} and every code for \mathcal{N} in \mathcal{L} codes a structure isomorphic to \mathcal{N} . Thus $\mathcal{N} \models \varphi$ if and only if $\mathcal{L} \models \theta$. This proves $\text{Th}(\mathcal{N}) \leq_1 \text{Th}(\mathcal{L})$. We always have $\text{Th}(\mathcal{L}) \leq_1 \text{Th}(\mathcal{L}; \leq_{\mathcal{L}})$ because the lattice operations sup and inf are first-order definable from the partial order. \square

3.3. Counting quantifiers

An analysis of the quantifier complexity of our coding scheme shows that to determine the truth of existential sentences in \mathcal{N} we only need to determine the truth of Π_3^0 sentences in \mathcal{L} .

We switch to coding models of PA^+ because the axioms of PA^+ are all of the form $\forall \vec{x} \psi(\vec{x})$ for quantifier-free ψ . Here code now means a code for a structure in the language of \mathcal{N}^+ . A code is now a sequence

$$\vec{w} = (w_0, w_1, w_2, m, \ell, p, t, d, z, o)$$

(with “ d ” for “difference”) interpreted as coding the structure

$$\mathcal{M}_{\vec{w}}^+ = (E(w_0); R_{\ell}^2, R_p^3, R_t^3, R_d^3, z, o).$$

As in Definition 3.7, sentences in the language of \mathcal{N}^+ translate to formulas in the language of lattices. The new $\dot{-}$ relation is replaced by the formula defining R_d^3 in the translation. A code for a model of PA^+ is a code \vec{w} such that $\mathcal{M}_{\vec{w}}^+ \models \text{PA}^+$.

In the language of lattices, “ s is meet-irreducible” is a Π_1^0 property and “ s meets to w ” is a Σ_1^0 property, so “ $s \in E(w)$ ” is a Δ_2^0 property. Hence $R_c^2(s_0, u_1)$ and $R_c^3(s_0, u_1, v_2)$ are both Δ_2^0 properties of s_0, u_1, v_2 , and the coding parameters w_0, w_1, w_2, m , and c . Therefore, our coding translates atomic formulas in the language of \mathcal{N}^+ to Δ_2^0 properties of lattices. Every Boolean combination of Δ_2^0 properties is again a Δ_2^0 property, so our coding also translates quantifier-free formulas in the language of \mathcal{N}^+ to Δ_2^0 properties of lattices. Thus if $\varphi = \exists \vec{x} \psi(\vec{x})$ is a sentence in the language of \mathcal{N}^+ where ψ is quantifier-free, then the translation $\varphi'(\vec{w})$ may be taken to be a Σ_2^0 formula in the language of lattices. Similarly, if $\varphi = \forall \vec{x} \psi(\vec{x})$, then the translation $\varphi'(\vec{w})$ is Π_2^0 . Thus “ $\mathcal{M}_{\vec{w}}^+ \models \text{PA}^+$ ” can be expressed by a Π_2^0 formula in the language of lattices. The axioms of PA^+ need to be unnested before they are translated, but this can be done in such a way that they all remain of the form $\forall \vec{x} \psi(\vec{x})$ for quantifier-free ψ .

In a lattice, the relations $\text{sup}(x, y) = z$ and $\text{inf}(x, y) = z$ are definable by Π_1^0 formulas in the language of partial orders. This translation increases the quantifier-complexities calculated in the previous paragraph by one alternation. Existential sentences in the language of \mathcal{N}^+ translate to Σ_3^0 formulas in the language of partial orders, and universal sentences in the language of \mathcal{N}^+ translate to Π_3^0 formulas in the language of partial orders. The property “ $\mathcal{M}_{\vec{w}}^+ \models \text{PA}^+$ ” is a Π_3^0 property of \vec{w} in the language of partial orders.

Lemma 3.15. *Let \mathcal{L} be a lattice in which there exists a code \vec{w} such that $\mathcal{M}_{\vec{w}}^+ \cong \mathcal{N}^+$. Then $\Sigma_3^0\text{-Th}(\mathcal{L})$ and $\Sigma_4^0\text{-Th}(\mathcal{L}; \leq_{\mathcal{L}})$ are undecidable.*

Proof. We prove

$$\{\exists \vec{x} \psi(\vec{x}) \mid \psi \text{ is quantifier-free} \wedge \mathcal{N} \models \exists \vec{x} \psi(\vec{x})\} \leq_1 \Pi_3^0\text{-Th}(\mathcal{L}).$$

It is well-known that the problem of determining whether $\mathcal{N} \models \exists \vec{x} \psi(\vec{x})$ for quantifier-free ψ is undecidable.² Clearly $\Sigma_3^0\text{-Th}(\mathcal{L}) \equiv_1 \Pi_3^0\text{-Th}(\mathcal{L})$.

² For example, undecidability is implied by Matiyasevich’s solution to Hilbert’s tenth problem [23]. It is a standard fact in computability theory that determining whether $\mathcal{N} \models \exists \vec{x} \psi(\vec{x})$ is undecidable if ψ is allowed bounded quantifiers, but allowing bounded quantifiers in ψ increases the quantifier complexity of the translated formula.

Let $\varphi = \exists \vec{x} \psi(\vec{x})$ be a sentence in the language of arithmetic where ψ is quantifier-free. Let θ be the sentence

$$\theta = \forall \vec{w} (\mathcal{M}_{\vec{w}}^{\dot{\cdot}} \models \text{PA}^{\dot{\cdot}} \rightarrow \mathcal{M}_{\vec{w}}^{\dot{\cdot}} \models \varphi)$$

in the language of lattices. As calculated above, $\mathcal{M}_{\vec{w}}^{\dot{\cdot}} \models \text{PA}^{\dot{\cdot}}$ is a Π_2^0 property of \vec{w} , and $\mathcal{M}_{\vec{w}}^{\dot{\cdot}} \models \varphi$ is a Σ_2^0 property of \vec{w} . Thus θ is a Π_3^0 sentence in the language of lattices. We need to show $\mathcal{N} \models \varphi$ if and only if $\mathcal{L} \models \theta$. Suppose $\mathcal{N} \models \varphi$. Then $\text{PA}^{\dot{\cdot}} \vdash \varphi$ by Lemma 2.6, which implies that $\mathcal{L} \models \theta$. Suppose $\mathcal{N} \not\models \varphi$. Then by assumption there is a code \vec{w} such that $\mathcal{M}_{\vec{w}}^{\dot{\cdot}} \cong \mathcal{N}^{\dot{\cdot}}$. For this \vec{w} , $\mathcal{M}_{\vec{w}}^{\dot{\cdot}} \models \text{PA}^{\dot{\cdot}}$ but $\mathcal{M}_{\vec{w}}^{\dot{\cdot}} \not\models \varphi$, which implies $\mathcal{L} \not\models \theta$.

The proof that $\Sigma_4^0\text{-Th}(\mathcal{L}; \leq_{\mathcal{L}})$ is undecidable is the same. The above sentence θ is Π_4^0 in the language of partial orders. \square

4. Meet-irreducibles in \mathcal{E}_s and r.e. separating degrees

In this section we present facts about meet-irreducibles in \mathcal{E}_s that allow us to implement our coding in \mathcal{E}_s . We begin with a characterization of the meet-irreducibles.

Lemma 4.1 ([1] Corollary 3.5). *Let Q be a Π_1^0 class. Then $\text{deg}_s(Q)$ is meet-irreducible if and only if for every clopen $C \subseteq 2^\omega$ either $Q \cap C \equiv_s Q$ or $Q \cap C^c \equiv_s Q$.*

Proof. We prove the contrapositive in both directions. First, suppose $C \subseteq 2^\omega$ is clopen, $Q \cap C \not\equiv_s Q$, and $Q \cap C^c \not\equiv_s Q$. $Q \cap C \geq_s Q$ and $Q \cap C^c \geq_s Q$ by the identity functional, so it must be that $Q \cap C >_s Q$ and $Q \cap C^c >_s Q$. C is clopen, so there is a finite set of strings $\{\sigma_i\}_{i < n} \subseteq 2^{<\omega}$ such that $C = \bigcup_{i < n} I(\sigma_i)$. Then $0^\frown(Q \cap C) \cup 1^\frown(Q \cap C^c) \leq_s Q$ by the functional

$$f \mapsto \begin{cases} 0^\frown f & \text{if } (\exists i < n)(\sigma_i \subset f) \\ 1^\frown f & \text{otherwise.} \end{cases}$$

So $\text{deg}_s(Q \cap C) >_s \text{deg}_s(Q)$, $\text{deg}_s(Q \cap C^c) >_s \text{deg}_s(Q)$, and $\inf(\text{deg}_s(Q \cap C), \text{deg}_s(Q \cap C^c)) \leq_s \text{deg}_s(Q)$. Hence $\text{deg}_s(Q)$ is meet-reducible.

Conversely, suppose $\text{deg}_s(Q)$ is meet-reducible, and let X and Y be Π_1^0 classes such that $X >_s Q$, $Y >_s Q$, and $Q \equiv_s 0^\frown X \cup 1^\frown Y$. Let Φ be such that $\Phi(Q) \subseteq 0^\frown X \cup 1^\frown Y$. Consider the set $\widehat{X} = \{f \in Q \mid \Phi(f)(0) = 0\}$. $\Phi(f)$ is total for all $f \in Q$, so we can write $\widehat{X} = Q \cap \{f \in 2^\omega \mid \Phi(f)(0) \neq 1\}$ (where $\Phi(f)(0) \neq 1$ includes the possibility that $\Phi(f)(0)$ diverges), which is the intersection of two closed subsets of 2^ω . Hence \widehat{X} is compact. Let $\Sigma = \{\sigma \in 2^{<\omega} \mid \Phi(\sigma)(0) = 0\}$. Then $\widehat{X} \subseteq \bigcup_{\sigma \in \Sigma} I(\sigma)$, so by compactness there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\widehat{X} \subseteq \bigcup_{\sigma \in \Sigma_0} I(\sigma)$. Let $C = \bigcup_{\sigma \in \Sigma_0} I(\sigma)$ be this clopen set. Φ witnesses that $Q \cap C \geq_s 0^\frown X$ and that $Q \cap C^c \geq_s 1^\frown Y$. As $0^\frown X \equiv_s X >_s Q$ and $1^\frown Y \equiv_s Y >_s Q$, we have the desired clopen set $C \subseteq 2^\omega$ such that $Q \cap C \not\equiv_s Q$ and $Q \cap C^c \not\equiv_s Q$. \square

Degrees of r.e. separating classes are the main examples of meet-irreducibles in \mathcal{E}_s .

Definition 4.2. For $A, B \subseteq \omega$, define

$$S(A, B) = \{f \in 2^\omega \mid \forall n((n \in A \rightarrow f(n) = 1) \wedge (n \in B \rightarrow f(n) = 0))\}.$$

An $f \in S(A, B)$ is said to separate A from B . $S \subseteq 2^\omega$ is an r.e. separating class if and only if there are disjoint r.e. sets $A, B \subseteq \omega$ such that $S = S(A, B)$.

From the definition, an r.e. separating class is always a Π_1^0 class. An $\mathfrak{s} \in \mathcal{E}_s$ is an r.e. separating degree if and only if $\mathfrak{s} = \text{deg}_s(S)$ for an r.e. separating class S .

Lemma 4.3 ([7] Lemma 6). *If S is an r.e. separating class and $C \subseteq 2^\omega$ is a clopen set such that $S \cap C \neq \emptyset$, then $S \cap C \equiv_s S$.*

Proof. Let $S = S(A, B)$ be an r.e. separating class and let $C \subseteq 2^\omega$ be a clopen set such that $S \cap C \neq \emptyset$. $S \leq_s S \cap C$ by the identity functional. To see $S \geq_s S \cap C$, let $I(\sigma)$ be such that $I(\sigma) \subseteq C$ and $S \cap I(\sigma) \neq \emptyset$. For any $f \in 2^\omega$, let f_σ be the function obtained from f by replacing the initial segment of f of length $|\sigma|$ by σ :

$$f_\sigma(n) = \begin{cases} \sigma(n) & \text{if } n < |\sigma| \\ f(n) & \text{if } n \geq |\sigma|. \end{cases}$$

The condition $S \cap I(\sigma) \neq \emptyset$ implies that σ separates $\{n \in A \mid n < |\sigma|\}$ from $\{n \in B \mid n < |\sigma|\}$. Thus if f separates A from B , then so does f_σ . Therefore the functional $f \mapsto f_\sigma$ witnesses $S \geq_s S \cap C$. \square

Lemmas 4.1 and 4.3 imply that every r.e. separating degree is meet-irreducible. It is important to note (as in [7]) that the r.e. separating classes are closed under \otimes and consequently that the r.e. separating degrees are closed under sup: if $S(A_0, B_0)$ and $S(A_1, B_1)$ are r.e. separating classes then $S(A_0, B_0) \otimes S(A_1, B_1) = S(A_0 \oplus A_1, B_0 \oplus B_1)$. Thus the sup of two r.e. separating degrees is meet-irreducible. In fact, the sup of any r.e. separating degree and any meet-irreducible degree is again meet-irreducible.

Lemma 4.4. *Let $\mathfrak{q} \in \mathcal{E}_s$ be meet-irreducible and let $\mathfrak{s} \in \mathcal{E}_s$ be an r.e. separating degree. Then $\text{sup}(\mathfrak{q}, \mathfrak{s})$ is meet-irreducible.*

Proof. Suppose $\sup(\mathbf{q}, \mathbf{s}) \geq_s \inf(\mathbf{x}, \mathbf{y})$ for some $\mathbf{x}, \mathbf{y} \in \mathcal{E}_s$. We show $\sup(\mathbf{q}, \mathbf{s}) \geq_s \mathbf{x}$ or $\sup(\mathbf{q}, \mathbf{s}) \geq_s \mathbf{y}$. Let Q, X , and Y be Π_1^0 classes such that $\deg_s(Q) = \mathbf{q}$, $\deg_s(X) = \mathbf{x}$, and $\deg_s(Y) = \mathbf{y}$, and let S be an r.e. separating class such that $\deg_s(S) = \mathbf{s}$. Let Φ be such that $\Phi(Q \otimes S) \subseteq 0^{\wedge}X \cup 1^{\wedge}Y$. By compactness, choose a $\sigma \in 2^{<\omega}$ such that $S \cap I(\sigma) \neq \emptyset$ and an $n \in \omega$ such that

$$(\forall \tau \in 2^n)((\exists f \in Q)(\tau \subset f) \rightarrow \Phi(\tau \oplus \sigma)(0) \downarrow).$$

Let $C = \bigcup \{I(\tau) \mid \tau \in 2^n \wedge \Phi(\tau \oplus \sigma)(0) = 0\}$. Then C is clopen, and Φ witnesses that $(Q \cap C) \otimes (S \cap I(\sigma)) \geq_s 0^{\wedge}X \equiv_s X$ and that $(Q \cap C^c) \otimes (S \cap I(\sigma)) \geq_s 1^{\wedge}Y \equiv_s Y$. Since $S \cap I(\sigma) \equiv_s S$ by Lemma 4.3 and either $Q \cap C \equiv_s Q$ or $Q \cap C^c \equiv_s Q$ by Lemma 4.1, we have either $Q \otimes S \geq_s X$ or $Q \otimes S \geq_s Y$ as desired. \square

Our proof that \mathcal{E}_s has the finite matching property uses the following lemma of Cole and Kihara. It is the main tool in their proof that the Σ_2^0 -theory of \mathcal{E}_s as a partial order is decidable.

Lemma 4.5 ([10] Lemma 1). *Let $\{\mathbf{q}_i\}_{i < n} \subseteq \mathcal{E}_s$ and let $m \in \omega$. Then there is a set $\{\mathbf{r}_i\}_{i < m} \subseteq \mathcal{E}_s$ such that*

$$\begin{aligned} & (\forall I \subseteq m) (\forall J, K \subseteq n) \\ & \left(J \cap K = \emptyset \wedge \sup_{i \in J} \mathbf{q}_i \not\geq_s \inf_{i \in K} \mathbf{q}_i \rightarrow \sup_{i \in J} (\sup_{i \in I} \mathbf{q}_i, \sup_{i \in I} \mathbf{r}_i) \not\geq_s \inf_{i \in K} (\inf_{i \in I} \mathbf{q}_i, \inf_{i \in I} \mathbf{r}_i) \right) \end{aligned}$$

(where $\sup_{i \in \emptyset} \mathbf{x}_i = \mathbf{0}_s$ and $\inf_{i \in \emptyset} \mathbf{x}_i = \mathbf{1}_s$).

Cole and Kihara note that the $\{\mathbf{r}_i\}_{i < m}$ that they construct in Lemma 4.5 are all r.e. separating degrees. Their proof of Lemma 4.5 is an elaboration of Cenzer and Hinman's proof that \mathcal{E}_s is dense [7]. Cenzer and Hinman prove that if $\mathbf{p}, \mathbf{q} \in \mathcal{E}_s$ are such that $\mathbf{q} \not\geq_s \mathbf{p}$, then there is an r.e. separating degree $\mathbf{r} \in \mathcal{E}_s$ such that $\inf(\mathbf{q}, \mathbf{r}) \not\geq_s \mathbf{p}$ and $\mathbf{q} \not\geq_s \sup(\mathbf{p}, \mathbf{r})$. Thus if $\mathbf{p} <_s \mathbf{q}$, then $\mathbf{p} <_s \inf(\sup(\mathbf{p}, \mathbf{r}), \mathbf{q}) <_s \mathbf{q}$, yielding density. To make Lemma 4.5 somewhat easier to read and apply, we note that we only need the following special case.

Lemma 4.6. *Let $\{\mathbf{q}_i\}_{i < n} \subseteq \mathcal{E}_s \setminus \{\mathbf{1}_s\}$ and let $m \in \omega$. Then there is a set of r.e. separating degrees $\{\mathbf{r}_i\}_{i < m} \subseteq \mathcal{E}_s$ such that*

- (i) $(\forall i, i' < n)(\forall j < m)(\mathbf{q}_i \not\geq_s \mathbf{q}_{i'} \rightarrow \sup(\mathbf{q}_i, \mathbf{r}_j) \not\geq_s \mathbf{q}_{i'})$ and
- (ii) $(\forall i < n)(\forall j, j' < m)(j \neq j' \rightarrow \sup(\mathbf{q}_i, \mathbf{r}_j) \not\geq_s \mathbf{r}_{j'})$.

We can now show that \mathcal{E}_s has the finite matching property.

Lemma 4.7. \mathcal{E}_s has the finite matching property. That is, if $\mathbf{q}, \mathbf{q}' \in \mathcal{E}_s$ are such that $|E(\mathbf{q})| = |E(\mathbf{q}')| = n$ for some $n \in \omega$, then there is an $\mathbf{r} \in \mathcal{E}_s$ such that $E(\mathbf{r})$ matches both $E(\mathbf{q})$ and $E(\mathbf{q}')$.

Proof. If $n = 0$, then let $\mathbf{r} = \mathbf{q}$. Any degree \mathbf{z} vacuously witnesses that $E(\mathbf{r})$ matches $E(\mathbf{q})$ and that $E(\mathbf{r})$ matches $E(\mathbf{q}')$. So suppose $n > 0$, let $E(\mathbf{q}) = \{\mathbf{q}_i\}_{i < n}$, and let $E(\mathbf{q}') = \{\mathbf{q}'_i\}_{i < n}$. Apply Lemma 4.6 to $\{\mathbf{q}_i\}_{i < n} \cup \{\mathbf{q}'_i\}_{i < n}$ with $m = n$, noting that $\{\mathbf{q}_i\}_{i < n}$ and $\{\mathbf{q}'_i\}_{i < n}$ are both antichains by Lemma 3.3, to get r.e. separating degrees $\{\mathbf{r}_i\}_{i < n}$ such that

- (i) $\sup(\mathbf{q}_i, \mathbf{r}_j) \not\geq_s \mathbf{q}_k$ and $\sup(\mathbf{q}'_i, \mathbf{r}_j) \not\geq_s \mathbf{q}'_k$ whenever $i, j, k < n$ are such that $i \neq k$, and
- (ii) $\sup(\mathbf{q}_i, \mathbf{r}_j) \not\geq_s \mathbf{r}_k$ and $\sup(\mathbf{q}'_i, \mathbf{r}_j) \not\geq_s \mathbf{r}_k$ whenever $i, j, k < n$ are such that $j \neq k$.

(Lemma 4.6 applies because, by definition, $\mathbf{1}_s$ does not meet to any degree and so cannot be in $E(\mathbf{q})$ or $E(\mathbf{q}')$.)

Put $\mathbf{r} = \inf_{i < n} \mathbf{r}_i$, $\mathbf{z} = \inf_{i < n} \sup(\mathbf{q}_i, \mathbf{r}_i)$, and $\mathbf{z}' = \inf_{i < n} \sup(\mathbf{q}'_i, \mathbf{r}_i)$. We show that \mathbf{z} witnesses that $E(\mathbf{r})$ matches $E(\mathbf{q})$. The proof that \mathbf{z}' witnesses that $E(\mathbf{r})$ matches $E(\mathbf{q}')$ is similar. Item (ii) implies that $\{\mathbf{r}_i\}_{i < n}$ and $\{\sup(\mathbf{q}_i, \mathbf{r}_i)\}_{i < n}$ are both antichains. Lemma 4.4 implies that $\sup(\mathbf{q}_i, \mathbf{r}_i)$ is meet-irreducible for each $i < n$. Therefore $E(\mathbf{r}) = \{\mathbf{r}_i\}_{i < n}$ and $E(\mathbf{z}) = \{\sup(\mathbf{q}_i, \mathbf{r}_i)\}_{i < n}$ by Lemma 3.4. Suppose $\sup(\mathbf{q}_i, \mathbf{r}_j) \geq_s \mathbf{z}$ for some $i, j < n$. Then $\sup(\mathbf{q}_i, \mathbf{r}_j) \geq_s \sup(\mathbf{q}_k, \mathbf{r}_k)$ for some $k < n$ because $\sup(\mathbf{q}_i, \mathbf{r}_j)$ is meet-irreducible by Lemma 4.4. Item (i) implies that $i = k$, and item (ii) implies that $j = k$. Thus for each $i < n$, \mathbf{r}_i is the unique $\mathbf{y} \in E(\mathbf{r})$ such that $\sup(\mathbf{q}_i, \mathbf{y}) \in E(\mathbf{z})$, and \mathbf{q}_i is the unique $\mathbf{y} \in E(\mathbf{q})$ such that $\sup(\mathbf{r}_i, \mathbf{y}) \in E(\mathbf{z})$. Thus \mathbf{z} witnesses that $E(\mathbf{r})$ matches $E(\mathbf{q})$. \square

We need one last fact about the r.e. separating classes to implement our coding in \mathcal{E}_s . Let $\{f_n\}_{n \in \omega} \subseteq 2^\omega$ be a sequence of functions, and let $m \in \omega$. Define $\bigoplus_{n \in \omega} f_n$ and $\bigoplus_{n \in \omega \setminus \{m\}} f_n$ by

$$\begin{aligned} & \left(\bigoplus_{n \in \omega} f_n \right) ((i, j)) = f_i(j) \text{ and} \\ & \left(\bigoplus_{n \in \omega \setminus \{m\}} f_n \right) ((i, j)) = \begin{cases} f_i(j) & \text{if } i \neq m \\ 0 & \text{if } i = m. \end{cases} \end{aligned}$$

Definition 4.8. A sequence of functions $\{f_n\}_{n \in \omega} \subseteq 2^\omega$ is *strongly independent* if and only if $\forall m (f_m \not\leq_T \bigoplus_{n \in \omega \setminus \{m\}} f_n)$. A sequence of Π_1^0 classes $\{X_n\}_{n \in \omega}$ is *strongly independent* if and only if $\{f_n\}_{n \in \omega}$ is strongly independent whenever $\forall n (f_n \in X_n)$.

Lemma 4.9 ([17] Theorem 4.1). *There is a recursive sequence $\{S_n\}_{n \in \omega}$ r.e. separating classes that is strongly independent.*

5. Interpreting true arithmetic in \mathcal{E}_s

In this section we prove that $\text{Th}(\mathcal{E}_s; \leq_s) \equiv_1 \text{Th}(\mathcal{M})$ and that $\Pi_3^0\text{-Th}(\mathcal{E}_s)$ and $\Pi_4^0\text{-Th}(\mathcal{E}_s; \leq_s)$ are undecidable. By Lemmas 3.14, 3.15 and 4.7 it suffices to find a code \vec{w} in \mathcal{E}_s such that $\mathcal{M}_{\vec{w}} \cong \mathcal{M}$. This section is analogous to [32] Section 5, in which it is proved that the first-order theories of $\mathcal{D}_{s,cl}$ and $\mathcal{D}_{s,cl}^{01}$ are recursively isomorphic to true second-order arithmetic.

Definition 5.1. Let Q be a Π_1^0 class with no recursive member. Let A be an infinite recursive set, and let $\{\sigma_n\}_{n \in A}$ be a recursive sequence of pairwise incomparable strings such that $\bigcup_{n \in A} I(\sigma_n) = 2^\omega \setminus Q$ (for example, let T be a recursive tree such that $Q = [T]$ and let $\{\sigma_n\}_{n \in A}$ be the strings $\sigma \notin T$ of minimal length). Let $\{S_n\}_{n \in A}$ be an infinite recursive sequence of Π_1^0 classes. Define $\text{spine}(Q, \{S_n\}_{n \in A})$ to be the Π_1^0 class

$$\text{spine}(Q, \{S_n\}_{n \in A}) = Q \cup \bigcup_{n \in A} \sigma_n \hat{\wedge} S_n.$$

The next lemma gives the analog of Lemma 3.4 for spines.

Lemma 5.2. Let Q be a Π_1^0 class with no recursive member. Let $\{S_n\}_{n \in A}$ be an infinite recursive sequence of r.e. separating classes (indexed by a recursive set A) that is an antichain and is such that $Q \not\leq_s S_n$ for all $n \in A$. Let $\mathbf{w} = \text{deg}_s(\text{spine}(Q, \{S_n\}_{n \in A}))$.

- (i) If $\mathbf{x} \in \mathcal{E}_s$ meets to \mathbf{w} , then $\mathbf{x} \leq_s \text{deg}_s(S_n)$ for some $n \in A$.
- (ii) $E(\mathbf{w}) = \{\text{deg}_s(S_n) \mid n \in A\}$.

Proof. Let $W = \text{spine}(Q, \{S_n\}_{n \in A})$.

(i) Let $\mathbf{x} \in \mathcal{E}_s$ be such that \mathbf{x} meets to \mathbf{w} . Suppose for a contradiction that $\mathbf{x} \not\leq_s \text{deg}_s(S_n)$ for all $n \in A$. Let X be a Π_1^0 class such that $\mathbf{x} = \text{deg}_s(X)$, and let Y be a Π_1^0 class such that $\text{deg}_s(Y)$ witnesses that \mathbf{x} meets to \mathbf{w} . That is, $Y >_s W$ and $W \equiv_s 0 \hat{\wedge} X \cup 1 \hat{\wedge} Y$. Let Φ be such that $\Phi(Q \cup \bigcup_{n \in A} \sigma_n \hat{\wedge} S_n) \subseteq 0 \hat{\wedge} X \cup 1 \hat{\wedge} Y$.

Claim.

- (a) $\Phi(\sigma_n \hat{\wedge} S_n) \subseteq 1 \hat{\wedge} Y$ for all $n \in A$ and
- (b) $\Phi(Q) \subseteq 1 \hat{\wedge} Y$.

Proof of claim. If item (a) fails, then for some $n \in A$ there is a clopen $C \subseteq 2^\omega$ such that $(\sigma_n \hat{\wedge} S_n) \cap C \neq \emptyset$ and $\Phi((\sigma_n \hat{\wedge} S_n) \cap C) \subseteq 0 \hat{\wedge} X$. So $(\sigma_n \hat{\wedge} S_n) \cap C \geq_s 0 \hat{\wedge} X \equiv_s X$. The class $\sigma_n \hat{\wedge} S_n$ is an r.e. separating class because S_n is, so $(\sigma_n \hat{\wedge} S_n) \cap C \equiv_s \sigma_n \hat{\wedge} S_n \equiv_s S_n$, where the first equivalence is by Lemma 4.3. Thus the contradiction $X \leq_s S_n$.

If item (b) fails, then there is an $f \in Q$ and a $\sigma \subset f$ such that $\Phi(\sigma)(0) \downarrow = 0$. Since $I(\sigma) \not\subseteq Q$, there is an $n \in A$ such that $\sigma_n \supseteq \sigma$. Hence $\Phi(\sigma_n \hat{\wedge} S_n) \not\subseteq 1 \hat{\wedge} Y$, contradicting item (a). \square

The claim shows that $\Phi(Q \cup \bigcup_{n \in \omega} \sigma_n \hat{\wedge} S_n) \subseteq 1 \hat{\wedge} Y$. Thus $Y \leq_s W$, which contradicts $Y >_s W$.

(ii) Let $n \in A$. To see that $\text{deg}_s(S_n) \in E(\mathbf{w})$, let $Y = Q \cup \bigcup_{i \in A \setminus \{n\}} \sigma_i \hat{\wedge} S_i$.

Claim. $S_n \not\leq_s Y$

Proof of claim. Suppose for a contradiction that Φ is such that $\Phi(S_n) \subseteq Y$. If there is an $i \in A \setminus \{n\}$ such that $\Phi(S_n) \cap (\sigma_i \hat{\wedge} S_i) \neq \emptyset$, then there is a clopen $C \subseteq 2^\omega$ such that $S_n \cap C \neq \emptyset$ and $\Phi(S_n \cap C) \subseteq \sigma_i \hat{\wedge} S_i$. Hence $S_n \equiv_s S_n \cap C$ by Lemma 4.3, and $S_n \cap C \geq_s \sigma_i \hat{\wedge} S_i \equiv_s S_i$. This contradicts that $\{S_n\}_{n \in A}$ is an antichain. Thus $\Phi(S_n) \cap (\sigma_i \hat{\wedge} S_i) = \emptyset$ for all $n \in A$. Therefore $\Phi(S_n) \subseteq Q$. This contradicts $Q \not\leq_s S_n$. \square

It is easy to check that $W \equiv_s 0 \hat{\wedge} S_n \cup 1 \hat{\wedge} Y$, so, by the claim, $\text{deg}_s(Y)$ witnesses that $\text{deg}_s(S_n)$ meets to \mathbf{w} . The degree $\text{deg}_s(S_n)$ is meet-irreducible because it is an r.e. separating degree. Thus $\text{deg}_s(S_n) \in E(\mathbf{w})$.

We have shown that $\{\text{deg}_s(S_n) \mid n \in A\} \subseteq E(\mathbf{w})$. To see equality, let $\mathbf{x} \in E(\mathbf{w})$. By item (i), $\mathbf{x} \leq_s \text{deg}_s(S_n)$ for some $n \in A$. $E(\mathbf{w})$ is an antichain by Lemma 3.3 and $\text{deg}_s(S_n) \in E(\mathbf{w})$, so it must be that $\mathbf{x} = \text{deg}_s(S_n)$. \square

We now have all the ingredients to find a code for \mathcal{M} in \mathcal{E}_s .

Lemma 5.3. There is a code \vec{w} in \mathcal{E}_s such that $\mathcal{M}_{\vec{w}} \cong \mathcal{M}$.

Proof. By Lemma 4.9, let Q be an r.e. separating class and let $\{S_{0,n}\}_{n \in \omega}$, $\{S_{1,n}\}_{n \in \omega}$, and $\{S_{2,n}\}_{n \in \omega}$ be recursive sequences of r.e. separating classes such that $\{Q\} \cup \{S_{0,n}\}_{n \in \omega} \cup \{S_{1,n}\}_{n \in \omega} \cup \{S_{2,n}\}_{n \in \omega}$ is strongly independent. Then let

$$\begin{aligned} \mathbf{w}_0 = \text{deg}_s(W_0) & \text{ for } W_0 = \text{spine}(Q, \{S_{0,n}\}_{n \in \omega}), \\ \mathbf{w}_1 = \text{deg}_s(W_1) & \text{ for } W_1 = \text{spine}(Q, \{S_{1,n}\}_{n \in \omega}), \\ \mathbf{w}_2 = \text{deg}_s(W_2) & \text{ for } W_2 = \text{spine}(Q, \{S_{2,n}\}_{n \in \omega}), \\ \mathbf{m} = \text{deg}_s(M) & \text{ for } M = \text{spine}(Q, \{S_{0,n} \otimes S_{1,n}\}_{n \in \omega} \cup \{S_{0,n} \otimes S_{2,n}\}_{n \in \omega}), \\ \ell = \text{deg}_s(L) & \text{ for } L = \text{spine}(Q, \{S_{0,i} \otimes S_{1,j} \mid i < j\}), \\ \mathbf{p} = \text{deg}_s(P) & \text{ for } P = \text{spine}(Q, \{S_{0,i} \otimes S_{1,j} \otimes S_{2,k} \mid i + j = k\}), \\ \mathbf{t} = \text{deg}_s(T) & \text{ for } T = \text{spine}(Q, \{S_{0,i} \otimes S_{1,j} \otimes S_{2,k} \mid i \times j = k\}), \\ \mathbf{d} = \text{deg}_s(D) & \text{ for } D = \text{spine}(Q, \{S_{0,i} \otimes S_{1,j} \otimes S_{2,k} \mid i \dot{-} j = k\}), \\ \mathbf{z} = \text{deg}_s(S_{0,0}), & \text{ and} \\ \mathbf{o} = \text{deg}_s(S_{0,1}). & \end{aligned}$$

By Lemma 5.2 item (ii), $E(\mathbf{w}_0) = \{\text{deg}_s(S_{0,n})\}_{n \in \omega}$. The map $\text{deg}_s(S_{0,n}) \mapsto n$ is the isomorphism witnessing $\mathcal{M}_{\vec{w}} \cong \mathcal{N}^{\dot{-}}$. Clearly $\mathbf{z} \mapsto 0$ and $\mathbf{o} \mapsto 1$. We show that the map preserves $<$. The proofs that the map preserves $+$, \times , and $\dot{-}$ are similar. Let $i, j \in \omega$. If $i < j$, then $\text{deg}_s(S_{1,j})$ meets to \mathbf{w}_1 by Lemma 5.2 item (ii), and it is easy to see that $\text{sup}(\text{deg}_s(S_{0,j}), \text{deg}_s(S_{1,j})) \geq_s \mathbf{m}$ and that $\text{sup}(\text{deg}_s(S_{0,i}), \text{deg}_s(S_{1,j})) \geq_s \ell$. Thus $R_s^2(\text{deg}_s(S_{0,i}), \text{deg}_s(S_{0,j}))$. Conversely, suppose that $R_s^2(\text{deg}_s(S_{0,i}), \text{deg}_s(S_{0,j}))$. Let $\mathbf{u}_1 \in \mathcal{E}_s$ be such that \mathbf{u}_1 meets to \mathbf{w}_1 , $\text{sup}(\text{deg}_s(S_{0,j}), \mathbf{u}_1) \geq_s \mathbf{m}$, and $\text{sup}(\text{deg}_s(S_{0,i}), \mathbf{u}_1) \geq_s \ell$. Since \mathbf{u}_1 meets to \mathbf{w}_1 , it must be that $\mathbf{u}_1 \leq_s \text{deg}_s(S_{1,k})$ for some $k \in \omega$ by Lemma 5.2 item (i). Thus $\text{sup}(\text{deg}_s(S_{0,j}), \text{deg}_s(S_{1,k})) \geq_s \mathbf{m}$. However, if $k \neq j$, then no member of $S_{0,j} \otimes S_{1,k}$ computes any member of M by strong independence. Thus $\mathbf{u}_1 \leq_s \text{deg}_s(S_{1,j})$, which implies that $\text{sup}(\text{deg}_s(S_{0,i}), \text{deg}_s(S_{1,j})) \geq_s \ell$. Again by strong independence, if $i \neq j$, then no member of $S_{0,i} \otimes S_{1,j}$ computes any member of L . Hence $i < j$. \square

Higuchi also used spines of recursive sequences of independent r.e. separating classes to prove that \mathcal{E}_s is not a Brouwer algebra [13].

Theorem 5.4. $\text{Th}(\mathcal{E}_s; \leq_s) \equiv_1 \text{Th}(\mathcal{N})$.

Proof. We first prove $\text{Th}(\mathcal{E}_s; \leq_s) \leq_1 \text{Th}(\mathcal{N})$. Let $\{T_e\}_{e \in \omega}$ be a uniformly recursive sequence of trees representing all Π_1^0 classes as in Lemma 2.3. Given a sentence θ in the language of partial orders, produce an equivalent sentence in the language of partial orders by replacing every atomic formula $x = y$ by the formula $x \leq y \wedge y \leq x$. Then produce a sentence φ in the language of arithmetic by replacing every atomic formula $x \leq y$ by the Σ_3^0 formula from Lemma 2.2 expressing $[T_x] \leq_s [T_y]$. Then $\mathcal{E}_s \models \theta$ if and only if $\mathcal{N} \models \varphi$.

For $\text{Th}(\mathcal{N}) \leq_1 \text{Th}(\mathcal{E}_s; \leq_s)$, by Lemma 5.3 let \vec{w} be a code in \mathcal{E}_s such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}^{\dot{-}}$. Removing the degree \mathbf{d} from the code \vec{w} gives a code \vec{v} such that $\mathcal{M}_{\vec{v}} \cong \mathcal{N}$. \mathcal{E}_s has the finite matching property by Lemma 4.7, thus $\text{Th}(\mathcal{N}) \leq_1 \text{Th}(\mathcal{E}_s; \leq_s)$ by Lemma 3.14. \square

Theorem 5.5. $\Sigma_3^0\text{-Th}(\mathcal{E}_s)$ and $\Sigma_4^0\text{-Th}(\mathcal{E}_s; \leq_s)$ are undecidable.

Proof. There is a code \vec{w} in \mathcal{E}_s such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}^{\dot{-}}$ by Lemma 5.3. The results then follow from Lemma 3.15. \square

Fragments of first-order theories were not considered in [32]. The refined coding scheme used here also shows that $\Sigma_3^0\text{-Th}(\mathcal{L})$ and $\Sigma_4^0\text{-Th}(\mathcal{L}; \leq_{\mathcal{L}})$ are undecidable for $\mathcal{L} = \mathcal{D}_s, \mathcal{D}_w, \mathcal{D}_{s,cl}, \mathcal{D}_{w,cl}, \mathcal{D}_{s,cl}^{01}$, and $\mathcal{D}_{w,cl}^{01}$.

6. The degree of \mathcal{E}_s is $\mathbf{0}''$

In this section, we consider the complexities of presentations of \mathcal{E}_s .

Definition 6.1. A presentation of \mathcal{E}_s as a partial order consists of a relation $\leq_{\mathcal{P}} \subseteq \omega \times \omega$ such that the structure $\mathcal{P} = (\omega; \leq_{\mathcal{P}})$ is isomorphic to $(\mathcal{E}_s; \leq_s)$. A presentation of \mathcal{E}_s as a lattice consists of a relation $\leq_{\mathcal{L}} \subseteq \omega \times \omega$ and functions $\text{sup}_{\mathcal{L}}: \omega \times \omega \rightarrow \omega$ and $\text{inf}_{\mathcal{L}}: \omega \times \omega \rightarrow \omega$ such that the structure $\mathcal{L} = (\omega; \leq_{\mathcal{L}}, \text{sup}_{\mathcal{L}}, \text{inf}_{\mathcal{L}})$ is isomorphic to \mathcal{E}_s .

We measure the complexities of presentations by their Turing degrees.

Definition 6.2. The degree of a presentation \mathcal{P} of \mathcal{E}_s as a partial order is $\text{deg}_T(\mathcal{P}) = \text{deg}_T(\leq_{\mathcal{P}})$. The degree of a presentation \mathcal{L} of \mathcal{E}_s as a lattice is $\text{deg}_T(\mathcal{L}) = \text{deg}_T(\leq_{\mathcal{L}} \oplus \text{sup}_{\mathcal{L}} \oplus \text{inf}_{\mathcal{L}})$.

Equivalently, the degree of a presentation is the Turing degree of its atomic diagram, suitably Gödel numbered.

Lemma 6.3. There is a presentation \mathcal{L} of \mathcal{E}_s as a lattice with $\text{deg}_T(\mathcal{L}) \leq_T \mathbf{0}''$.

Proof. Let $\{T_e\}_{e \in \omega}$ be a uniformly recursive sequence of trees representing all Π_1^0 classes as in Lemma 2.3. Since $[T_i] \leq_s [T_j]$ is a Σ_3^0 property of $\langle i, j \rangle$ by Lemma 2.2, we can use $\mathbf{0}''$ to make a new sequence of trees $\{T'_e\}_{e \in \omega}$ such that $\{[T'_e]\}_{e \in \omega}$ contains exactly one representative for each degree in \mathcal{E}_s . Inductively, let T'_e be T_i for the least $i \in \omega$ such that $(\forall j < e)([T_i] \not\leq_s [T'_j])$. Again using $\mathbf{0}''$, for $i, j \in \omega$ define $i \leq_{\mathcal{L}} j$ if and only if $[T'_i] \leq_s [T'_j]$, define $\text{sup}_{\mathcal{L}}(i, j)$ to be the $k \in \omega$ such that $[T'_k] \equiv_s [T'_i \otimes T'_j]$, and define $\text{inf}_{\mathcal{L}}(i, j)$ to be the $k \in \omega$ such that $[T'_k] \equiv_s [0 \dot{\wedge} T'_i \cup 1 \dot{\wedge} T'_j]$. Then $\mathcal{L} \cong \mathcal{E}_s$ and $\text{deg}_T(\mathcal{L}) \leq_T \mathbf{0}''$. \square

We prepare to show that every presentation of \mathcal{E}_s as a lattice computes $\mathbf{0}''$. Let $\{X_n\}_{n \in \omega}$ be a recursive sequence of Π_1^0 classes, and let $m \in \omega$. Define $\bigotimes_{n \in \omega} X_n$ and $\bigotimes_{n \in \omega \setminus \{m\}} X_n$ by

$$\bigotimes_{n \in \omega} X_n = \left\{ \bigoplus_{n \in \omega} f_n \mid \forall n (f_n \in X_n) \right\} \text{ and}$$

$$\bigotimes_{n \in \omega \setminus \{m\}} X_n = \left\{ \bigoplus_{n \in \omega \setminus \{m\}} f_n \mid \forall n (n \neq m \rightarrow f_n \in X_n) \right\}.$$

The predicates $\forall n(f_n \in X_n)$ and $\forall n(n \neq m \rightarrow f_n \in X_n)$ are Π_1^0 because the sequence $\{X_n\}_{n \in \omega}$ is recursive. Hence $\bigotimes_{n \in \omega} X_n$ and $\bigotimes_{n \in \omega \setminus \{m\}} X_n$ are Π_1^0 classes. If $\{S(A_n, B_n)\}_{n \in \omega}$ is a recursive sequence of r.e. separating classes, then one checks that

$$\begin{aligned} \bigotimes_{n \in \omega} S(A_n, B_n) &= S\left(\bigoplus_{n \in \omega} A_n, \bigoplus_{n \in \omega} B_n\right) \text{ and} \\ \bigotimes_{n \in \omega \setminus \{m\}} S(A_n, B_n) &= S\left(\bigoplus_{n \in \omega \setminus \{m\}} A_n, \left(\bigoplus_{n \in \omega \setminus \{m\}} B_n\right) \cup \{(m, k) \mid k \in \omega\}\right). \end{aligned}$$

These two Π_1^0 classes are in fact r.e. separating classes because any Π_1^0 class that is a separating class must be an r.e. separating class. If T is a recursive tree such that $[T] = S(A, B)$ for $A, B \subseteq \omega$, then $A = \{n \mid (\exists s > n)(\forall \sigma \in 2^s)(\sigma \in T \rightarrow \sigma(n) = 1)\}$ and $B = \{n \mid (\exists s > n)(\forall \sigma \in 2^s)(\sigma \in T \rightarrow \sigma(n) = 0)\}$, both of which are r.e.

Lemma 6.4. *Let Q be an r.e. separating class, and let $\varphi(e, m, k, \ell)$ be a recursive predicate. Then there is a recursive sequence of Π_1^0 classes $\{X_{(e,m)}\}_{(e,m) \in \omega}$ such that for all $e, m \in \omega$*

$$\text{deg}_s(X_{(e,m)}) = \begin{cases} \mathbf{0}_s & \text{if } \forall k \exists \ell \varphi(e, m, k, \ell) \\ \text{deg}_s(Q) & \text{if } \exists k \forall \ell \neg \varphi(e, m, k, \ell). \end{cases}$$

Proof. Let A and B be disjoint r.e. sets such that $Q = S(A, B)$. Let $\{A_s\}_{s \in \omega}$ and $\{B_s\}_{s \in \omega}$ be recursive stage enumerations of A and B respectively. For $e, m \in \omega$, let $X_{(e,m)}$ be the r.e. separating class $X_{(e,m)} = S(C_{(e,m)}, D_{(e,m)})$ where

$$\begin{aligned} C_{(e,m)} &= \{(k, x) \mid \exists s(x \in A_s \wedge (\forall \ell < s)(\neg \varphi(e, m, k, \ell)))\} \text{ and} \\ D_{(e,m)} &= \{(k, x) \mid \exists s(x \in B_s \wedge (\forall \ell < s)(\neg \varphi(e, m, k, \ell)))\}. \end{aligned}$$

For all $k \in \omega$, the k th column of $C_{(e,m)}$ is a subset of A , and the k th column of $D_{(e,m)}$ is a subset of B . Thus $C_{(e,m)}$ and $D_{(e,m)}$ are disjoint. The sequences $\{C_{(e,m)}\}_{(e,m) \in \omega}$ and $\{D_{(e,m)}\}_{(e,m) \in \omega}$ are uniformly r.e., which implies that the sequence $\{X_{(e,m)}\}_{(e,m) \in \omega}$ is a recursive sequence of Π_1^0 classes.

To see that $X_{(e,m)}$ has the desired degree, first suppose that $\forall k \exists \ell \varphi(e, m, k, \ell)$. In this case, the set $C_{(e,m)}$ is recursive. To determine if $(k, x) \in C_{(e,m)}$, search for the least ℓ such that $\varphi(e, m, k, \ell)$, which must exist by assumption. Once ℓ is found, enumerate A up to stage ℓ . Then $(k, x) \in C_{(e,m)}$ if and only if $x \in A_\ell$. $X_{(e,m)}$ contains the characteristic function of $C_{(e,m)}$, which we have just shown is recursive, so $\text{deg}_s(X_{(e,m)}) = \mathbf{0}_s$. On the other hand, if $\exists k \forall \ell \neg \varphi(e, m, k, \ell)$, then fix a witnessing k . In this case, the k th column of $C_{(e,m)}$ is A , and the k th column of $D_{(e,m)}$ is B . Given $f \in 2^\omega$, let f_k be the function $f_k(x) = f((k, x))$. If f separates $C_{(e,m)}$ from $D_{(e,m)}$, then f_k separates A from B . Thus the functional $f \mapsto f_k$ witnesses $X_{(e,m)} \geq_s Q$. The functional $f \mapsto g$ where $g((i, x)) = f(x)$ always witnesses $Q \geq_s X_{(e,m)}$. Hence $\text{deg}_s(X_{(e,m)}) = \text{deg}_s(Q)$. \square

Lemma 6.5. *If \mathcal{L} is a presentation of \mathcal{E}_s as a lattice, then $\mathbf{0}''' \leq_T \text{deg}_T(\mathcal{L})$.*

Proof. Let $\mathcal{L} = (\omega; \leq_{\mathcal{L}}, \sup_{\mathcal{L}}, \inf_{\mathcal{L}})$ be a presentation of \mathcal{E}_s . Let $f: \mathcal{E}_s \rightarrow \mathcal{L}$ be an isomorphism. Fix a Σ_3^0 -complete set $C \subseteq \omega$. We show how to compute C from $\leq_{\mathcal{L}} \oplus \sup_{\mathcal{L}} \oplus \inf_{\mathcal{L}}$.

By Lemma 4.9, let Q be an r.e. separating class and let $\{S_{0,n}\}_{n \in \omega}$ and $\{S_{1,n}\}_{n \in \omega}$ be recursive sequences of r.e. separating classes such that $\{Q\} \cup \{S_{0,n}\}_{n \in \omega} \cup \{S_{1,n}\}_{n \in \omega}$ is strongly independent. Then let

$$\begin{aligned} \mathbf{w}_0 &= \text{deg}_s(W_0) & \text{for } W_0 &= \text{spine}(Q, \{S_{0,n}\}_{n \in \omega}), \\ \mathbf{w}_1 &= \text{deg}_s(W_1) & \text{for } W_1 &= \text{spine}(Q, \{S_{1,n}\}_{n \in \omega}), \\ \mathbf{v} &= \text{deg}_s(V) & \text{for } V &= \bigotimes_{n \in \omega} S_{0,n}, \\ \mathbf{r} &= \text{deg}_s(R) & \text{for } R &= \text{spine}(Q, \{R_n\}_{n \in \omega}), \text{ where } R_n = \bigotimes_{m \in \omega \setminus \{n\}} S_{0,m}, \\ \mathbf{m} &= \text{deg}_s(M) & \text{for } M &= \text{spine}(Q, \{S_{0,n} \otimes S_{1,n}\}_{n \in \omega}), \text{ and} \\ \mathbf{p} &= \text{deg}_s(P) & \text{for } P &= \text{spine}(Q, \{S_{0,n} \otimes S_{1,n+1}\}_{n \in \omega}). \end{aligned}$$

Let $\{T_e\}_{e \in \omega}$ be a uniformly recursive sequence of trees representing all Π_1^0 classes as in Lemma 2.3, and, for notational ease, let $\{Z_e\}_{e \in \omega}$ denote the corresponding recursive sequence of Π_1^0 classes (i.e., $Z_e = [T_e]$ for each $e \in \omega$). Let $D \subseteq \omega$ be the set

$$D = \{e \mid \exists n(n \in C \wedge Z_e \leq_s S_{0,n} \wedge \forall V \leq_s Z_e \otimes R_n)\}.$$

D is Σ_3^0 because C is Σ_3^0 , the sequences $\{Z_e\}_{e \in \omega}$, $\{S_{0,n}\}_{n \in \omega}$, and $\{R_n\}_{n \in \omega}$ are recursive, and \leq_s is Σ_3^0 by Lemma 2.2. Let $\varphi(e, m, k, \ell)$ be a recursive predicate such that $D = \{e \mid \exists m \forall k \exists \ell \varphi(e, m, k, \ell)\}$. By Lemma 6.4, let $\{X_{(e,m)}\}_{(e,m) \in \omega}$ be a recursive sequence of Π_1^0 classes such that for all $e, m \in \omega$

$$\text{deg}_s(X_{(e,m)}) = \begin{cases} \mathbf{0}_s & \text{if } \forall k \exists \ell \varphi(e, m, k, \ell) \\ \text{deg}_s(Q) & \text{if } \exists k \forall \ell \neg \varphi(e, m, k, \ell). \end{cases}$$

Let $\mathbf{x} = \text{deg}_s(X)$ for $X = \text{spine}(Q, \{Z_e \otimes X_{(e,m)}\}_{(e,m) \in \omega})$.

The procedure for determining whether $n \in C$ from $\leq_{\mathcal{L}} \oplus \sup_{\mathcal{L}} \oplus \inf_{\mathcal{L}}$ uses the fixed parameters $f(\mathbf{w}_0), f(\mathbf{w}_1), f(\mathbf{r}), f(\mathbf{v}), f(\mathbf{m}), f(\mathbf{p}), f(\text{deg}_s(S_{0,0}))$, and $f(\mathbf{x})$. Given $n \in \omega$ search \mathcal{L} for elements $a_{i,j}$ for $i < 2$ and $1 \leq j \leq n$ and for an element b satisfying the conditions

- (i) $a_{i,j}$ meets to $f(\mathbf{w}_i)$ for all $i < 2$ and all $1 \leq j \leq n$,
- (ii) $\sup_{\mathcal{L}}(a_{0,j}, a_{1,j}) \geq_{\mathcal{L}} f(\mathbf{m})$ for all $1 \leq j \leq n$,
- (iii) $\sup_{\mathcal{L}}(a_{0,j}, a_{1,j+1}) \geq_{\mathcal{L}} f(\mathbf{p})$ for all $0 \leq j \leq n-1$ (where $a_{0,0} = f(\text{deg}_s(S_{0,0}))$),
- (iv) b meets to $f(\mathbf{r})$, and
- (v) $\sup_{\mathcal{L}}(a_{0,n}, b) \geq_{\mathcal{L}} f(\mathbf{v})$.

When the search is completed, output “yes” if $f(\mathbf{x}) \leq_{\mathcal{L}} a_{0,n}$ and output “no” otherwise.

First, observe that the above search is recursive in $\leq_{\mathcal{L}} \oplus \sup_{\mathcal{L}} \oplus \inf_{\mathcal{L}}$ because the “meets to” relation is r.e. in $\leq_{\mathcal{L}} \oplus \sup_{\mathcal{L}} \oplus \inf_{\mathcal{L}}$. Furthermore, the search will always terminate because the elements $a_{i,j} = f(\text{deg}_s(S_{i,j}))$ for all $i < 2$ and all $1 \leq j \leq n$ and the element $b = f(\text{deg}_s(R_n))$ satisfy conditions (i)–(v), and the search will eventually find them. Conditions (i) and (iv) follow from Lemma 5.2 item (ii), which says that the meet-irreducibles that meet to \mathbf{w}_i are exactly the $\text{deg}_s(S_{i,j})$ and that the meet-irreducibles that meet to \mathbf{r} are exactly the $\text{deg}_s(R_n)$. Notice that Q and $\{R_j\}_{j \in \omega}$ satisfy the hypothesis of Lemma 5.2 because $\{Q\} \cup \{S_{0,j}\}_{j \in \omega}$ is strongly independent. Conditions (ii) and (iii) are easy to see. For condition (v), it is also easy to see that $S_{0,n} \otimes R_n \equiv_s V$.

We need to show that the procedure outputs “yes” on input n if and only if $n \in C$. Let $a_{i,j}$ for $i < 2$ and $1 \leq j \leq n$ and b be the elements found in the search performed on input n .

Claim. For all $i < 2$ and all $1 \leq j \leq n$, $a_{i,j} \leq_{\mathcal{L}} f(\text{deg}_s(S_{i,j}))$.

Proof of claim. For each $i < 2$ and each $1 \leq j \leq n$, let $A_{i,j}$ be a Π_1^0 class such that $\text{deg}_s(A_{i,j}) = f^{-1}(a_{i,j})$. By condition (i) of the search and Lemma 5.2 item (i), $A_{0,1} \leq_s S_{0,m}$ and $A_{1,1} \leq_s S_{1,k}$ for some $m, k \in \omega$. Condition (iii) implies that $S_{0,0} \otimes S_{1,k} \geq_s P$, which is false by strong independence unless $k = 1$. So $A_{1,1} \leq_s S_{1,1}$. Knowing this, condition (ii) implies that $S_{0,m} \otimes S_{1,1} \geq_s M$, which is false by strong independence unless $m = 1$. So $A_{0,1} \leq_s S_{0,1}$. Now proceed by induction. Let $1 \leq j < n$ and assume that $A_{0,j} \leq_s S_{0,j}$ and that $A_{1,j} \leq_s S_{1,j}$. Just as in the argument for the base case, $A_{0,j+1} \leq_s S_{0,m}$ and $A_{1,j+1} \leq_s S_{1,k}$ for some $m, k \in \omega$. $S_{0,j} \otimes S_{1,k} \geq_s P$ by condition (iii), which implies that $k = j+1$. $S_{0,m} \otimes S_{1,j+1} \geq_s M$ by condition (ii), which implies that $m = j+1$. \square

At the end of the search, $a_{0,n} \leq_{\mathcal{L}} f(\text{deg}_s(S_{0,n}))$ by the claim, b meets to $f(\mathbf{r})$ by condition (iv), and $\sup_{\mathcal{L}}(a_{0,n}, b) \geq_{\mathcal{L}} f(\mathbf{v})$ by condition (v). By Lemma 5.2 item (i), $b \leq_{\mathcal{L}} f(\text{deg}_s(R_m))$ for some $m \in \omega$. However, if $m \neq n$, then $S_{0,n} \leq_s R_m$, in which case $S_{0,n} \otimes R_m \equiv_s R_m \not\geq_s V$. Thus it must be that $b \leq_{\mathcal{L}} f(\text{deg}_s(R_n))$.

Suppose $n \in C$. Since $\{Z_e\}_{e \in \omega}$ lists all the Π_1^0 classes, there is an $e \in \omega$ such that $\text{deg}_s(Z_e) = f^{-1}(a_{0,n})$. This e satisfies $\exists n (n \in C \wedge Z_e \leq_s S_{0,n} \wedge V \leq_s Z_e \otimes R_n)$. Thus $e \in D$, which means $\exists m \forall k \exists \ell \varphi(e, m, k, \ell)$. If m is such that $\forall k \exists \ell \varphi(e, m, k, \ell)$, then we have that $\text{deg}_s(X_{(e,m)}) = \mathbf{0}_s$ and $Z_e \otimes X_{(e,m)} \equiv_s Z_e$. Thus $X \leq_s Z_e$, which means $f(\mathbf{x}) \leq_{\mathcal{L}} a_{0,n}$. Thus “yes” was the output.

Suppose $n \notin C$. We show $X \not\leq_s S_{0,n}$.

Claim. For all $e, m \in \omega$, $S_{0,n} \not\leq_s Z_e \otimes X_{(e,m)}$.

Proof of claim. If $\text{deg}_s(X_{(e,m)}) = \text{deg}_s(Q)$, then $S_{0,n} \not\leq_s Z_e \otimes X_{(e,m)}$ because $S_{0,n} \not\leq_s Q$ by strong independence. If $\text{deg}_s(X_{(e,m)}) = \mathbf{0}_s$, then $\forall k \exists \ell \varphi(e, m, k, \ell)$. Therefore $e \in D$, so there is an $n' \in C$, $Z_e \leq_s S_{0,n'}$, and $V \leq_s Z_e \otimes R_{n'}$. Notice that $n \neq n'$ because $n \notin C$ and $n' \in C$. Therefore $S_{0,n} \leq_s R_{n'}$. So if $S_{0,n} \geq_s Z_e$, then $R_{n'} \geq_s Z_e$. So $V \leq_s R_{n'} \equiv_s Z_e \otimes R_{n'}$, a contradiction. Hence $S_{0,n} \not\leq_s Z_e \otimes X_{(e,m)}$. \square

Suppose for a contradiction that Φ is such that $\Phi(S_{0,n}) \subseteq X$. If there are $n, m \in \omega$ such that $\Phi(S_{0,n}) \cap (\sigma_{(e,m)} \wedge (Z_e \otimes X_{(e,m)})) \neq \emptyset$, then there is a clopen $C \subseteq 2^\omega$ such that $S_{0,n} \cap C \neq \emptyset$ and $\Phi(S_{0,n} \cap C) \subseteq \sigma_{(e,m)} \wedge (Z_e \otimes X_{(e,m)})$. $S_{0,n} \equiv_s S_{0,n} \cap C$ by Lemma 4.3, and $S_{0,n} \cap C \geq_s (\sigma_{(e,m)} \wedge (Z_e \otimes X_{(e,m)})) \equiv_s Z_e \otimes X_{(e,m)}$. This contradicts the claim. Thus $\Phi(S_{0,n}) \cap (\sigma_{(e,m)} \wedge (Z_e \otimes X_{(e,m)})) = \emptyset$ for all $e, m \in \omega$. Therefore $\Phi(S_{0,n}) \subseteq Q$. This contradicts $Q \not\leq_s S_n$. Hence $X \not\leq_s S_{0,n}$. It follows that $f(\mathbf{x}) \not\leq_{\mathcal{L}} a_{0,n}$ because $a_{0,n} \leq_{\mathcal{L}} f(\text{deg}_s(S_{0,n}))$. Thus “no” was the output. \square

Theorem 6.6. The degree of \mathcal{E}_s as a lattice is $\mathbf{0}'''$. That is, there is a presentation of \mathcal{E}_s as a lattice recursive in $\mathbf{0}'''$ and $\mathbf{0}'''$ is recursive in every presentation of \mathcal{E}_s as a lattice.

Proof. Lemma 6.3 proves that there is a presentation recursive in $\mathbf{0}'''$, and Lemma 6.5 proves that $\mathbf{0}'''$ is recursive in every presentation. \square

Corollary 6.7. \mathcal{E}_s has no presentation as a partial order recursive in $\mathbf{0}'$.

Proof. In any lattice, the relations $\sup(x, y) = z$ and $\inf(x, y) = z$ are definable from the partial order by Π_1^0 formulas. Thus if \mathcal{E}_s had a presentation as a partial order recursive in $\mathbf{0}'$, it would have a presentation as a lattice recursive in $\mathbf{0}'$. This contradicts the theorem. \square

Of course the same argument shows that \mathcal{E}_s cannot have a presentation as a partial order recursive in any degree \mathbf{d} such that $\mathbf{d}' <_{\mathcal{T}} \mathbf{0}'''$.

7. Undecidability in \mathcal{E}_w

In this section, we code $\mathcal{N}^{\dot{-}}$ in \mathcal{E}_w , thereby showing that $\Sigma_3^0\text{-Th}(\mathcal{E}_w)$ and $\Sigma_4^0\text{-Th}(\mathcal{E}_w; \leq_w)$ are undecidable. In place of separating classes, our coding of $\mathcal{N}^{\dot{-}}$ in \mathcal{E}_w uses Simpson’s Σ_3^0 embedding lemma and his embedding of \mathcal{R} into \mathcal{E}_w .

Lemma 7.1 (Σ_3^0 Embedding Lemma [38] Lemma 3.3). *Let $S \subseteq \omega^\omega$ be Σ_3^0 and let $P \subseteq 2^\omega$ be a Π_1^0 class. Then there is a Π_1^0 class $Q \subseteq 2^\omega$ such that $Q \equiv_w S \cup P$.*

In the Muchnik case, $\inf(\text{deg}_w(S), \text{deg}_w(P)) = \text{deg}_w(S \cup P)$ for any $S, P \subseteq \omega^\omega$. For this reason, the Σ_3^0 embedding lemma may be phrased as “if S is Σ_3^0 and P is a Π_1^0 class then $\inf(\text{deg}_w(S), \text{deg}_w(P)) \in \mathcal{E}_w$.” For our purposes, P is always DNR_2 , so $\text{deg}_w(P) = \text{deg}_w(\text{DNR}_2) = \mathbf{1}_w$, the greatest element of \mathcal{E}_w .

If A is an r.e. set, then $\{A\}$ is a Σ_3^0 (in fact a Π_2^0) subset of 2^ω . One of Simpson’s original applications of his Σ_3^0 embedding lemma is to show that the map $\text{deg}_T(A) \mapsto \inf(\text{deg}_w(\{A\}), \mathbf{1}_w)$ is an upper-semilattice embedding of \mathcal{R} into \mathcal{E}_w preserving the least and greatest elements [38]. To show that this map is indeed an embedding, Simpson uses the following variant of the Arslanov completeness criterion, which we also employ.

Lemma 7.2 (See [16] Lemma 4.1 and [40] Theorem V.5.1). *If A is an r.e. set, then $\text{DNR}_2 \leq_w \{A\}$ if and only if $A \equiv_T 0'$.*

Proof. It is easy to compute a function in DNR_2 from $0'$. Conversely, if A computes a function in DNR_2 , then A computes a function f such that $\forall e (W_{f(e)} \neq W_e)$, where here $\{W_e\}_{e \in \omega}$ is the standard enumeration of the r.e. sets (such an f is called *fixed-point free*; see [16] Lemma 4.1). Thus $A \equiv_T 0'$ by the Arslanov completeness criterion (see [40] Theorem V.5.1). \square

For comparison, it is not known whether \mathcal{R} embeds into \mathcal{E}_s . See [5] for further results concerning embedding distributive lattices in \mathcal{E}_s and \mathcal{E}_w .

For us, the key property of the degrees $\inf(\text{deg}_w(\{A\}), \mathbf{1}_w)$ for r.e. sets A is that they are all meet-irreducible in \mathcal{E}_w (of course these degrees are generally meet-reducible in \mathcal{D}_w).

Lemma 7.3. *If A is an r.e. set, then $\inf(\text{deg}_w(\{A\}), \mathbf{1}_w)$ is meet-irreducible in \mathcal{E}_w .*

Proof. Suppose that $\mathbf{x}, \mathbf{y} \in \mathcal{E}_w$ are such that $\inf(\text{deg}_w(\{A\}), \mathbf{1}_w) \geq_w \inf(\mathbf{x}, \mathbf{y})$. Either $\text{deg}_w(\{A\}) \geq_w \mathbf{x}$ or $\text{deg}_w(\{A\}) \geq_w \mathbf{y}$ because $\text{deg}_w(\{A\})$ is the degree of a singleton. As $\mathbf{1}_w \geq_w \mathbf{x}$ and $\mathbf{1}_w \geq_w \mathbf{y}$, either $\inf(\text{deg}_w(\{A\}), \mathbf{1}_w) \geq_w \mathbf{x}$ or $\inf(\text{deg}_w(\{A\}), \mathbf{1}_w) \geq_w \mathbf{y}$. \square

If $\{A_n\}_{n \in B}$ is a uniformly r.e. sequence of r.e. sets indexed by a recursive set B (i.e., the set $\{\langle n, m \rangle \mid n \in B \wedge m \in A_n\}$ is r.e.), then $\{A_n\}_{n \in B}$ is a Σ_3^0 subset of 2^ω and it follows that $\inf(\text{deg}_w(\{A_n\}_{n \in B}), \mathbf{1}_w) \in \mathcal{E}_w$. In place of Lemma 4.9, we use the following simpler fact.

Lemma 7.4 (See [40] Section VII.2). *There is a uniformly r.e. sequence of r.e. sets $\{A_n\}_{n \in \omega}$ that is strongly independent.*

Notice that Lemma 7.4 is also a consequence of Lemma 4.9. If $\{S(A_n, B_n)\}_{n \in \omega}$ is a recursive sequence of r.e. separating classes that is strongly independent, then $\{A_n\}_{n \in \omega}$ and $\{B_n\}_{n \in \omega}$ are both uniformly r.e. sequences of r.e. sets that are strongly independent.

Now we have the following analog of Lemma 5.2.

Lemma 7.5. *Let $\{A_n\}_{n \in B}$ be an infinite uniformly r.e. sequence of r.e. sets (indexed by a recursive set B) that is a \leq_T -antichain. Let $\mathbf{w} = \inf(\text{deg}_w(\{A_n\}_{n \in B}), \mathbf{1}_w)$.*

- (i) *If $\mathbf{x} \in \mathcal{E}_w$ meets to \mathbf{w} , then $\mathbf{x} \leq_w \inf(\text{deg}_w(\{A_n\}), \mathbf{1}_w)$ for some $n \in B$.*
- (ii) *$E(\mathbf{w}) = \{\inf(\text{deg}_w(\{A_n\}), \mathbf{1}_w) \mid n \in B\}$.*

Proof. (i) Let $\mathbf{x} \in \mathcal{E}_w$ be such that \mathbf{x} meets to \mathbf{w} , and suppose $\mathbf{x} \not\leq_w \inf(\text{deg}_w(\{A_n\}), \mathbf{1}_w)$ for all $n \in B$ for a contradiction. Since $\mathbf{x} \leq_w \mathbf{1}_w$, it must be that $\mathbf{x} \not\leq_w \text{deg}_w(\{A_n\})$ for all $n \in B$. Let $\mathbf{y} \in \mathcal{E}_w$ witness that \mathbf{x} meets to \mathbf{w} . That is, $\mathbf{y} >_w \mathbf{w}$ and $\inf(\mathbf{x}, \mathbf{y}) = \mathbf{w}$. Let X and Y be Π_1^0 classes such that $\mathbf{x} = \text{deg}_w(X)$ and $\mathbf{y} = \text{deg}_w(Y)$. Then $X \cup Y \leq_w \{A_n\}$ for all $n \in B$. Thus $Y \leq_w \{A_n\}$ for all $n \in B$ because $X \not\leq_w \{A_n\}$ for all $n \in B$. Therefore $Y \leq_w \{A_n\}_{n \in B}$, which implies that $\mathbf{y} \leq_w \mathbf{w}$, a contradiction.

(ii) Let $n \in B$. To see that $\inf(\text{deg}_w(\{A_n\}), \mathbf{1}_w) \in E(\mathbf{w})$, let

$$\mathbf{y} = \inf(\text{deg}_w(\{A_i\}_{i \in B \setminus \{n\}}), \mathbf{1}_w).$$

It is easy to check that

$$\inf(\inf(\text{deg}_w(\{A_n\}), \mathbf{1}_w), \mathbf{y}) = \mathbf{w}.$$

Moreover, $\inf(\text{deg}_w(\{A_n\}), \mathbf{1}_w) \not\leq_w \mathbf{y}$. This is because $\{A_n\} \not\leq_w \{A_i\}_{i \in B \setminus \{n\}}$ as $\{A_i\}_{i \in B}$ is a \leq_T -antichain and because $\{A_n\} \not\leq_w \text{DNR}_2$ by Lemma 7.2 (note that $A_n <_T 0'$ because $\{A_i\}_{i \in B}$ is a \leq_T -antichain). Thus $\mathbf{y} >_w \mathbf{w}$, and therefore \mathbf{y} witnesses that $\inf(\text{deg}_w(\{A_n\}), \mathbf{1}_w)$ meets to \mathbf{w} . The degree $\inf(\text{deg}_w(\{A_n\}), \mathbf{1}_w)$ is meet-irreducible in \mathcal{E}_w by Lemma 7.3. Thus $\inf(\text{deg}_w(\{A_n\}), \mathbf{1}_w) \in E(\mathbf{w})$.

We have shown that $\{\inf(\text{deg}_w(\{A_n\}), \mathbf{1}_w) \mid n \in B\} \subseteq E(\mathbf{w})$. To see equality, let $\mathbf{x} \in E(\mathbf{w})$. By item (i), $\mathbf{x} \leq_w \inf(\text{deg}_w(\{A_n\}), \mathbf{1}_w)$ for some $n \in B$. $E(\mathbf{w})$ is an antichain by Lemma 3.3 and $\inf(\text{deg}_w(\{A_n\}), \mathbf{1}_w) \in E(\mathbf{w})$, so it must be that $\mathbf{x} = \inf(\text{deg}_w(\{A_n\}), \mathbf{1}_w)$. \square

We are now able to code $\mathcal{N}^{\dot{+}}$ in \mathcal{E}_w .

Lemma 7.6. *There is a code \vec{w} in \mathcal{E}_w such that $\mathcal{M}_{\vec{w}}^{\dot{+}} \cong \mathcal{N}^{\dot{+}}$.*

Proof. The proof is very similar to the proof of Lemma 5.3. By Lemma 7.4, let $\{A_{0,n}\}_{n \in \omega}$, $\{A_{1,n}\}_{n \in \omega}$, and $\{A_{2,n}\}_{n \in \omega}$ be uniformly r.e. sequences of r.e. sets such that $\{A_{0,n}\}_{n \in \omega} \cup \{A_{1,n}\}_{n \in \omega} \cup \{A_{2,n}\}_{n \in \omega}$ is strongly independent. Let

$$\begin{aligned} \mathbf{w}_0 &= \inf(\deg_w(W_0), \mathbf{1}_w) & \text{for } W_0 &= \{A_{0,n}\}_{n \in \omega}, \\ \mathbf{w}_1 &= \inf(\deg_w(W_1), \mathbf{1}_w) & \text{for } W_1 &= \{A_{1,n}\}_{n \in \omega}, \\ \mathbf{w}_2 &= \inf(\deg_w(W_2), \mathbf{1}_w) & \text{for } W_2 &= \{A_{2,n}\}_{n \in \omega}, \\ \mathbf{m} &= \inf(\deg_w(M), \mathbf{1}_w) & \text{for } M &= \{A_{0,n} \oplus A_{1,n}\}_{n \in \omega} \cup \{A_{0,n} \oplus A_{2,n}\}_{n \in \omega}, \\ \ell &= \inf(\deg_w(L), \mathbf{1}_w) & \text{for } L &= \{A_{0,i} \oplus A_{1,j} \mid i < j\}, \\ \mathbf{p} &= \inf(\deg_w(P), \mathbf{1}_w) & \text{for } P &= \{A_{0,i} \oplus A_{1,j} \oplus A_{2,k} \mid i + j = k\}, \\ \mathbf{t} &= \inf(\deg_w(T), \mathbf{1}_w) & \text{for } T &= \{A_{0,i} \oplus A_{1,j} \oplus A_{2,k} \mid i \times j = k\}, \\ \mathbf{d} &= \inf(\deg_w(D), \mathbf{1}_w) & \text{for } D &= \{A_{0,i} \oplus A_{1,j} \oplus A_{2,k} \mid i \dot{-} j = k\}, \\ \mathbf{z} &= \inf(\deg_w(\{A_{0,0}\}), \mathbf{1}_w), & \text{and} & \\ \mathbf{o} &= \inf(\deg_w(\{A_{0,1}\}), \mathbf{1}_w). \end{aligned}$$

To aid readability, let $\mathbf{a}_{i,j} = \deg_w(\{A_{i,j}\})$ for all $i < 3$ and $j \in \omega$. By Lemma 7.5 item (ii), $E(\mathbf{w}_0) = \{\inf(\mathbf{a}_{0,n}, \mathbf{1}_w)\}_{n \in \omega}$. The map $\inf(\mathbf{a}_{0,n}, \mathbf{1}_w) \mapsto n$ is the isomorphism witnessing $\mathcal{M}_{\vec{w}}^{\dot{+}} \cong \mathcal{N}^{\dot{+}}$. Clearly $\mathbf{z} \mapsto 0$ and $\mathbf{o} \mapsto 1$. We show that the map preserves $<$. The proofs that the map preserves $+$, \times , and $\dot{-}$ are similar. Let $i, j \in \omega$. If $i < j$, then $\inf(\mathbf{a}_{1,j}, \mathbf{1}_w)$ meets to \mathbf{w}_1 by Lemma 7.5 item (ii), and by distributivity

$$\begin{aligned} \sup(\inf(\mathbf{a}_{0,j}, \mathbf{1}_w), \inf(\mathbf{a}_{1,j}, \mathbf{1}_w)) &= \inf(\sup(\mathbf{a}_{0,j}, \mathbf{a}_{1,j}), \mathbf{1}_w) \\ &= \inf(\deg_w(\{A_{0,j} \oplus A_{1,j}\}), \mathbf{1}_w) \\ &\geq_w \mathbf{m}, \text{ and} \\ \sup(\inf(\mathbf{a}_{0,i}, \mathbf{1}_w), \inf(\mathbf{a}_{1,j}, \mathbf{1}_w)) &= \inf(\sup(\mathbf{a}_{0,i}, \mathbf{a}_{1,j}), \mathbf{1}_w) \\ &= \inf(\deg_w(\{A_{0,i} \oplus A_{1,j}\}), \mathbf{1}_w) \\ &\geq_w \ell. \end{aligned}$$

Thus $R_\ell^2(\inf(\mathbf{a}_{0,i}, \mathbf{1}_w), \inf(\mathbf{a}_{0,j}, \mathbf{1}_w))$. Conversely, suppose $R_\ell^2(\inf(\mathbf{a}_{0,i}, \mathbf{1}_w), \inf(\mathbf{a}_{0,j}, \mathbf{1}_w))$. Let $\mathbf{u}_1 \in \mathcal{E}_w$ be such that \mathbf{u}_1 meets to \mathbf{w}_1 , $\sup(\inf(\mathbf{a}_{0,j}, \mathbf{1}_w), \mathbf{u}_1) \geq_w \mathbf{m}$, and also $\sup(\inf(\mathbf{a}_{0,i}, \mathbf{1}_w), \mathbf{u}_1) \geq_w \ell$. Since \mathbf{u}_1 meets to \mathbf{w}_1 , it must be that $\mathbf{u}_1 \leq_w \inf(\mathbf{a}_{1,k}, \mathbf{1}_w)$ for some $k \in \omega$ by Lemma 7.5 item (i). Thus $\sup(\inf(\mathbf{a}_{0,j}, \mathbf{1}_w), \inf(\mathbf{a}_{1,k}, \mathbf{1}_w)) \geq_w \mathbf{m}$, so by distributivity

$$\begin{aligned} \inf(\deg_w(\{A_{0,j} \oplus A_{1,k}\}), \mathbf{1}_w) &= \inf(\sup(\mathbf{a}_{0,j}, \mathbf{a}_{1,k}), \mathbf{1}_w) \\ &= \sup(\inf(\mathbf{a}_{0,j}, \mathbf{1}_w), \inf(\mathbf{a}_{1,k}, \mathbf{1}_w)) \\ &\geq_w \mathbf{m}. \end{aligned}$$

However, if $k \neq j$, then $\{A_{0,j} \oplus A_{1,k}\} \not\leq_w M$ by strong independence and $\{A_{0,j} \oplus A_{1,k}\} \not\leq_w \text{DNR}_2$ by Lemma 7.2. This implies that $\inf(\deg_w(\{A_{0,j} \oplus A_{1,k}\}), \mathbf{1}_w) \not\geq_w \mathbf{m}$, so it must be that $k = j$. Thus $\mathbf{u}_1 \leq_w \inf(\mathbf{a}_{1,j}, \mathbf{1}_w)$, implying that $\sup(\inf(\mathbf{a}_{0,i}, \mathbf{1}_w), \inf(\mathbf{a}_{1,j}, \mathbf{1}_w)) \geq_w \ell$. Then

$$\begin{aligned} \inf(\deg_w(\{A_{0,i} \oplus A_{1,j}\}), \mathbf{1}_w) &= \inf(\sup(\mathbf{a}_{0,i}, \mathbf{a}_{1,j}), \mathbf{1}_w) \\ &= \sup(\inf(\mathbf{a}_{0,i}, \mathbf{1}_w), \inf(\mathbf{a}_{1,j}, \mathbf{1}_w)) \\ &\geq_w \ell. \end{aligned}$$

So if $i \neq j$, then $\{A_{0,i} \oplus A_{1,j}\} \not\leq_w L$ by strong independence and $\{A_{0,i} \oplus A_{1,j}\} \not\leq_w \text{DNR}_2$ by Lemma 7.2, giving the contradiction $\inf(\deg_w(\{A_{0,i} \oplus A_{1,j}\}), \mathbf{1}_w) \not\geq_w \ell$. Hence $i < j$. \square

Theorem 7.7. $\Sigma_3^0\text{-Th}(\mathcal{E}_w)$ and $\Sigma_4^0\text{-Th}(\mathcal{E}_w; \leq_w)$ are undecidable.

Proof. There is a code \vec{w} in \mathcal{E}_w such that $\mathcal{M}_{\vec{w}}^{\dot{+}} \cong \mathcal{N}^{\dot{+}}$ by Lemma 7.6. The results then follow from Lemma 3.15. \square

Clearly then $\text{Th}(\mathcal{E}_w; \leq_w)$ is undecidable. Unfortunately we do not yet know how to prove anything like the finite matching property for \mathcal{E}_w to obtain $\text{Th}(\mathcal{N}) \leq_1 \text{Th}(\mathcal{E}_w; \leq_w)$. The proof of the finite matching property for \mathcal{E}_s (Lemma 4.7 above) appeals to a lemma of Cole and Kihara that grew out of Cenzer and Hinman's proof that \mathcal{E}_s is dense. By analogy, perhaps progress must be made on the density of \mathcal{E}_w before further progress is made on the complexity of $\text{Th}(\mathcal{E}_w; \leq_w)$.

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