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ELSEVIER International Journal of Approximate Reasoning 26 (2001) 145–152
INTERNATIONAL JOURNAL OF APPROXIMATE REASONING
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Elkan's theoretical argument, reconsidered ☆

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Received 1 November 1999; received in revised form 1 September 2000; accepted 1 October 2000

Abstract

This paper takes part in the discussion motivated by Elkan's paper "The Paradoxical Success of Fuzzy Logic" printed in 1993, whose main theoretical point was that Fuzzy Logic does not properly deal with a specific Law of Classical Logic: $\neg(p \wedge \neg q) = q \vee (\neg p \wedge \neg q)$. The given answer can be summarized, like in other previous cases, by the sentence "Yes it can but, of course, at some cost". As it is shown this cost is, basically, duality. But without De Morgan laws there are uncountable many theories of Fuzzy Sets on which that classical law holds. On the way it is observed that, this equation, is not universally verified in De Morgan lattices and the solution given by Elkan, in a particular case, is incorrect. © 2001 Elsevier Science Inc. All rights reserved.

Keywords: Fuzzy sets; Laws of fuzzy sets; t-Norms; t-Conorms; Strong-negations

1. Introduction

1.1. Given a reference set X, the functions in $[0,1]^X$ are called *fuzzy sets* when ordered pointwise ($A \leqslant B$ if $A(x) \leqslant B(x)$ for any x in X) and, consequently, identified by the definition

☆ This paper was partially supported by CICYT Spain, under project TIC96-1393-C06, and it is dedicated to all those that contributed to the theoretical aspects of Fuzzy Set Theory.

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$$A = B \text{ iff } A(x) = B(x) \quad \text{for all } x \text{ in } X.$$

Each time a triple of operations $\cap : [0,1]^X \times [0,1]^X \to [0,1]^X$, $\cup : [0,1]^X \times [0,1]^X \to [0,1]^X$ and $' : [0,1]^X \to [0,1]^X$ are defined in such a way that if a, b are in $\{0,1\}^X$ then $a \cap b$, $a \cup b$, a' and b' are in $\{0,1\}^X$ (that is, the theory of classical subsets is preserved), it is said that $([0,1]^X, \cap, \cup, ')$ is a Theory of Fuzzy Sets. In general, theories of fuzzy sets are taken to be Functionally Expressible, i.e., there are numerical functions $T : [0,1] \times [0,1] \to [0,1], S : [0,1] \times [0,1] \to [0,1]$ and $N : [0,1] \to [0,1]$, such that $(A \cap B)(x) = T(A(x), B(x))$, $(A \cup B)(x) = S(A(x), B(x))$ and $A'(x) = N(A(x))$, for any pair A, B in $[0,1]^X$ and any x in X, or, for short, $A \cap B = T \circ (A \times B)$, $A \cup B = S \circ (A \times B)$ and $A' = N \circ A$ (see [1,14]).

When T is a t-norm, S a t-conorm and N a strong-negation (see [14]), the corresponding theory $([0,1]^X, T, S, N)$ is called a *Standard Theory of Fuzzy Sets* (STFS) (see [1]).

In general a theory of fuzzy sets does not verify all the laws of Set Theory. However, each time one of such laws was considered, either a theory of fuzzy sets or a different approach was found to guarantee it within fuzzy sets. The basic cases of the classical laws $a \cap b^c = \emptyset$ (Non-contradiction) and $a \cup b^c = X$ (Excluded-Middle) illustrate quite well this last statement, and there is also a Non-Functionally Expressible theory that is a Boolean Algebra (see [1]).

The laws of Non-Contradiction and Excluded-Middle are not generally verified in an STFS. More precisely:

1. It is $T \circ (A \times A') = T \circ (A \times (N \circ A)) = \mu_\emptyset$ if and only if $T = \varphi^{-1} \circ W \circ (\varphi \times \varphi)$ and $N \leqslant N_\varphi$, with φ an order-automorphism of the unit interval $[0,1]$, $N_\varphi = \varphi^{-1} \circ (1 - id) \circ \varphi$ the strong-negation associated to T (see [4]) and W the t-norm of Lukasiewicz.
2. It is $S \circ (A \times A') = S \circ (A \times (N \circ A)) = \mu_X$ if and only if $S = \varphi^{-1} \circ W^* \circ (\varphi \times \varphi)$ and $N \geqslant N_\varphi$, with $W^*(x,y) = 1 - W(1-x, 1-y)$ the dual t-conorm of W.

Then, the only STFS that verify these two laws are those $([0,1]^X, T, S, N)$ in which $T = \varphi_1^{-1} \circ W \circ (\varphi_1 \times \varphi_1)$, $S = \varphi_2^{-1} \circ W^* \circ (\varphi_2 \times \varphi_2)$ and $N_{\varphi_2} \leqslant N \leqslant N_{\varphi_1}$, for two automorphism φ_1 and φ_2 of $[0,1]$.

Nevertheless, it should be pointed out that in classical set theory $a \cap a^c = \emptyset$ is equivalent to $a \cap a^c \subset (a \cap a^c)^c$, because $\emptyset$ is the only selfcontradictory set. From this point of view, since $A \leqslant A'$ i.e., $A \leqslant N \circ A = \varphi^{-1} \circ (1 - id) \circ \varphi \circ A$, is equivalent to $A(x) \leqslant \varphi^{-1}(1/2)$ for any x in X ($\varphi^{-1}(1/2) \in (0,1)$ is the only fixed point of N), in an SFST one has always the fact that $A \cap A'$ is selfcontradictoy, indeed

$$\begin{aligned}(A \cap A')(x) &= T(A(x), N(A(x))) \leqslant \mathrm{Min}(A(x), \varphi^{-1}(1 - \varphi(A(x)))) \\ &= \varphi^{-1}(\mathrm{Min}(\varphi(A(x)), 1 - \varphi(A(x)))) \leqslant \varphi^{-1}(1/2).\end{aligned}$$

Analogously, $a \cup a^c = X$ is equivalent to $(a \cup a^c)^c = \emptyset$ and to $(a \cup a^c)^c \subset a \cup a^c$, and in any SFST $(A \cup A')$ is selfcontradictoy, because one has

$$\begin{aligned}(A \cup A')(x) &= S(A(x), N(A(x))) \geqslant \mathrm{Max}(A(x), \varphi^{-1}(1 - \varphi(A(x)))) \\ &= \varphi^{-1}(\mathrm{Max}(\varphi(A(x)), 1 - \varphi(A(x)))) \geqslant \varphi^{-1}(1/2),\end{aligned}$$

and therefore $[(A \cup A')'](x) \leqslant N(\varphi^{-1}(1/2)) = \varphi^{-1}(1 - 1/2) = \varphi^{-1}(1/2)$.

Hence, looking at the laws of Non-Contradiction and Excluded-Middle not from the point of view of "incompatibility" but from the "selfcontradiction" approach, both laws are verified by any STFS, regardless of the N-duality of the pair T, S (see [2]). Perhaps, if this is not the reason for the theoretical success of STFS it is certainly a reason for its success.

1.2. These kinds of problems were not forgotten at all in theoretical fuzzy logic. As early as in 1973, in [3] it was proved that only with $T = \mathrm{Min}$ and $S = \mathrm{Max}$ in an SFST one can preserve a large number of laws of classical Set Theory. The preservation of the law of idempotency and distributivity were studied in [4], the preservation of the law $a = (a \cap b) \cup (a \cap b^c)$ was studied in [5] and that of the law $a^c \cup b = a^c \cup (a \cap b)$ in [6,7].

Then, if within SFST a Boolean Algebra structure is never reached, for any given classical law, an SFST verifying it has been obtained. And this paper will show that this is exactly the case with the law $\neg(a \cap \neg b) = b \cup (\neg a \cap \neg b)$ considered by Elkan [8,9], and maintained by himself in [10]. But before doing this let us review Elkan's theoretical result.

2. Elkan's theorem

2.1. An equivalent formulation in fuzzy logic to what Professor Elkan claimed in Refs. [8,9] is:

If in the SFST $([0,1]^X, \mathrm{Min}, \mathrm{Max}, 1 - id)$ the law

$$(A \cap B')' = B \cup (A' \cap B') \tag{1}$$

for any A, B in $[0,1]^X$ is imposed, then either $A = B$ or $A = 1 - B$.

First of all, law (1) cannot be forced to universally hold in $([0,1]^X, \mathrm{Min}, \mathrm{Max}, 1 - id)$ because taking $A = \emptyset$ it would follow $\emptyset' = B \cup B'$ or $X = B \cup B'$ for any B in $[0,1]^X$ and since Max is not a t-conorm in the Lukasiewicz family, (1) does not universally hold in the given SFST. For example if B is such that $B(x_0) = 0.5$ for some $x_0 \in X$, it is $(B \cup B')(x_0) = \mathrm{Max}(B(x_0), 1 - B(x_0)) = 0.5$ but $X(x_0) = 1$. There are also A, B in $[0,1]^X$ that verify (1) but both $A \neq B$ and $A \neq 1 - B$, for example, $A(x) = x$ and

$$B(x) = \begin{cases} 1 & \text{if } x \in [0, 1/2], \\ 1 - x & \text{if } x \in (1/2, 1]. \end{cases}$$

Then, what happens?

As it was said in [11], Elkan's proof is incorrect. Let us study when (1) holds. When the law (1) is translated into $([0,1]^X, \mathrm{Min}, \mathrm{Max}, 1 - id)$ as

$$1 - \mathrm{Min}(A(x), 1 - B(x)) = \mathrm{Max}(B(x), \mathrm{Min}(1 - A(x), 1 - B(x))),$$

which is equivalent to

$$\mathrm{Min}(A(x), 1 - B(x)) = \mathrm{Min}(1 - B(x), \mathrm{Max}(A(x), B(x))), \tag{2}$$

we can prove the following.

Theorem 1. (2) *holds if and only if* $A(x) \geqslant \mathrm{Min}(1 - B(x), B(x))$, *for any* $x \in X$.

Proof. Just note that (2) may be presented in the form

$$\mathrm{Min}(A(x), 1 - B(x)) = \mathrm{Max}[\mathrm{Min}(1 - B(x), A(x)), \mathrm{Min}(1 - B(x)), B(x)],$$

which says

$$\mathrm{Min}(A(x), 1 - B(x)) \geqslant \mathrm{Min}(1 - B(x), B(x)),$$

and this is equivalent to $A(x) \geqslant \mathrm{Min}(1 - B(x), B(x))$ for all x in X. □

Corollary 1. *Equation* $(A \cap B')' = B \cup (A' \cap B')$ *is verified in* $([0,1]^X$, Min, Max, $1 - id)$ *if and only if* $A \geqslant B \cap B'$.

Note that if $B \leqslant A$ or $B' \leqslant A$ then $\mathrm{Min}(B'(x), B(x)) \leqslant A(x)$ but, obviously the converse is not true (e.g. consider $X = [0,1]$, $B(x) = x$, $B'(x) = 1 - x$ and $A(x) = 1/2$ whenever $0 \leqslant x \leqslant 1/2$ and $A(x) = B'(x)$ elsewhere. Then $A(x) \geqslant \mathrm{Min}(B(x), B'(x))$ but A is neither comparable with B nor with B').

2.2. Of course, because Eq. (1) is not symmetrical in A, B, it is possible to interpret Elkan's own theorem as concerning both the pairs (A, B) and (B, A), as it is done in [11]. Then, one needs to deal with the verification of both equations $(A \cap B')' = B \cup (A' \cap B')$ and $(B \cap A')' = A \cup (B' \cap A')$. By Corollary 1, one needs to have $A \geqslant B \cap B'$ and $B \geqslant A \cap A'$.

But these two inequalities do not imply the conditions $A = B$ or $A \leqslant B'$, which is the wrong solution given in [11]. For example if $X = [0,1]$, $B(x) = 1$, $B'(x) = 0$ and $A(x) = 1/2$, for any x in $[0,1]$ then $B \cap B' = 0 \leqslant A$ and $A \cap A' = 1/2 \leqslant B$ but neither $A = B$ nor $A \leqslant B'$.

Nevertheless, if this interpretation can seem reasonable by looking at the proof that Elkan offers (see [8]), it is hard to accept it by reading Elkan's justifications for the election of (1). Namely, to the letter, Elkan says:

"The equivalence used in Theorem 1 is rather complicated, but it is plausible intuitively, and it is natural to apply it in reasoning about a set of fuzzy rules, since $\neg(A \wedge \neg B)$ and $B \vee (\neg A \wedge \neg B)$ are both reexpressions of the classical implication $A \to B$" (see [8]).

"The equivalence between $\neg(A \wedge \neg B)$ and $B \vee (\neg A \wedge \neg B)$ is a natural one to use perhaps inadvertently in compiling a knowledge base of fuzzy logic sentences" (see [10]).

But in Elkan's context, both statements "$A \to B$ is the same as $B \to A$" and "the fuzzy sentences $B \cup (\neg A \cap \neg B)$ and $A \cup (\neg B \cap \neg A)$ are equal" appear to be incorrect.

3. The case of lattices

If $(L, \wedge, \vee)$ is a lattice endowed with an involution $\neg : L \to L$ verifying the property $\neg(p \wedge q) = \neg p \vee \neg q$, and hence, $\neg(p \vee q) = \neg p \wedge \neg q$, the expression

$$\neg(p \wedge \neg q) = q \vee (\neg p \wedge \neg q)$$

is equivalent to

$$\neg p \vee q = \neg p \vee (p \wedge q),$$

but this may not hold. Since $p \wedge q \leqslant q$, the inequality $\neg p \vee (p \wedge q) \leqslant \neg p \vee q$ is always true, but the equality is difficult to be reached. For example, if the lattice has greatest element 1 and the involution a fixed point $z = \neg z < 1$ (as it happens in the case $L = [0,1]^X$, $\wedge = \mathrm{Min}$, $\vee = \mathrm{Max}$, $\neg = 1 - id$, with $z = 0.5$), with $q = 1$ and $p = z$ it results:

$$\neg p \vee q = z \vee 1 = 1, \quad \neg p \vee (p \wedge q) = z \vee (z \wedge 1) = z \vee z = z,$$

but $z = 1$ is a contradiction.

In general, an analogous situation happens in Orthocomplemented Lattices (see [12]). For example, it is well known that in Orthomodular Lattices (typical of the logics of Quantum Physics) the implication is not modeled by means of the classical material implication $p \to_M q = \neg p \vee q$ but by means of the so-called Suzuki Operator $p \to_S q = \neg p \vee (p \wedge q)$ that, because of the above-mentioned inequality, is weaker than the classical operator: $p \to_S q \leqslant b \to_M q$.

The case of Boolean Algebras is special, since they verify both the laws of distributivity and the law of Excluded-Middle. Hence, in any Boolean Algebra it is

$$\neg p \vee (p \wedge q) = (\neg p \vee p) \wedge (\neg p \vee q) = 1 \wedge (\neg p \vee q) = \neg p \vee q.$$

Hence, it seems that in the framework of lattices the law $\neg(p \wedge \neg q) = q \vee (\neg p \wedge \neg q)$ mainly appears as a particular property of Boolean Algebras,

that is, of classical logic. In enough general types of lattices the only that can be asserted is the inequality $q \vee (\neg p \wedge \neg q) \leqslant \neg(p \wedge \neg q)$.

4. The case of the standard theories of fuzzy sets

The only STFS that are lattices are $([0,1]^X, \mathrm{Min}, \mathrm{Max}, N)$. When $T \neq \mathrm{Min}$ or $S \neq \mathrm{Max}$, $([0,1]^X, T, S, N)$ is not a lattice and $A \cap B < \mathrm{Min}(A,B) \leqslant \mathrm{Max}(A,B) < A \cup B$. Nevertheless, when T and S are N-duals, since $T(A'(x), B'(x)) \leqslant A'(x)$ it follows that $S(B(x), T(A'(x), B'(x))) \leqslant S(B(x), A'(x)) = N(T(B'(x), A(x)))$, so at least it is possible to have the inequality $B \cup (A' \cap C') \leqslant (A \cap B')'$.

Theorem 2. *The law $B \cup (A' \cap B') = (A \cap B')'$ cannot be universally verified in an SFST $([0,1]^X, T, S, N)$ when T and S are N-duals.*

Proof. Duality and the equality would yield

$$N(T(r, N(t))) = S(t, T(N(r), N(t))),$$

for all r, t in $[0,1]$. We will see immediately that this equation is not possible. If it would hold, consider $x_N \in (0,1)$ be the unique fixed point of N, i.e., $N(x_N) = x_N$ and substitute $r = t = x_N$ to obtain

$$S(x_N, T(x_N, x_N)) = N(T(x_N, x_N)) = S(x_N, x_N). \tag{3}$$

The substitution $r = 0$ in (3) yields $1 = S(t, N(t))$ for all t so we obtain $T(t, N(t)) = N(S(N(t), t)) = 0$, $T(x_N, x_N) = 0$ so $x_N = S(x_N, 0) = S(x_N, T(x_n, x_N)) = S(x_N, x_N) = S(x_N, N(x_N)) = 1$, which is a contradiction.

When T and S are not duals the above functional equation admits, as we will see immediately for some unknowns S, T and N, a lot of solutions. Note for example that in $([0,1]^X, \mathrm{Min}, W^*, 1-j)$ it is $A' \cup B = A' \cup (A \cap B)$ and $A' \cup B = B \cup (A' \cap B')$ but it does not follow $(A \cap B')' = B \cup (A' \cap B')$ because $A' \cup B$ is not $(A \cap B')'$. $\square$

5. The modeling of the general case

Let us consider the functional model

$$N_3(T_1(r, N_2(t))) = S_2(t, T(N_1(r), N(t))), \tag{4}$$

where N, N_1, N_2, N_3 are strong-negations, T, T_1 are continuous t-norms and S_2 is a continuous t-conorm. Thus (4) is a Pexider functional equation in two variables r and t running in $[0,1]$ and six unknown functions. Our aim here is to show that in this case there are infinite solutions for (4).

Note that a modeling of type (4) may appear for example in the case that $(r \cdot t')' = t + (r' \cdot t')$ means that r, t are random variables and the "equality" in Elkan's condition means that the random variables $1 - \mathrm{Min}(r, 1-t)$ and $\mathrm{Max}(t, \mathrm{Min}(1-r, 1-t))$ have equal pointwise distribution functions.

First observe that the substitution $t = 0$ yields $N_3 = N_1$ and $r = 1$ implies $N_3(N_2(t)) = t$, i.e., $N_3 = N_2$.

Call $N_4 := N_1 = N_2 = N_3$ and introduce the continuous t-conorm $S_1(r,t) = N_4(T_1(N_4(r), N_4(t)))$. Then we can present (1) in the form

$$S_1(r,t) = S_2(t, T(r, N(t))). \tag{5}$$

This equation has been completely studied by the authors (see [7], [13]). While in the case of S_1 and S_2 non-strict Archimedean t-conorms and T an ordinal sum of Archimedean t-norms, Eq. (5) does not determine T, in all other cases we have the following:

Theorem 3. *Let $S_2(r,t) = s_2^{(-1)}(s_2(r) + s_2(t))$ a non-strict Archimedean t-conorm with associated strong negation $N_{S_2}(r) = s_2^{-1}(1 - s_2(r))$, where $s_2 : [0,1] \to [0,1]$ is continuous strictly increasing $s_2(0) = 0$, $s_2(1) = 1$ and $s_2^{(-1)}(r) = s_2^{-1}(r)$ on $[0,1]$ but $s_2^{(-1)}(r) = 1$ if $r \geqslant 1$. Let N be a strong-negation, let S_1 be a continuous t-conorm and let T be a continuous t-norm which is not an ordinal sum of Archimedean t-norms.*

Then (5) holds if and only if one of the following conditions holds:
(i) $S_1 = \mathrm{Max}$, $T = s_2^{(-1)} \circ W \circ s_2 \times s_2$, $N = N_{S_2}$;
(ii) $S_1 = s_2^{-1} \circ \mathrm{Prod}^* \circ s_2 \times s_2$, $T = s_2^{-1} \circ \mathrm{Prod} \circ s_2 \times s_2$, $N = N_{S_2}$;
(iii) $S_1 = s_2^{-1} \circ T^*_{1/\lambda} \circ s_2 \times s_2$, $T = s_2^{-1} \circ T_\lambda \circ s_2 \times s_2$, $N = N_{S_2}$, where for any α, $0 < \alpha < \infty$, $\alpha \neq 1$,

$$T_\alpha(r,t) = \log_\alpha[1 + (\alpha^r - 1)(\alpha^t - 1)/(\alpha - 1)],$$

$$T^*_\alpha(r,t) = 1 - T_\alpha(1-r, 1-t);$$

(iv) $S_1 = S_2$, $T = \mathrm{Min}$, $N \geqslant N_{S_2} = N_{S_1}$.

With the appropriate changes one obtains from this theorem the corresponding solutions of (4). For example, (4) is satisfied with $S_1 = S_2 = W^*$, $N = 1 - id$, $T = \mathrm{Min}$ and $T_1 = N_3 \circ W^* \circ (N_3 \times N_3)$ for any strong-negation $N_3 = N_2 = N_1$.

To sum up this section: when mixing a connective $\cup$ with connectives $\cap_i$, $i = 1, 2$, and $'j$, $j = 1, 2, 3, 4$, Eq. (1) takes the form

$$(A \cap_1 B'^1)'^2 = B \cup (A'^3 \cap_2 B'^4),$$

and Theorem 3 gives uncountable many solutions of it.

6. Conclusion

Like in [11], the first conclusion of this paper is that Elkan was wrong in solving the equation

$$(A \cap B')' = B \cup (A' \cap B') \qquad (*)$$

in the De Morgan's Lattice $([0,1]^X, \mathrm{Min}, \mathrm{Max}, 1 - id)$. But, by going further than [11], a second conclusion arises: from a theoretical point of view Elkan did not think that, among the wide possibilities open by Standard Fuzzy Set Theories, equation (*) should be universally valid in some of them.

Nevertheless, the authors are very grateful to Charles Elkan in at least two aspects. The first one is because seven years ago he was writing down his opinions on "the paradoxical success of fuzzy logic". His paper opened a discussion that helped to shed light on some interesting issues, and we hope that the present paper will contribute to clarify some ideas behind Fuzzy Set Theory.

The second one is that because of the excitement Elkan's awarded paper provoked, we have had the opportunity of looking at the equation (*) from a wide perspective that leads us to posing and solving the equation $N_3(T_1(r, N_2(t))) = S_2(t, T(N_1(r), N(t)))$, perhaps the most challenging functional equation until today arising from a question on Fuzzy Logic.

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