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Existence of solutions for nonlocal impulsive partial functional integrodifferential equations via fractional operators

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ABSTRACT

In this paper, by using the Leray–Schauder alternative, we have investigated the existence of mild solutions to first-order impulsive partial functional integrodifferential equations with nonlocal conditions in an α-norm. We assume that the linear part generates an analytic compact bounded semigroup, and that the nonlinear part is a Lipschitz continuous function with respect to the fractional power norm of the linear part. An example is also given to illustrate our main results.

1. Introduction

The theory of impulsive differential equations has found wide applications in many branches of physics and technical sciences; see the monographs of Lakshmikantham et al. [1], Bainov and Simeonov [2], Benchohra et al. [3], and the papers of Rogovchenko [4] and the survey papers of Rogovchenko [5], Bainov [6] and the references therein. Recently, much attention has been paid to existence results for the impulsive differential and integrodifferential equations in abstract spaces; for example, see [7–13]. In this paper, we are concerned with the following impulsive partial functional integrodifferential equations with nonlocal conditions

$$x'(t) = Ax(t) + F\left(t, x(\sigma_1(t)), \ldots, x(\sigma_n(t)), \int_0^t h(t, s, x(\sigma_{n+1}(s)))ds\right),$$
$$t \in J = [0, b],\ t \neq t_k,\ k = 1, \ldots, m, \tag{1.1}$$
$$x(0) + g(x) = x_0, \tag{1.2}$$
$$\Delta x(t_k) = I_k(x(t_k)), \quad k = 1, \ldots, m, \tag{1.3}$$

where the unknown $x(\cdot)$ takes values in the Banach space X, and A is the infinitesimal generator of a compact, analytic semigroup $T(t), t > 0$; $0 < t_1 < \cdots < t_m < b$, are prefixed points and the symbol $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^-)$ and $x(t_k^+)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. F, h, g, I_k and $\sigma_i, i = 1, \ldots, n+1$, are given functions to be specified later.

Nonlocal conditions were initiated in [14,15] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked in [15,16], the nonlocal condition can be more useful than the standard initial

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condition to describe some physical phenomena. For other contributions on the nonlocal problems; see [17–23] and the references therein. Very recently, there has been extensive study of impulsive differential equations with nonlocal conditions, and concerning this matter we cite the pioneer works; Liang et al. [24], and Anguraj and Karthikeyan [25] have studied the existence, uniqueness and continuous dependence of a mild solution of a nonlocal Cauchy problem for an impulsive neutral functional differential evolution equation. The purpose of this paper is to continue the study of these authors. We get the existence results for mild solutions of problem (1.1)–(1.3) with an α-norm as in [19], assuming that F is defined on $J \times X_\alpha^{n+1}$, the nonlocal item g only depends upon the continuous properties on $C(J, X_\alpha)$, where $X_\alpha = D(A^\alpha)$, for some $0 < \alpha < 1$, the domain of the fractional power of A. Our results are based on the analytic semigroup theory of linear operators, the Banach contraction principle and the Leray–Schauder alternative.

The rest of this paper is organized as follows: In Section 2 we recall briefly some basic definitions and preliminary facts which will be used throughout this paper. The existence theorems for problem (1.1)–(1.3) and their proofs are arranged in Section 3. Finally, in Section 4 an example is presented to illustrate the applications of the obtained result.

2. Preliminaries

In this section, we shall introduce some notations, definitions and lemmas which are used throughout this paper.

Let $(X, \|\cdot\|)$ be a Banach space. $C(J, X)$ is the Banach space of continuous functions from J into X with the norm

$$\|x\|_J = \sup\{\|x(t)\| : t \in J\}$$

and let $L(X)$ denote the Banach space of bounded linear operators from X to X. A measurable function $x : J \to X$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable (for properties of the Bochner integral; see [26]). $L^1(J, X)$ denotes the Banach space of measurable functions $x : J \to X$ which are Bochner integrable normed by

$$\|x\|_{L^1} = \int_0^b \|x(t)\|\mathrm{d}t \quad \text{for all } x \in L^1(J, X).$$

The notation $B_r[x, X]$ stands for the closed ball with center at x and radius $r > 0$ in X.

Throughout this paper, $A : D(A) \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $T(t)$. Let $0 \in \rho(A)$. Then it is possible to define the fractional power A^α, for $0 < \alpha \le 1$, as a closed linear operator on its domain $D(A^\alpha)$ (see [27]). Furthermore, the subspace $D(A^\alpha)$ is dense in X and the expression

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x \in D(A^\alpha),$$

defines a norm on $D(A^\alpha)$. Let X_α be the Banach space $D(A^\alpha)$ endowed with the norm $\|x\|_\alpha$, and in the following, we use $\|\cdot\|_\alpha$ to denote the operator norm in X_α. Then for each $0 < \alpha \le 1$, X_α is a Banach space, and $X_\alpha \hookrightarrow X_\beta$ for $0 < \beta < \alpha \le 1$ and the imbedding is compact whenever the resolvent operator of A is compact. For semigroup $\{T(t)_{t\ge 0}\}$, the following properties will be used:

(a) there is an $M \ge 1$ such that $\|T(t)\| \le M$, for all $0 \le t \le b$;
(b) for any $0 \le \alpha \le 1$, there exists a constant $M_\alpha > 0$ such that

$$\|A^\alpha T(t)\| \le \frac{M_\alpha}{t^\alpha}, \quad 0 < t \le b.$$

In order to define the solution of (1.1)–(1.3), we introduce the space $PC([0, b], X_\alpha) = \{x : J \to X_\alpha : x(t)$ is continuous at $t \ne t_k$ and left continuous at $t = t_k$ and the right limit $x(t_k^+)$ exists for $k = 1, 2, \ldots, m\}$, which is a Banach space with the norm

$$\|x\|_{PC} := \sup_{t\in J} \|x(t)\|_\alpha.$$

Then $PC(J, X_\alpha)$ is a Banach space.

To simplify the notations, we put $t_0 = 0,\ t_{m+1} = b$ and for $x \in PC([0, b], X_\alpha)$ we denote by $\hat{x}_k \in C([t_k, t_{k+1}]; X_\alpha),\ k = 0, 1, \ldots, m$, the function given by

$$\hat{x}_k(t) := \begin{cases} x(t) & \text{for } t \in (t_k, t_{k+1}], \\ x(t_k^+) & \text{for } t = t_k. \end{cases}$$

Moreover, for $B \subseteq PC([0, b], X_\alpha)$ we denote by $\hat{B}_k,\ k = 0, 1, \ldots, m$, the set $\hat{B}_k = \{\hat{x}_k : x \in B\}$.

Definition 2.1. A function $x(\cdot) \in PC(J, X_\alpha)$ is said to be a mild solution to problem (1.1)–(1.3) if it satisfies the following integral equation

$$x(t) = T(t)[x_0 - g(x)] + \sum_{0<t_k<t} T(t - t_k)I_k(x(t_k)) + \int_0^t T(t - s)F\left(s, x(\sigma_1(s)), \ldots, x(\sigma_n(s)), \int_0^s h(s, \tau, x(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\mathrm{d}s, \quad 0 \le t \le b. \tag{2.1}$$

Lemma 2.1. *A set $B \subseteq PC([0,b], X_\alpha)$ is relatively compact in $PC([0,b], X_\alpha)$ if and only if, the set $\hat{B}_k$ is relatively compact in $C([t_k, t_{k+1}]; X_\alpha)$, for every $k = 0, 1, \ldots, m$.*

Lemma 2.2 (*Leray–Schauder Nonlinear Alternative [28]*)**.** *Let X be a Banach space with $Z \subset X$ convex. Assume that U is a relatively open subset of Z with $0 \in U$ and $P : \overline{U} \to Z$ is a compact map. Then either*

(i) *P has a fixed point in $\overline{U}$, or*
(ii) *there exists a point $v \in \partial U$ such that $v \in \lambda P(v)$ for some $\lambda \in (0,1)$.*

3. Main results

In this section, we state and prove the existence theorem for problem (1.1)–(1.3). Let us list the following hypothesis: for some $\alpha \in (0,1)$,

(H1) The function $F : J \times X_\alpha^{n+1} \to X$ is continuous and there exist constants $L > 0, L_1 \geq 0$, such that for all $x_i, y_i \in X_\alpha, i = 1, \ldots, n+1$, we have

$$\|F(t, x_1, x_2, \ldots, x_{n+1}) - F(t, y_1, y_2, \ldots, y_{n+1})\| \leq L\left[\sum_{i=1}^{n+1} \|x_i - y_i\|_\alpha\right],$$

and

$$L_1 = \max_{t\in J} \|F(t, 0, \ldots, 0)\|.$$

(H2) The function $h : J \times J \times X_\alpha \to X_\alpha$ is continuous and there exist constants $N > 0, N_1 \geq 0$, such that for all $x, y \in X$,

$$\|h(t,s,x) - h(t,s,y)\|_\alpha \leq N\|x-y\|_\alpha,$$

and

$$N_1 = \max_{0\le s\le t\le b} \|h(t,s,0)\|_\alpha.$$

(H3) $\sigma_i : J \to J, i = 1, \ldots, n+1$, are continuous functions such that $\sigma_i(t) \leq t, i = 1, \ldots, n+1$.
(H4) $I_k \in C(X_\alpha, X_\alpha), k = 1, \ldots, m$ are all compact operators, and there exist continuous nondecreasing functions $\Psi_k : [0,\infty) \to (0,\infty), k = 1, \ldots, m$, such that

$$\|I_k(x)\|_\alpha \leq \Psi_k(\|x\|_\alpha), \quad \text{for each } x \in X_\alpha.$$

(H5) (i) The function $g(\cdot) : PC(J, X_\alpha) \to X_\alpha$ is continuous and there exists a $\delta \in (0, t_1)$ such that $g(\phi) = g(\psi)$ for any $\phi, \psi \in PC(J, X_\alpha)$ with $\phi = \psi$ on $[\delta, b]$.
(ii) There is a continuous nondecreasing function $\Lambda : [0,\infty) \to (0,\infty)$ such that
$$\|g(\phi)\|_\alpha \leq \Lambda(\|\phi\|_{PC}), \quad \phi \in PC(J, X_\alpha).$$
(H6) There exists a constant $M^* > 0$ such that

$$\frac{M^*}{\left[M_* + M\Lambda(M^*) + M\sum_{k=1}^{m}\Psi_k(M^*)\right]e^\eta} > 1, \tag{3.1}$$

where $\eta = \frac{M_\alpha L(n+Nb)b^{1-\alpha}}{1-\alpha}$, $M_* = M\|x_0\|_\alpha + \frac{M_\alpha b^{1-\alpha}(bLN_1+L_1)}{1-\alpha}$.

Theorem 3.1. *Let $x_0 \in X_\alpha$. If assumptions* (H1)–(H6) *are satisfied, then the impulsive nonlocal Cauchy problem (1.1)–(1.3) has at least one mild solution on J.*

Proof. Let $L_0 > 0$ be a constant chosen such that

$$q := \sup_{t\in J}\left\{LM_\alpha(n+Nb)\int_0^t e^{-L_0(t-s)}(t-s)^{-\alpha}ds\right\} < 1,$$

and we introduce in the space $PC(J, X_\alpha)$ the equivalent norm defined as

$$\|\phi\|_V := \sup_{t\in J} e^{-L_0 t}\|\phi(t)\|_\alpha.$$

Then, it is easy to see that $V := (PC(J, X_\alpha), \|\cdot\|_V)$ is a Banach space. Fix $v \in PC(J, X_\alpha)$ and for $t \in J, \phi \in V$, we now define an operator

$$(Q_v\phi)(t) = T(t)[x_0 - g(v)] + \sum_{0<t_k<t} T(t-t_k)I_k(v(t_k)) + \int_0^t T(t-s)F\left(s, \phi(\sigma_1(s)), \ldots, \phi(\sigma_n(s)), \int_0^s h(s, \tau, \phi(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\mathrm{d}s. \quad (3.2)$$

Since $T(\cdot)\ (x_0 - g(v)) \in PC(J, X_\alpha)$, it follows from (H1)–(H3) that $(Q_v\phi)(t) \in V$ for all $\phi \in V$. Let $\phi, \psi \in V$, we have

$$\begin{aligned}
\mathrm{e}^{-L_0t}\|(Q_v\phi)(t) - (Q_v\psi)(t)\|_\alpha &\le \mathrm{e}^{-L_0t}\int_0^t \Bigg\|A^\alpha T(t-s)\Bigg[F\left(s, \phi(\sigma_1(s)), \ldots, \phi(\sigma_n(s)), \int_0^s h(s, \tau, \phi(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right) \\
&\quad - F\left(s, \psi(\sigma_1(s)), \ldots, \psi(\sigma_n(s)), \int_0^s h(s, \tau, \psi(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\Bigg]\Bigg\| \mathrm{d}s \\
&\le LM_\alpha \int_0^t \mathrm{e}^{-L_0t}(t-s)^{-\alpha}\Bigg[\|\phi(\sigma_1(s)) - \psi(\sigma_1(s))\|_\alpha \\
&\quad + \cdots + \|\phi(\sigma_n(s)) - \psi(\sigma_n(s))\|_\alpha + \Bigg\|\int_0^s h(s, \tau, \phi(\sigma_{n+1}(\tau)))\mathrm{d}\tau \\
&\quad - \int_0^s h(s, \tau, \psi(\sigma_{n+1}(\tau)))\mathrm{d}\tau\Bigg\|_\alpha\Bigg]\mathrm{d}s \\
&\le LM_\alpha \int_0^t \mathrm{e}^{-L_0t}(t-s)^{-\alpha}\Bigg[\mathrm{e}^{L_0\sigma_1(s)}\sup_{s\in J}\mathrm{e}^{-L_0s}\|\phi(s) - \psi(s)\|_\alpha \\
&\quad + \cdots + \mathrm{e}^{L_0\sigma_n(s)}\sup_{s\in J}\mathrm{e}^{-L_0s}\|\phi(s) - \psi(s)\|_\alpha \\
&\quad + N\int_0^s \|\phi(\sigma_{n+1}(\tau)) - \psi(\sigma_{n+1}(\tau))\|_\alpha \mathrm{d}\tau\Bigg]\mathrm{d}s \\
&\le LM_\alpha \int_0^t \mathrm{e}^{-L_0t}(t-s)^{-\alpha}\Bigg[n\mathrm{e}^{L_0s}\sup_{s\in J}\mathrm{e}^{-L_0s}\|\phi(s) - \psi(s)\|_\alpha \\
&\quad + Nb\mathrm{e}^{L_0\sigma_{n+1}(s)}\sup_{s\in J}\mathrm{e}^{-L_0s}\|\phi(s) - \psi(s)\|_\alpha\Bigg]\mathrm{d}s \\
&\le LM_\alpha \int_0^t \mathrm{e}^{-L_0(t-s)}(t-s)^{-\alpha}\Bigg[n\sup_{s\in J}\mathrm{e}^{-L_0s}\|\phi(s) - \psi(s)\|_\alpha \\
&\quad + Nb\sup_{s\in J}\mathrm{e}^{-L_0s}\|\phi(s) - \psi(s)\|_\alpha\Bigg]\mathrm{d}s \\
&\le LM_\alpha(n + Nb)\int_0^t \mathrm{e}^{-L_0(t-s)}(t-s)^{-\alpha}\mathrm{d}s\|\phi - \psi\|_V \\
&\le q\|\phi - \psi\|_V, \quad t \in J,
\end{aligned}$$

which implies that

$$\mathrm{e}^{-L_0t}\|(Q_v\phi)(t) - (Q_v\psi)(t)\|_\alpha \le q\|\phi - \psi\|_V, \quad t \in J.$$

Thus

$$\|Q_v\phi - Q_v\psi\|_V \le q\|\phi - \psi\|_V, \quad \phi, \psi \in V.$$

Therefore, Q_v is a strict contraction. By the Banach contraction principle we conclude that Q_v has a unique fixed point $\phi_v \in V$ and Eq. (3.2) has a unique mild solution on $[0, b]$. Set

$$\tilde{v}(t) := \begin{cases} v(t) & \text{if } t \in (\delta, b], \\ v(\delta) & \text{if } t \in [0, \delta]. \end{cases}$$

From (3.2), we have

$$\phi_{\tilde{v}}(t) = T(t)[x_0 - g(\tilde{v})] + \sum_{0<t_k<t} T(t-t_k)I_k(v(t_k)) + \int_0^t T(t-s)F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), \ldots, \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s, \tau, \phi_{\tilde{v}}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\mathrm{d}s. \quad (3.3)$$

Consider the map $\Gamma : PC_\delta = PC([\delta, b], X_\alpha) \to PC_\delta$ defined by

$$(\Gamma v)(t) = \phi_{\tilde{v}}(t), \quad t \in [\delta, b]. \tag{3.4}$$

We shall show that Γ satisfies all conditions of Lemma 2.2. The proof will be given in several steps.

Step 1. Γ maps bounded sets into bounded sets in PC_δ.

Indeed, it is enough to show that there exists a positive constant $\mathcal{L}$ such that for each $v \in B_r(\delta) := \{\phi \in PC_\delta; \sup_{\delta \le t \le b} \|\phi(t)\|_\alpha \le r\}$ one has $\|\Gamma v\|_{PC} \le \mathcal{L}$.

Let $v \in B_r(\delta)$, then for $t \in (0, b]$, we have

$$\begin{aligned}
\|\phi_{\tilde{v}}(t)\|_\alpha &\le \|T(t)[x_0 - g(\tilde{v})]\|_\alpha + \left\| \sum_{0<t_k<t} T(t-t_k) I_k(v(t_k)) \right\|_\alpha \\
&\quad + \int_0^t \left\| T(t-s) F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), \ldots, \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right) \right\|_\alpha \mathrm{d}s \\
&\le M[\|x_0 + g(\tilde{v})\|_\alpha] + M \sum_{k=1}^m \|I_k(v(t_k))\|_\alpha + M_\alpha \int_0^t (t-s)^{-\alpha} \Bigg[\Bigg\| F\Big(s, \phi_{\tilde{v}}(\sigma_1(s)), \ldots, \phi_{\tilde{v}}(\sigma_n(s)), \\
&\qquad \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))\mathrm{d}\tau \Big) - F(s, 0, \ldots, 0) \Bigg\| + \|F(s,0,\ldots,0)\| \Bigg] \mathrm{d}s \\
&\le M[\|x_0\|_\alpha + \|g(\tilde{v})\|_\alpha] + M \sum_{k=1}^m \Psi_k(\|v(t_k)\|_\alpha) \\
&\quad + M_\alpha \int_0^t (t-s)^{-\alpha} \Bigg\{ L \Bigg[\sup_{s\in(0,b]} \|\phi_{\tilde{v}}(s)\|_\alpha + \cdots + \sup_{s\in(0,b]} \|\phi_{\tilde{v}}(s)\|_\alpha \\
&\quad + \int_0^s [\|h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau))) - h(s,\tau,0)\|_\alpha + \|h(s,\tau,0)\|_\alpha]\mathrm{d}\tau \Bigg] + L_1 \Bigg\} \mathrm{d}s \\
&\le M[\|x_0\|_\alpha + \Lambda(\|\tilde{v}\|_{PC})] + M \sum_{k=1}^m \Psi_k(\|v(t_k)\|_\alpha) \\
&\quad + M_\alpha \int_0^t (t-s)^{-\alpha} \left\{ L[n \sup_{s\in(0,b]} \|\phi_{\tilde{v}}(s)\|_\alpha + b(N \sup_{s\in(0,b]} \|\phi_{\tilde{v}}(s)\|_\alpha + N_1)] + L_1 \right\} \mathrm{d}s \\
&\le M_* + M\Lambda(r) + M \sum_{k=1}^m \Psi_k(r) + M_\alpha L(n+Nb) \int_0^t (t-s)^{-\alpha} \sup_{s\in(0,b]} \|\phi_{\tilde{v}}(s)\|_\alpha \mathrm{d}s,
\end{aligned}$$

where $M_* = M\|x_0\|_\alpha + \frac{M_\alpha b^{1-\alpha}(bLN_1+L_1)}{1-\alpha}$. Using the Gronwall inequality we get

$$\sup_{t\in(0,b]} \|\phi_{\tilde{v}}(t)\|_\alpha \le \left[M_* + M\Lambda(r) + \sum_{k=1}^m \Psi_k(r) \right] \mathrm{e}^{\frac{M_\alpha L(n+Nb)b^{1-\alpha}}{1-\alpha}}.$$

Thus

$$\|\Gamma v\|_{PC} \le \left[M_* + M\Lambda(r) + \sum_{k=1}^m \Psi_k(r) \right] \mathrm{e}^{\frac{M_\alpha L(n+Nb)b^{1-\alpha}}{1-\alpha}} := \mathcal{L}.$$

Step 2. Γ is continuous on $B_r(\delta)$.

From (3.2) and (H1)–(H5), we deduce that for $v_1, v_2 \in B_r(\delta),\ t \in (0, b]$,

$$\begin{aligned}
\|\phi_{\tilde{v}_1}(t) - \phi_{\tilde{v}_2}(t)\|_\alpha &\le \|T(t)[g(\tilde{v}_1) - g(\tilde{v}_2)]\|_\alpha + \left\| \sum_{0<t_k<t} T(t-t_k) I_k(v_1(t_k)) - \sum_{0<t_k<t} T(t-t_k) I_k(v_2(t_k)) \right\|_\alpha \\
&\quad + \int_0^t \Bigg\| A^\alpha T(t-s) \Bigg[F\left(s, \phi_{\tilde{v}_1}(\sigma_1(s)), \ldots, \phi_{\tilde{v}_1}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}_1}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right) \\
&\quad - F\left(s, \phi_{\tilde{v}_2}(\sigma_1(s)), \ldots, \phi_{\tilde{v}_2}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}_2}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right) \Bigg] \Bigg\| \mathrm{d}s \\
&\le M\|g(\tilde{v}_1) - g(\tilde{v}_2)\|_\alpha + M \sum_{k=1}^m \|I_k(v_1(t_k)) - I_k(v_2(t_k))\|_\alpha
\end{aligned}$$

$$
\begin{aligned}
&+LM_\alpha \int_0^t (t-s)^{-\alpha}\Bigg[\|\phi_{\tilde v_1}(\sigma_1(s))-\phi_{\tilde v_2}(\sigma_1(s))\|_\alpha+\cdots+\|\phi_{\tilde v_1}(\sigma_n(s))-\phi_{\tilde v_2}(\sigma_n(s))\|_\alpha\\
&+\left\|\int_0^s h(s,\tau,\phi_{\tilde v_1}(\sigma_{n+1}(\tau)))\mathrm{d}\tau-\int_0^s h(s,\tau,\phi_{\tilde v_2}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right\|_\alpha\Bigg]\mathrm{d}s\\
\le{}& M\|g(\tilde v_1)-g(\tilde v_2)\|_\alpha+M\sum_{k=1}^m\|I_k(v_1(t_k))-I_k(v_2(t_k))\|_\alpha\\
&+LM_\alpha\int_0^t(t-s)^{-\alpha}\Bigg[\sup_{s\in[0,b]}\|\phi_{\tilde v_1}(s)-\phi_{\tilde v_2}(s)\|_\alpha\\
&+\cdots+\sup_{s\in[0,b]}\|\phi_{\tilde v_1}(s)-\phi_{\tilde v_2}(s)\|_\alpha+N\int_0^s[\|\phi_{\tilde v_1}(\sigma_{n+1}(\tau))-\phi_{\tilde v_2}(\sigma_{n+1}(\tau))\|_\alpha]\mathrm{d}\tau\Bigg]\mathrm{d}s\\
\le{}& M\|g(\tilde v_1)-g(\tilde v_2)\|_\alpha+M\sum_{k=1}^m\|I_k(v_1(t_k))-I_k(v_2(t_k))\|_\alpha\\
&+LM_\alpha\int_0^t(t-s)^{-\alpha}\Bigg[n\sup_{s\in[0,b]}\|\phi_{\tilde v_1}(s)-\phi_{\tilde v_2}(s)\|_\alpha+Nb\sup_{s\in[0,b]}\|\phi_{\tilde v_1}(s)-\phi_{\tilde v_2}(s)\|_\alpha\Bigg]\mathrm{d}s\\
\le{}& M\|g(\tilde v_1)-g(\tilde v_2)\|_\alpha+M\sum_{k=1}^m\|I_k(v_1(t_k))-I_k(v_2(t_k))\|_\alpha\\
&+LM_\alpha(n+Nb)\int_0^t(t-s)^{-\alpha}\sup_{s\in[0,b]}\|\phi_{\tilde v_1}(s)-\phi_{\tilde v_2}(s)\|_\alpha\,\mathrm{d}s.
\end{aligned}
$$

Using again the Gronwall inequality, that for t, v_1, v_2 as above

$$\sup_{s\in[0,b]}\|\phi_{\tilde v_1}(t)-\phi_{\tilde v_2}(t)\|_\alpha\le M\mathrm{e}^{\frac{LM_\alpha(n+Nb)b^{1-\alpha}}{1-\alpha}}\left[\|g(\tilde v_1)-g(\tilde v_2)\|_\alpha+\sum_{k=1}^m\|I_k(v_1(t_k))-I_k(v_2(t_k))\|_\alpha\right],$$

for all $t\in[0,b]$, which implies that

$$\|\Gamma v_1-\Gamma v_2\|_{PC}\le M\mathrm{e}^{\frac{LM_\alpha(n+Nb)b^{1-\alpha}}{1-\alpha}}\left[\|g(\tilde v_1)-g(\tilde v_2)\|_\alpha+\sum_{k=1}^m\|I_k(v_1(t_k))-I_k(v_2(t_k))\|_\alpha\right]$$

for all $t\in[\delta,b],\ v_1,v_2\in B_r(\delta)$. Therefore, Γ is continuous.

Step 3. Γ is a compact operator.

To this end, we consider the decomposition $\Gamma=\Gamma_1+\Gamma_2$, where Γ_1, Γ_2 are the operators on $B_r(\delta)$ defined respectively by

$$
\begin{aligned}
(\Gamma_1 v)(t) ={}& T(t)[x_0-g(\tilde v)]+\int_0^t T(t-s)\\
&\times F\left(s,\phi_{\tilde v}(\sigma_1(s)),\ldots,\phi_{\tilde v}(\sigma_n(s)),\int_0^s h(s,\tau,\phi_{\tilde v}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\mathrm{d}s,\quad t\in[\delta,b],\\
(\Gamma_2 v)(t) ={}& \sum_{0<t_k<t}T(t-t_k)I_k(v(t_k)),\quad t\in[\delta,b].
\end{aligned}
$$

We first show that Γ_1 is a compact operator.

(i) $\Gamma_1(B_r(\delta))$ is equicontinuous.

Let $\delta\le\tau_1<\tau_2\le b$, and $\varepsilon>0$ be small, note that

$$
\begin{aligned}
&\left\|F\left(s,\phi_{\tilde v}(\sigma_1(s)),\ldots,\phi_{\tilde v}(\sigma_n(s)),\int_0^s h(s,\tau,\phi_{\tilde v}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\right\|\\
&\le\left\|F\left(s,\phi_{\tilde v}(\sigma_1(s)),\ldots,\phi_{\tilde v}(\sigma_n(s)),\int_0^s h(s,\tau,\phi_{\tilde v}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)-F(s,0,\ldots,0)\right\|+\|F(s,0,\ldots,0)\|\\
&\le L\left[\|\phi_{\tilde v}(\sigma_1(s))\|_\alpha+\cdots+\|\phi_{\tilde v}(\sigma_n(s))\|_\alpha+\left\|\int_0^s h(s,\tau,\phi_{\tilde v}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right\|_\alpha\right]+L_1\\
&\le L\left[\sup_{s\in[\delta,b]}\|\phi_{\tilde v}(s)\|_\alpha+\cdots+\sup_{s\in[\delta,b]}\|\phi_{\tilde v}(s)\|_\alpha+\int_0^s[\|h(s,\tau,\phi_{\tilde v}(\tau))-h(s,\tau,0)\|_\alpha+\|h(s,\tau,0)\|_\alpha]\,\mathrm{d}\tau\right]+L_1
\end{aligned}
$$

$$
\begin{aligned}
&\le L\left[n\sup_{s\in[\delta,b]}\|\phi_{\tilde{v}}(s)\|_\alpha+b\left[N\sup_{s\in[\delta,b]}\|\phi_{\tilde{v}}(s)\|_\alpha+N_1\right]\right]+L_1\\
&\le L\left[(n+Nb)\sup_{s\in[\delta,b]}\|\phi_{\tilde{v}}(s)\|_\alpha+bN_1\right]+L_1\\
&\le L[(n+Nb)r+bN_1]+L_1:=M^{**}.
\end{aligned}
$$

We have

$$
\begin{aligned}
\|\Gamma_1 v(\tau_2)-\Gamma_1 v(\tau_1)\|_\alpha &\le \|[T(\tau_2)-T(\tau_1)][x_0-g(\tilde{v})]\|_\alpha\\
&\quad+\left\|\int_0^{\tau_2}T(\tau_2-s)F\left(s,\phi_{\tilde{v}}(\sigma_1(s)),\ldots,\phi_{\tilde{v}}(\sigma_n(s)),\int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\mathrm{d}s\right.\\
&\quad\left.-\int_0^{\tau_1}T(\tau_1-s)F\left(s,\phi_{\tilde{v}}(\sigma_1(s)),\ldots,\phi_{\tilde{v}}(\sigma_n(s)),\int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\mathrm{d}s\right\|_\alpha\\
&\le \|[T(\tau_2)-T(\tau_1)][x_0-g(\tilde{v})]\|_\alpha+\int_0^{\tau_1-\varepsilon}\|[T(\tau_2-s)-T(\tau_1-s)]\\
&\quad\times F\left(s,\phi_{\tilde{v}}(\sigma_1(s)),\ldots,\phi_{\tilde{v}}(\sigma_n(s)),\int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\Big\|_\alpha\mathrm{d}s\\
&\quad+\int_{t_1-\varepsilon}^{t_1}\|[T(\tau_2-s)-T(\tau_1-s)]\ F\left(s,\phi_{\tilde{v}}(\sigma_1(s)),\ldots,\phi_{\tilde{v}}(\sigma_n(s)),\int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\Big\|_\alpha\mathrm{d}s\\
&\quad+\int_{\tau_1}^{\tau_2}\left\|T(\tau_2-s)F\left(s,\phi_{\tilde{v}}(\sigma_1(s)),\ldots,\phi_{\tilde{v}}(\sigma_n(s)),\int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\right\|_\alpha\mathrm{d}s\\
&\le \|[T(\tau_2)-T(\tau_1)][x_0-g(\tilde{v})]\|_\alpha+\|T(\tau_2-\tau_1+\varepsilon)-T(\varepsilon)\|\int_0^{\tau_1-\varepsilon}\|A^\alpha T(\tau_1-s-\varepsilon)\|M^{**}\mathrm{d}s\\
&\quad+\int_{\tau_1-\varepsilon}^{\tau_1}\|A^\alpha[T(\tau_2-s)-T(\tau_1-s)]\|M^{**}\mathrm{d}s+\int_{\tau_1}^{\tau_2}\|A^\alpha T(\tau_2-s)\|M^{**}\mathrm{d}s\\
&\le \|[T(\tau_2)-T(\tau_1)][x_0-g(\tilde{v})]\|_\alpha+\frac{M_\alpha}{1-\alpha}M^{**}(\tau_1-\varepsilon)^{1-\alpha}\|T(\tau_2-\tau_1+\varepsilon)-T(\varepsilon)\|\\
&\quad+\frac{M_\alpha}{1-\alpha}M^{**}[(\tau_2-\tau_1)^{1-\alpha}-(\tau_2-\tau_1-\varepsilon)^{1-\alpha}+\varepsilon^{1-\alpha}]+\frac{M_\alpha}{1-\alpha}M^{**}(\tau_2-\tau_1)^{1-\alpha}.
\end{aligned}
$$

We see that $\|\Gamma_1 v(\tau_2)-\Gamma_1 v(\tau_1)\|$ tends to zero independently of $v\in B_r(\delta)$ as $\tau_2-\tau_1\to 0$, since the compactness of $T(t)$ for $t>0$, implies the continuity in the uniform operator topology. Thus Γ_1 maps $B_r(\delta)$ into an equicontinuous family of functions.

(ii) The set $\Gamma_1(B_r(\delta))(t)$ is precompact in X_α.

Let $\delta<t\le s\le b$ be fixed and ε a real number satisfying $0<\varepsilon<t$. For $v\in B_r(\delta)$, we define

$$
\begin{aligned}
(\Gamma_{1,\varepsilon}v)(t)&=T(t)[x_0-g(\tilde{v})]+\int_0^{t-\varepsilon}T(t-s)F\left(s,\phi_{\tilde{v}}(\sigma_1(s)),\ldots,\phi_{\tilde{v}}(\sigma_n(s)),\int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\mathrm{d}s\\
&=T(t)[x_0-g(\tilde{v})]+T(\varepsilon)\int_0^{t-\varepsilon}T(t-s-\varepsilon)\\
&\qquad\times F\left(s,\phi_{\tilde{v}}(\sigma_1(s)),\ldots,\phi_{\tilde{v}}(\sigma_n(s)),\int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\mathrm{d}s.
\end{aligned}
$$

Using the compactness of $T(t)$ for $t>0$, we deduce that the set $\{(\Gamma_{1,\varepsilon}v)(t):v\in B_r(\delta)\}$ is precompact $v\in B_r(\delta)$ for every ε, $0<\varepsilon<t$. Moreover, for every $v\in B_r(\delta)$ we have

$$
\begin{aligned}
\|(\Gamma_1 v)(t)-(\Gamma_{1,\varepsilon}v)(t)\|_\alpha &\le \int_{t-\varepsilon}^{t}\left\|A^\alpha T(t-s)F\left(s,\phi_{\tilde{v}}(\sigma_1(s)),\ldots,\phi_{\tilde{v}}(\sigma_n(s)),\int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\right\|\mathrm{d}s\\
&\le M_\alpha\int_{t-\varepsilon}^{t}(t-s)^{-\alpha}M^{**}\mathrm{d}s\\
&\le \frac{M_\alpha M^{**}}{1-\alpha}\varepsilon^{1-\alpha}.
\end{aligned}
$$

Therefore, there are precompact sets arbitrarily close to the set $\{(\Gamma_1 v):v\in B_r(\delta)\}$. Hence the set $\{(\Gamma_1 v):v\in B_r(\delta)\}$ is a precompact in X_α. It is easy to see that $\Gamma_1(B_r(\delta))$ is uniformly bounded. Since we have shown that $\Gamma_1(B_r(\delta))$ is an equicontinuous collection, by the Arzela–Ascoli theorem it suffices to show that Γ_1 maps $B_r(\delta)$ into a precompact set in X_α.

Next, it remains to verify that Γ_2 is also a compact operator.

We begin by showing $\Gamma_2(B_r(\delta))$ is equicontinuous, for any $\varepsilon > 0$ and $0 < t < b$. Since the functions I_k, $k = 1, 2, \ldots, m$, are compact in X_α, we find that the set $W = \{I_k(v(t_k)) : v \in B_r(\delta)\}$ is precompact in X_α. From the strong continuity of $(T(t))_{t\geq 0}$, for $\varepsilon > 0$, we can choose $0 < \xi < b - t$ such that

$$\|(T(t+h) - T(t))x\|_\alpha < \frac{\varepsilon}{m}, \quad x \in W,$$

when $|h| < \xi$. For each $v \in B_r(\delta)$, $t \in (0, b)$ be fixed, $t \in [t_i, t_{i+1}]$, such that

$$\begin{aligned}\|[\widehat{(\Gamma_2 v)}]_i(t+h) - [\widehat{(\Gamma_2 v)}]_i(t)\|_\alpha &\leq \sum_{k=1}^{m} \|(T(t+h-t_k) - T(t-t_k))I_k(v(t_k))\|_\alpha \\ &< \varepsilon.\end{aligned}$$

As $h \to 0$, and ε is sufficiently small, the right-hand side of the above inequality tends to zero independently of v, so that $[\Gamma_2\widehat{(B_r(\delta))}]_i$, $i = 1, 2, \ldots, m$, are equicontinuous.

Now we prove that $[\Gamma_2\widehat{(B_r(\delta))}]_i$, $i = 1, 2, \ldots, m$, is precompact for every $t \in [\delta, b]$.

From the following relations

$$[\widehat{(\Gamma_2 v)}]_i(t) = \sum_{0<t_k<t} T(t-t_k)I_k(v(t_k)) \in \sum_{k=1}^{m} T(t-t_k)I_k(B_r(\delta)[0, X_\alpha]).$$

We conclude that $[\Gamma_2\widehat{(B_r(\delta))}]_i$, $i = 1, 2, \ldots, m$, is precompact for every $t \in [t_i, t_{i+1}]$.

By Lemma 2.1, we infer that $\Gamma_2(B_r(\delta))$ is precompact. Now an application of the Arzela–Ascoli theorem justifies the precompactness of $\Gamma_2(B_r(\delta))$. Therefore, Γ_2 is a compact operator, and hence Γ is a compact operator.

Step 4. We now show there exists an open set $U \subseteq PC_\delta$ with $v \notin \lambda \Gamma v$ for $\lambda \in (0, 1)$ and $v \in \partial U$. Let $\lambda \in (0, 1)$ and $v \in PC_\delta$ be a possible solution of $v = \lambda \Gamma(v)$ for some $0 < \lambda < 1$. Thus, for each $t \in (0, b]$,

$$\begin{aligned} v(t) &= \lambda \phi_{\tilde{v}}(t) = \lambda T(t)[x_0 - g(\tilde{v})] + \lambda \sum_{0<t_k<t} T(t-t_k)I_k(v(t_k)) \\ &\quad + \lambda \int_0^t T(t-s)F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), \ldots, \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s, \tau, \phi_{\tilde{v}}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\mathrm{d}s. \end{aligned} \tag{3.5}$$

This implies by (H1)–(H5) that for each $t \in (0, b]$ we have $\|v(t)\|_\alpha \leq \|\phi_{\tilde{v}}(t)\|_\alpha$ and

$$\begin{aligned} \|\phi_{\tilde{v}}(t)\|_\alpha &\leq \|T(t)[x_0 - g(\tilde{v})]\|_\alpha + \sum_{k=1}^{m} \|T(t-t_k)I_k(v(t_k))\|_\alpha \\ &\quad + \int_0^t \left\|T(t-s)F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), \ldots, \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s, \tau, \phi_{\tilde{v}}(\sigma_{n+1}(\tau)))\mathrm{d}\tau\right)\right\|_\alpha \mathrm{d}s \\ &\leq M_* + M\Lambda(\|\tilde{v}\|_{PC}) + M\sum_{k=1}^{m} \Psi_k(\|v(t_k)\|_\alpha) + M_\alpha L(n+Nb)\int_0^t (t-s)^{-\alpha} \sup_{s\in(0,b]} \|\phi_{\tilde{v}}(s)\|_\alpha \mathrm{d}s. \end{aligned}$$

Making use of the Gronwall inequality, such that

$$\sup_{t\in(0,b]} \|\phi_{\tilde{v}}(t)\|_\alpha \leq \left[M_* + M\Lambda(\|\tilde{v}\|_{PC}) + M\sum_{k=1}^{m} \Psi_k(\|v\|_{PC})\right] \mathrm{e}^{\frac{M_\alpha L(n+Nb)b^{1-\alpha}}{1-\alpha}},$$

and the previous inequality holds. Consequently,

$$\|v\|_{PC} \leq \left[M_* + M\Lambda(\|\tilde{v}\|_{PC}) + M\sum_{k=1}^{m} \Psi_k(\|v\|_{PC})\right] \mathrm{e}^{\eta},$$

and therefore

$$\frac{\|v\|_{PC}}{\left[M_* + M\Lambda(\|\tilde{v}\|_{PC}) + M\sum_{k=1}^{m} \Psi_k(\|v\|_{PC})\right] \mathrm{e}^{\eta}} \leq 1.$$

Then, by (H6), there exists M^* such that $\|v\| \neq M^*$. Set

$$U = \left\{ v \in PC[\delta, b] : \sup_{\delta \le t \le b} \|v(t)\|_\alpha < M^* \right\}.$$

As a consequence of Steps 1–3 in Theorem 3.1, it suffices to show that $\Gamma : \overline{U} \to PC_\delta$ is a compact map.

From the choice of U, there is no $x \in \partial U$ such that $v \in \lambda \Gamma v$ for $\lambda \in (0, 1)$. As a consequence of Lemma 2.2, we deduce that Γ has a fixed point $\tilde{v}_* \in \overline{U}$. Then, we have

$$\begin{aligned} x(t) &= T(t)[x_0 - g(\tilde{v}_*)] + \sum_{0<t_k<t} T(t-t_k) I_k(v(t_k)) \\ &\quad + \int_0^t T(t-s) F\left(s, x(\sigma_1(s)), \ldots, x(\sigma_n(s)), \int_0^s h(s, \tau, x(\sigma_{n+1}(\tau))) \mathrm{d}\tau\right) \mathrm{d}s. \end{aligned} \tag{3.6}$$

Noting that $x = \phi_{\tilde{v}_*} = (\Gamma \tilde{v}_*)(t) = \tilde{v}_*$, $t \in [\delta, b]$. By (H5)(i), we obtain

$$g(x) = g(\tilde{v}_*) \quad \text{and} \quad v_*(t_k) = x(t_k).$$

This implies, combined with (3.6), that $x(t)$ is a mild solution of problem (1.1)–(1.3) and the proof of Theorem 3.1 is complete. □

Remark 3.1. (H4) is satisfied if there exist constants $a_k > 0$, $b_k > 0$, $\alpha_k \in [0, 1)$, $k = 1, \ldots, m$, such that

$$\|I_k(x)\|_\alpha \le a_k + b_k \|x\|_\alpha^{\alpha_k}, \quad k = 1, \ldots, m,\ x \in X_\alpha,$$

and (H5) is satisfied if there exist constants d_1 and d_2, $\mu \in [0, 1)$ such that

$$\|g(\phi)\|_\alpha \le d_1 + d_2 \|\phi\|_{PC}^{\mu}, \quad \phi \in PC(J, X_\alpha),$$

or (H4) is satisfied if there exist constants $\bar{a}_k > 0$, $\bar{b}_k > 0$, $k = 1, \ldots, m$, such that

$$\|I_k(x)\|_\alpha \le \bar{a}_k + \bar{b}_k \|x\|_\alpha, \quad k = 1, \ldots, m,\ x \in X_\alpha,$$

and (H5) is satisfied if there exist constants $\bar{d}_1$ and $\bar{d}_2$ such that

$$\|g(\phi)\|_\alpha \le \bar{d}_1 + \bar{d}_2 \|\phi\|_{PC}, \quad \phi \in PC(J, X_\alpha).$$

4. Application

In this section, we shall give an example to illustrate our results. Consider the following impulsive partial functional integrodifferential equation of the form:

$$z_t(t, x) = \frac{\partial^2}{\partial x^2} a_0(t, x) z(t, x) + a_1(t) z(\sin t, x) + \sin z(t, x) + \frac{1}{1+t^2} \int_0^t a_2(s) z(\sin s, x) \mathrm{d}s, \tag{4.1}$$

$$\Delta z(t_k, x) = \int_0^\pi p_k(x, y) z(t_k, y) \mathrm{d}y, \quad k = 1, \ldots, m, \tag{4.2}$$

$$z(t, 0) = z(t, \pi) = 0, \tag{4.3}$$

$$z(0, x) + \int_\delta^1 [z(s, x) + \log(1 + |z(s, x)|)] \mathrm{d}s = z_0(x), \quad 0 \le t \le 1,\ 0 \le x \le \pi, \tag{4.4}$$

where $\delta > 0$, $z_0(x) \in X = L^2([0, \pi])$ and $z_0(0) = z_0(\pi) = 0$.

Let $X = L^2([0, \pi])$ and the operators $A : D(A) \subset X \to X$ given by $Au = u''$ with

$$D(A) := H_0^2([0, \pi]) = \{u \in X : u'' \in X, u(0) = u(\pi) = 0\}.$$

It is well known that A generates a strongly continuous semigroup $T(\cdot)$ which is compact, analytic and self-adjoint. Furthermore, A has a discrete spectrum; the eigenvalues are $-n^2$, $n \in N$, with the corresponding normalized eigenvectors $z_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$. Then the following properties hold:

(i) If $u \in D(A)$, then

$$Au = \sum_{n=1}^{\infty} n^2 \langle u, z_n \rangle z_n.$$

(ii) For each $u \in X$,

$$A^{\frac{1}{2}}u = \sum_{n=1}^{\infty} \frac{1}{n}\langle u, z_n\rangle z_n.$$

(iii) The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}u = \sum_{n=1}^{\infty} n\langle u, z_n\rangle z_n$$

on the space $D(A^{\frac{1}{2}}) = \{u(\cdot) \in X, \sum_{n=1}^{\infty} n\langle u, z_n\rangle z_n \in X\}$ and $\|A^{-\frac{1}{2}}\| = 1$.

Lemma 4.1 (*[29]*). *If $m \in D(A^{\frac{1}{2}})$, then m is absolutely continuous, $m' \in X$ and $\|m'\| = \|A^{\frac{1}{2}}m\|$.*
We assume that

(a) *the functions $a_i(\cdot)$, $i = 1, 2, 3$, is continuous on $[0, 1]$, and $l_i = \sup_{0\le s\le 1} |a_i(s)| < 1$, $i = 1, 2, 3$.*
(b) *the functions $p_k : [0, \pi] \times [0, \pi] \to \mathbb{R}$, $k = 1, 2, \ldots, m$, are continuously differentiable and*

$$\gamma_k = \left(\int_0^{\pi}\int_0^{\pi}\left(\frac{\partial}{\partial x}p_k(x, y)\right)^2 \mathrm{d}x\mathrm{d}y\right)^{1/2} < \infty$$

for every $k = 1, 2, \ldots, m$.

According to paper [29], we know that, if $z \in X_{\frac{1}{2}}$, then z is absolutely continuous, $z' \in X$, and $z(0) = z(\pi) = 0$. Then, for $(t, z) \in [0, 1]\times X_{\frac{1}{2}}$, and $z \in C([0, 1], X_{\frac{1}{2}})$, we can define respectively $F : [0, 1]\times X_{\frac{1}{2}} \times X_{\frac{1}{2}} \to X$, $h : [0, 1]\times[0, 1]\times X_{\frac{1}{2}} \to X_{\frac{1}{2}}$ and $g : PC([0, 1], X_{\frac{1}{2}}) \to X_{\frac{1}{2}}$ by

$$F\left(t, z(\sigma(t)), \int_0^t h(t, s, z(\sigma(s)))\mathrm{d}s\right)(x) = a_1(t)z(\sin t, x) + a_2(t)\sin z(t, x) + \int_0^t \frac{a_3(s)}{1+t^2}z(\sin s, x)\mathrm{d}s,$$
$$\int_0^t h(t, s, \sigma(z(s)))(x)\mathrm{d}s = \int_0^t \frac{a_3(s)}{1+t^2}z(\sin s, x)\mathrm{d}s,$$
$$I_k(z)(x) = \int_0^{\pi} p_k(x, y)z(t_k, y)\mathrm{d}y, \quad k = 1, 2, \ldots, m$$

and

$$g(z)\ (x) = \int_{\delta}^1 [z(s, x) + \log(1 + |z(s, x)|)]\mathrm{d}s, \quad z \in PC([0, 1], X_{\frac{1}{2}}).$$

Then Eqs. (4.1)–(4.4) takes the abstract form (1.1)–(1.3). Moreover, for $z_i, y_i \in X_{\frac{1}{2}}$, $i = 1, 2$ and $x \in [0, \pi]$, we have

$$\begin{aligned}
\|F(t, z_1, y_1) - F(t, z_2, y_2)\| &\le \left(\int_0^{\pi}(a_1(t)\ (z_1(\sin t, x) - z_2(\sin t, x)))^2\mathrm{d}x\right)^{\frac{1}{2}} \\
&\quad + \left(\int_0^{\pi}(a_2(t)\ (\sin z_1(t, x) - \sin z_2(t, x)))^2\mathrm{d}x\right)^{\frac{1}{2}} \\
&\quad + \left(\int_0^{\pi}\left(\int_0^t \frac{a_3(s)}{1+t^2}(z_1(\sin s, x) - z_2(\sin s, x))\mathrm{d}s\right)^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le (l_1 + l_2 + l_3)\ \|A^{-\frac{1}{2}}\|\ \|A^{\frac{1}{2}}(z_1 - z_2)\| \\
&= (l_1 + l_2 + l_3)\ \|z_1 - z_2\|_{\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
\|I_k(z)\|_{\frac{1}{2}} &= \|A^{\frac{1}{2}}I_k(z)\ (\cdot)\ \| = \|I_k(z)'(\cdot)\| \\
&\le \gamma_k\|z\| \le \gamma_k\|A^{-\frac{1}{2}}\|\ \|A^{\frac{1}{2}}z\| = \gamma_k\|z\|_{\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
\|g(z)\|_{\frac{1}{2}} &= \|A^{\frac{1}{2}}g(z)(\cdot)\| = \|g(z)'(\cdot)\| \\
&\le (1-\delta)\|z\|_{\frac{1}{2}}\left(1 + \sqrt{\pi} + \frac{1}{1+\|z\|}\right)
\end{aligned}$$

$$\leq (1-\delta)\|z\|_{\frac{1}{2}}\left(2+\sqrt{\pi}+\frac{1}{1+\|A^{-\frac{1}{2}}\|\,\|A^{\frac{1}{2}}z\|}\right)$$
$$=(1-\delta)\|z\|_{\frac{1}{2}}\left(2+\sqrt{\pi}+\frac{1}{1+\|z\|_{\frac{1}{2}}}\right).$$

It is easy to see that with these choices, assumptions (H1)–(H5) of Theorem 3.1 are satisfied. In particular, the constants are $L=l_1+l_2+l_3, N=l_3, L_1=N_1=0$ and $c_k=\gamma_k$, $c=1-\delta$. If we assume that

$$M\left[(1-\delta)(2+\sqrt{\pi})+\sum_{k=1}^{m}\gamma_k\right]\mathrm{e}^{\eta_1}<\frac{1}{2},$$

where $\eta_1=2M_{\frac{1}{2}}(l_1+l_2+l_3)\,(1+l_3)$, and if we can choose the constant $M^*=\max\{8M\|z_0\|_{\frac{1}{2}}\mathrm{e}^{\eta_1},1\}$, then

$$1>\frac{M\left[\|z_0\|_{\frac{1}{2}}+M^*\left(2+\sqrt{\pi}+\frac{1}{1+M^*}\right)(1-\delta)+M^*\sum_{k=1}^{m}\gamma_k\right]\mathrm{e}^{\eta_1}}{M^*}.$$

Now the condition (H6) in Section 3 holds and hence by Theorem 3.1, we deduce that nonlocal Cauchy problem (4.1)–(4.4) has a mild solution on [0, 1].

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