Semigroups of Order Preserving Partial Transformations of a Totally Ordered Set*

N. R. Reilly

Simon Fraser University, Burnaby, British Columbia, Canada

Communicated by G. B. Preston

Received June 1, 1970

INTRODUCTION

If \( S \) is an inverse semigroup and we define the relation \( \theta \) on the lattice, \( \Lambda(S) \), of congruences on \( S \) by saying that two congruences are \( \theta \)-equivalent if and only if they induce the same partition of the idempotents; then \( \theta \) is a congruence relation on the lattice \( \Lambda(S) \), and we write \( \Theta(S) \) for \( \Lambda(S)/\theta \). The relationship between \( \Theta(S) \) and certain equivalence relations on a partially ordered set \( X \), when \( S \) is an inverse semigroup of order preserving partial transformations of \( X \) was considered in [10]. Here we continue that study in the special case where \( X \) is totally ordered. We denote by \( J_X \) the subsemigroup of the symmetric inverse semigroup on \( X \) consisting of those elements \( \alpha \) for which the domain, \( \Delta(\alpha) \), and the range, \( \nabla(\alpha) \), of \( \alpha \) are ideals of \( X \), and \( \alpha \) is an order isomorphism of \( \Delta(\alpha) \) onto \( \nabla(\alpha) \). Let \( S \) be a subsemigroup of \( J_X \); then a convex equivalence relation \( \rho \) on \( X \) is a convex congruence if, for \( x, y \in \Delta(\alpha) \), \( \alpha \in S \), \( (\alpha x, \alpha y) \in \rho \) if and only if \( (x, y) \in \rho \). We call \( S \) \( o \)-primitive if the universal and identity convex congruences are the only convex congruences.

In the main theorem of Section 2, it is shown that if \( |X| \geq 2 \) and \( S \) is an \( o \)-primitive inverse subsemigroup of \( J_X \) then either \( X \) is anti-isomorphic to the natural numbers or isomorphic to the set of integers (in either of these cases \( S \) is transitive), or every transitivity class \( Y \) in the Dedekind completion \( \bar{X} \) of \( X \) is dense in \( \bar{X} \). (In certain circumstances, the converse will also hold.) The induced representation of \( S \) on any transitivity class is faithful and \( o \)-primitive.

The results in this section extend to certain semigroups of mappings results of Holland’s and McCleary’s in [3] and [4] with a view to applying them in Section 3 to \( \Theta(S) \) for certain semigroups \( S \).

* The research for this paper was supported, in part, by N.R.C. Grant No. A-4044

Copyright © 1972 by Academic Press, Inc.
All rights of reproduction in any form reserved.
In Section 3 we combine the results of Section 2 and of [10] with Munn's representation (Lemma 3.1, [7]) of an inverse semigroup $S$ on the semilattice of idempotents of $S$ to obtain certain facts about $\Theta(S)$ for certain inverse semigroups $S$. For instance, if $E_S$ is totally ordered, $|E_S| > 2$ and $|\Theta(S)| = 2$ then either $S$ is a bisimple $\omega$-semigroup or $S$ is a bisimple $Z$-semigroup or each \( \mathcal{D} \)-class of idempotents is dense in $E_S$. Moreover, each \( \mathcal{D} \)-class $D$ of $S$ is a bisimple inverse semigroup such that $E_D$ is totally ordered and $|\Theta(D)| = 2$.

I. Basic Results

We adopt the notation and terminology of [2]. In particular, a semigroup $S$ is called an inverse semigroup if $a \in aSa$, for all $a \in S$, and the idempotents of $S$ commute. Then there is a unique element $x$ such that $a = axa$ and $x = xax$. We call $x$ the inverse of $a$ and write $x = a^{-1}$. For any inverse semigroup $S$, we denote $E_S$ the subsemigroup of $S$ consisting of the idempotents of $S$. There is a natural partial ordering on $S$, compatible with the multiplication, defined by $a \leq b$ if and only if $ab^{-1} = a$. With respect to this partial ordering $E_S$ is a semilattice where, by a semilattice, we mean a partially ordered set in which any two elements $x$ and $y$ have a greatest lower bound, which we denote by $x \land y$. By a lattice we mean a semilattice in which any two elements also have a least upper bound, which we denote by $x \lor y$. For the basic results on inverse semigroups the reader is referred to [2].

Denote by $\Lambda(S)$ the lattice of congruences on an inverse semigroup $S$; that is, the lattice of equivalence relations $\rho$ such that, for $a, b, c \in S$, $(a, b) \in \rho$ implies that $(ca, cb) \in \rho$ and $(ac, bc) \in \rho$. Define the relation $\theta$ (cf. [11]) on $\Lambda(S)$ by $(\rho_1, \rho_2) \in \theta$ if and only if $\rho_1 \upharpoonright E_S = \rho_2 \upharpoonright E_S$, where $\rho_i \upharpoonright E_S$ denotes the restriction of the congruence $\rho_i$ to $E_S$. Then

**Lemma 1.1** ([11] Theorem 5.1). Let $S$ be an inverse semigroup and the relation $\theta$ be defined as above. Then

(i) $\theta$ is a congruence on $\Lambda(S)$;

(ii) each $\theta$-class is a complete modular sublattice of $\Lambda(S)$ (with a greatest and least element).

We shall denote the lattice of $\theta$-classes of an inverse semigroup $S$ by $\Theta(S)$. Now each congruence on an inverse semigroup $S$ determines a normal partition of $E_S$; that is a partition $P = \{E_\alpha : \alpha \in J\}$ such that

- $E(i)$ $\alpha, \beta \in J$ implies that there exists a $\gamma \in J$ such that $E_\alpha E_\beta \subseteq E_\gamma$;
E(ii) $\alpha \in J$ and $a \in S$ implies that there exists a $\beta \in J$ such that $aE_a a^{-1} \subseteq E_\beta$.

We call an equivalence relation $\rho$ on $E_S$ a normal equivalence if its classes constitute a normal partition of $E_S$.

Conversely, if $P$ is a normal partition of $E_S$ then $P$ is induced by some congruence on $S$. Thus the lattice of normal partitions of $E_S$ is just (isomorphic to) $\Theta(S)$.

The least congruence in the $\theta$-class corresponding to the normal partition $P$ can be characterized as follows (cf. [10]).

**Lemma 1.2.** Let $P = \{E_\alpha : \alpha \in J\}$ be a normal partition of the semilattice of idempotents of an inverse semigroup $S$. Let $\sigma = \{(a, b) \in S \times S :$ there exists an $\alpha \in J$ with $aa^{-1}, bb^{-1} \in E_\alpha$ and $ea = eb$, for some $e \in E_\alpha\}$. Then $\sigma$ is the smallest congruence on $S$ inducing the partition $P$.

By a one-to-one partial transformation of a set $X$, we mean a one-to-one mapping $\alpha$ of a subset $Y$ of $X$ onto a subset $Y' = Y\alpha$ of $X$. We call $Y$ the domain of $\alpha$, $Y'$ the range of $\alpha$ and write $\Delta(\alpha) = Y$, $\nabla(\alpha) = Y'$. If we denote by $\mathcal{J}_X$ the set of all one-to-one partial transformations of $X$ then, with respect to the natural multiplication of mappings, $\mathcal{J}_X$ is an inverse semigroup, called the symmetric inverse semigroup on $X$ [2].

Let $X$ be a totally ordered set. By an ideal of $X$, we mean a subset $Y$ of $X$ such that $x < y, y \in Y$ implies that $x \in Y$. By a principal ideal we mean an ideal of the form $\{x : x \leq y\}$, for some fixed element $y$. Then we call $\{x : x \leq y\}$ the (principal) ideal generated by $y$ and denote it by $\langle y \rangle$. Let $J_X$ denote the set of all $\alpha \in \mathcal{J}_X$ such that

(i) $\Delta(\alpha)$ and $\nabla(\alpha)$ are ideals of $X$;

(ii) $\alpha$ is an order isomorphism of $\Delta(\alpha)$ onto $\nabla(\alpha)$; that is, a one-to-one mapping of $\Delta(\alpha)$ onto $\nabla(\alpha)$ such that, for $x, y \in \Delta(\alpha)$, $x \leq y$ if and only if $x\alpha \leq y\alpha$.

Then $J_X$ is an inverse subsemigroup of $\mathcal{J}_X$. If $T_X$ denotes the set of $\alpha$ in $J_X$ such that $\Delta(\alpha)$ is a principal ideal then $T_X$ is an inverse subsemigroup of $J_X$. The semigroup $T_X$, in the more general situation when $X$ is a semilattice, was introduced and studied by Munn, [6] and [7].

If $X$ is a totally ordered set, $a \in X$, $B$, and $C$ are subsets of $X$ then "$a < B$" shall mean that $a < b$ for all $b \in B$, and "$B < C$" shall mean that $b < c$, for all $b \in B, c \in C$, and similarly with $\leq$ replacing $\leq$. Also $B \setminus C$ shall denote $\{b : b \in B, b \notin C\}$.

For an arbitrary set $A$, we denote by $|A|$, the cardinality of $A$.

We denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{Z}$ the set of all
integers, under their natural ordering. We denote by $N'$ a set order anti-isomorphic with $N$.

II. X Totally Ordered

Throughout this section we shall be concerned exclusively with subsemigroups of $J_X$ where $X$ is a totally ordered set.

Let $X$ be a totally ordered set. We call an equivalence relation $\rho$ on $X$ a convex equivalence if

(i) $x \leq y \leq z$, $(x, z) \in \rho$ implies that $(x, y) \in \rho$.

We denote the lattice of convex equivalence relations on $X$ by $E(X)$.

If $S$ is a subsemigroup of $J_X$, then by a convex congruence $\rho$ on $X$ we mean a convex equivalence relation $\rho$ on $X$ such that

(ii) if $x, y \in \Delta(a)$ and $a \in S$ then $(x, y) \in \rho$ if and only if $(xa, ya) \in \rho$.

We observe that, if $S$ is an inverse subsemigroup of $J_X$ then, since we have inverse mappings at our disposal, condition (ii) above on $\rho$ is equivalent to

(ii)' if $x, y \in \Delta(a)$, $a \in S$ and $(x, y) \in \rho$ then $(xa, ya) \in \rho$.

In the terminology of [10] a convex congruence on $X$ is an $s$-congruence. Clearly the definition of a convex congruence depends on $S$; however, since it should not lead to any confusion we generally omit any indication of this dependence. We shall denote the lattice (as it clearly is) of convex congruences on $X$ by $C(X)$ or by $C_s(X)$, when we have occasion to discuss the lattice of convex congruences with respect to distinct semigroups. We denote the maximum and minimum element of $C(X)$, that is, the universal and identity congruences by $\omega_X$ and $\iota_X$, respectively. Thus, if $X$ is nontrivial, $C(X)$ has at least two distinct elements. If $C(X)$ has exactly two distinct elements then we say that $S$ is o-primitive.

We say that $S$ is transitive, or is a transitive subsemigroup of $J_X$, if, for all $x, y \in X$, there exists an $a \in S$ such that $x \in \Delta(a)$ and $xa = y$.

The following lemma is an extension of Holland and McCleary’s Theorem 3 in [4], for ordered permutation groups, and may be established in an analogous manner.

**Lemma 2.1.** Let $X$ be a totally ordered set and $S$ be a transitive subsemigroup of $J_X$. Then $C(X)$ is a totally ordered set.

In the absence of the assumption that $S$ is transitive, $C(X)$ will still be a distributive lattice, as we now demonstrate.
We shall say that a lattice $L$ satisfies the infinite distributive law $D$ if, for any $\rho \in L$ and any subset $\{\sigma_i : i \in I\}$ of $L$,

$$\rho \land \left( \bigvee_i \sigma_i \right) = \bigvee_i (\rho \land \sigma_i).$$

**Lemma 2.2** [1, p. 24]. Let $X$ be a totally ordered set. Then $E(X)$ satisfies the infinite distributive law $D$.

**Proposition 2.3.** Let $X$ be a totally ordered set and $S$ be a subsemigroup of $J_X$. Then $C(X)$ is a complete sublattice of $E(X)$ and hence satisfies the infinite distributive law $D$.

**Proof.** Let $A = \{\sigma_i : i \in I\} \subseteq C(X)$. Then clearly the greatest lower bound of $A$ in both $E(X)$ and $C(X)$ is the relation

$$\{(x, y) : (x, y) \in \sigma_i , \text{ for all } i\}.$$

The least upper bound $\tau$ of $A$ in $E(X)$ is obtained as follows: $(x, y) \in \tau$ if and only if there exist $x = x_0 , x_1 ,..., x_n = y$ in $X$ such that $(x_{j-1} , x_j) \in \sigma_j$, $j = 1,..., n$, for some $j \in I$. Without loss of generality, let $x \geq y$. Then

$$(x , x \land x_1) = (x_0 \land x_0 , x_0 \land x_1) \in \sigma_1$$

and

$$(x_0 \land x_{j-1} , x_0 \land x_j) \in \sigma_j$$

for $j = 2,..., n$. Hence we may assume that $x_j \leq x$, for $j = 0,..., n$.

If $x \in \Delta(a)$, for some $a \in S$, then $x_j \in \Delta(a)$, $j = 1,..., n$, and, since $\sigma_j \in C(X)$, for all $j$, $(x_{j-1}a , x_ja) \in \sigma_j$, for $j = 1,..., n$. Thus $(xa, ya) \in \tau$.

Conversely, suppose that $(x,y) \in \Delta(a)$, $a \in S$ and that $(xa, ya) \in \tau$. Let $xa \geq ya$. Then there exist $y_k , k = 0,..., m$, such that $y_0 = xa,..., y_m = ya$, $y_k \leq xa$ and, for each $k$, $(y_{k-1} , y_k) \in \sigma_k$, for some $\sigma_k$. Let $x_k , k = 0,..., n$ be such that $x_ka = y_k$ (in particular $x = x_0 , y = x_n$). Then $(x_{k-1} , x_k) \in \sigma_k$, for $k = 1,..., n$, since $\sigma_k \in C(X)$. Hence $(x,y) \in \tau$. Thus $\tau \in C(X)$ and $C(X)$ is a complete sublattice of $E(X)$.

For any totally ordered set $X$, let $\bar{X}$ denote the Dedekind completion of $X$, without end points.

**Lemma 2.4.** Let $X$ be a totally ordered set. For $x \in f_X$ let $\bar{x} \in f_X$ be such that $\Delta(\bar{x}) = \{ \bar{x} \in \bar{X} : \bar{x} \leq x_1 , \text{ for some } x_1 \in \Delta(x) \}$ and, for $\bar{x} \in \Delta(\bar{x})$, $\bar{x}x = \sup\{yx : y \leq x, y \in X\}$. Then the mapping $x \mapsto \bar{x}$ is an isomorphism of $f_X$ into $f_X$. 


Proof. Let \( x \in \Delta(\alpha \beta) \). Then \( x \leq y \), for some \( y \in \Delta(\alpha \beta) \). Thus \( y \in \Delta(\alpha) \) and \( y\alpha \in \Delta(\beta) \). Also \( \overline{x} \overline{\alpha} \leq \overline{y} \overline{\alpha} = y\alpha \). Hence \( x \in \Delta(\overline{\alpha} \overline{\beta}) \).

Conversely, if \( x \in \Delta(\overline{\alpha} \overline{\beta}) \) then \( x \in \Delta(\overline{\alpha}) \) and \( \overline{x} \overline{\alpha} \in \Delta(\overline{\beta}) \). Thus there exist \( u \in \Delta(\alpha) \), \( v \in \Delta(\beta) \) such that \( x \leq u \), \( \overline{x} \overline{\alpha} \leq v \). Let \( w = \min\{u\alpha, v\} \), and \( z \) be such that \( z\alpha = w \). Then \( z \leq u \), \( z\alpha \leq v \), and so \( z \in \Delta(\alpha) \), and \( z\alpha \in \Delta(\beta) \). Thus \( z \in \Delta(\overline{\alpha} \overline{\beta}) \), \( \overline{x} \leq z \), and so \( x \in \Delta(\overline{\alpha} \overline{\beta}) \). Hence \( \Delta(\overline{\alpha} \overline{\beta}) = \Delta(\overline{\alpha} \overline{\beta}) \). The remainder of the proof follows routinely.

On account of the natural embedding of the above lemma we henceforth consider \( J_X \) as a subsemigroup of \( J_X \).

If \( A \) and \( B \) are semilattices and \( \alpha \) is a mapping of \( A \) into \( B \), then \( \alpha \) is a semilattice homomorphism if, for any \( u, v \in A \), \((u \wedge v)\alpha = u\alpha \wedge v\alpha \).

**Lemma 2.5.** Let \( X \) be a totally ordered set and \( S \) be a subsemigroup of \( J_X \subseteq J_X \). Let \( \alpha \) be the mapping of \( C(X) \) into \( C(X) \) defined by \( \rho \alpha = \rho|_{X \times X} \). Then \( \alpha \) is a semilattice homomorphism of \( C(X) \) onto \( C(X) \). Also \( \rho \alpha = \omega_X \) if and only if \( \rho = \omega_X \); and \( \rho \alpha = \iota_X \) if and only if \( \rho = \iota_X \). Thus \( S \) is \( o \)-primitive on \( X \) if and only if \( S \) is \( o \)-primitive on \( X \).

Proof. If \( \rho_1, \rho_2 \in C(X) \) then it is clear that \( (\rho_1 \wedge \rho_2)\alpha = \rho_1\alpha \wedge \rho_2\alpha \). Thus, \( \alpha \) is a semilattice homomorphism.

Let \( \sigma \in C(X) \) and let \( \rho = \{(u, v): \text{either } x = y \text{ or there exist } u, v \in X \text{ with } u \leq x, y \leq v \text{ and } (u, v) \in \sigma \} \). Clearly \( \rho \) is a convex equivalence on \( X \). Let \((x, y) \in \rho, x, y \in X \) and \( x, y \in \Delta(a) \). There exist \( u, v \in X \) such that \( u \leq x, y \leq v \) and \( (u, v) \in \sigma \). Without loss of generality, let \( x \leq y \).

Since \( y \in \Delta(a) \), there exists an element \( w \in X \) such that \( y \leq w \) and \( w \in \Delta(a) \). Let \( z = \min\{v, w\} \). Then \( (u, z) \in \sigma, z \in \Delta(a) \) and \( u \leq x \leq y \leq z \). Thus \( u\alpha \leq x\alpha \leq y\alpha \leq z\alpha \) and \( (u\alpha, z\alpha) \in \sigma \). Hence \( (x\alpha, y\alpha) \in \rho \) and \( \rho \) is a convex congruence. Since \( \rho \alpha = \sigma \), it follows that \( \alpha \) is a semilattice homomorphism of \( C(X) \) onto \( C(X) \).

Let \( \rho \alpha = \omega_X \). Let \( x, y \in X \), then there exist \( u, v \in X \) such that \( u \leq x, y \leq v \). Since \( (u, v) \in \omega_X \), we must have \( (x, y) \in \rho \). Thus \( \rho = \omega_X \).

Let \( \rho \alpha = \iota_X \) and \( (x, y) \in \rho \). If \( x \neq y \), say \( x < y \), then there must exist \( u, v \in X \) such that \( u \neq v \) and \( x \leq u, v \leq y \). Then \( (u, v) \in \rho \alpha = \iota_X \), a contradiction. Thus \( x = y \) and \( \rho = \iota_X \).

In general, even if \( S \) is the group of order preserving permutations of \( X \), the mapping \( \alpha \) of Lemma 2.5 need not be a lattice homomorphism, as Example 3 in Section 4 illustrates.

Let \( Y \) be a subset of \( X \). We say that \( Y \) is dense in \( X \) if, for all \( u, v \in X \) with \( u < v \), there exists \( y \in Y \) with \( u < y < v \). If, for all \( x, y \in Y \) there exists an \( a \in S \) such that \( x \in \Delta(a) \) and \( xa = y \) and if, for \( u \in \Delta(a) \), \( a \in S \) we have \( u \in Y \) if and only if \( u\alpha \in Y \), then we call \( Y \) a transitivity class of \( X \). If \( S \) is an inverse semigroup, then clearly, since we have inverse mappings at our disposal, or
see Chapter 7 of [2], $X$ is a disjoint union of transitivity classes. If $S$ is not an inverse semigroup, then some elements of $X$ may not belong to a transitivity class.

**Theorem 2.6.** Let $X$ be a totally ordered set such that $|X| > 2$ and let $S$ be an $o$-primitive inverse subsemigroup of $J_X$, then either (i) $X$ is isomorphic to $N'$ or $Z$ and $S$ is transitive or (ii) every transitivity class $Y$ of $X$ is dense in $X$.

*Proof.* By Lemma 2.5, $S$ is $o$-primitive on $X$ if and only if $S$ is $o$-primitive on $X$ and so, without loss of generality, we assume that $X = X$.

Since $S$ is an inverse semigroup, $X$ is a disjoint union of transitivity classes.

If $X$ is finite then clearly any convex equivalence relation on $X$ is a convex congruence. Hence, since $|X| > 2$ and $S$ is $o$-primitive, $X$ is not finite.

Let $Y$ be a trivial transitivity class, that is with just a single element $y$, say. Define the relation $\rho$ as follows:

$$(u, v) \in \rho \iff \text{either } (i) \ u = v = y;$$

or (ii) $y < u, v$;

or (iii) $y > u, v$.

Then $\rho$ is clearly a nontrivial convex congruence, contradicting the hypothesis that $S$ is $o$-primitive. Hence all transitivity classes are nontrivial.

Let $Y$ be any transitivity class. Define the relation $\sigma$ on $X$ as follows:

$$(u, v) \in \sigma \iff \text{either } (i) \ u = v$$

or (ii) $u < v$ and $[u, v] \cap Y \subseteq \{u\}$;

or (iii) $v < u$ and $[v, u] \cap Y \subseteq \{v\}$;

where, of course, $[u, v] = \{x \in X : u \leq x \leq v\}$. Then $\sigma \in C(X)$.

Similarly define the relation $\tau$ on $X$ as follows:

$$(u, v) \in \tau \iff \text{either } (i) \ u = v$$

or (ii) $u < v$ and $[u, v] \cap Y \subseteq \{v\}$;

or (iii) $v < u$ and $[v, u] \cap Y \subseteq \{u\}$.

Then $\tau \in C(X)$.

Since $\sigma$ and $\tau$ are both the identity relation when restricted to $Y$ and since $S$ is $o$-primitive, we must have $\sigma = \tau = \iota_X$.

Hence, either $Y$ is dense or there exist $y_1, y_2 \in Y$ with $y_1 < y_2$ and $[y_1, y_2] = \{y_1, y_2\}$. 
Define the relation \( \nu \) on \( X \) as follows:

\[(u, v) \in \nu \iff \text{(i) } u = v;\]

or \( \text{(ii) } u, v \in Y \) and either

\[(a) \ u < v, [u, v] \subseteq Y \quad \text{and} \quad [u, v] \text{ is finite;}\]

or \( \text{(b) } v < u, [v, u] \subseteq Y \quad \text{and} \quad [v, u] \text{ is finite.}\]

Then \( \nu \in C(X) \). Since \((y_1, y_2) \in \nu, \nu \neq \iota_X \) and hence \( \nu = \omega_X \). Thus \( X = Y \) and \( S \) is transitive. For \( x \in X \), if \( x > y_1 \), then \([y_1, x]\) is finite; if \( x < y_1 \) then \([x, y_1]\) is finite. As observed above \( X \) must be infinite.

If \( X \) has a largest element, then clearly \( X \) is isomorphic with \( N' \); otherwise \( X \) is isomorphic with \( Z \).

Thus, either \( X \) is isomorphic with \( N' \) or \( Z \) and \( S \) is transitive or, for all \( u, v \in X \) with \( u < v \), there exists a \( y \in Y \) with \( u < y < v \) that is, \( Y \) is dense in \( X \).

**Corollary 2.7.** Let \( X \) be a totally ordered set and let \( S \) be an \( o \)-primitive subsemigroup of \( J_X \). If there exists a nontrivial transitivity class \( Y \) in \( X \) then either \( X = Y \) and is isomorphic to \( N' \) or \( Z \), or \( Y \) is dense in \( X \).

**Proof.** In Theorem 2.6, the fact that \( S \) is an inverse subsemigroup of \( J_X \) is only used to establish that every transitivity class is nontrivial. Thereafter, the proof is entirely concerned with an arbitrary (nontrivial) transitivity class, and does not require that \( S \) be an inverse semigroup.

If \( S \) is a subsemigroup of \( J_X \) and \( Y \) is a transitivity class of \( X \), then we can define a homomorphism \( \phi \) of \( S \) into \( J_Y \), as follows: for \( a \in S \) let \( \phi_a \in J_Y \) be such that \( \Delta(\phi_a) = \Delta(a) \cap Y \) and for \( y \in \Delta(\phi_a), y\phi_a = ya \). Then the mapping \( \phi : a \rightarrow \phi_a \) is the desired homomorphism. We call \( \phi \) the *induced representation* of \( S \) on \( Y \).

**Lemma 2.8.** Let \( Y \) be a transitivity class of \( X \) which is dense in \( X \). If either \( S \) is a group of order preserving permutations of \( X \) or \( S \subseteq T_X \), then the homomorphism \( \phi \) of \( S \) into \( J_Y \) is an isomorphism.

**Proof.** Suppose that \( S \subseteq T_X \) and \( \phi_a = \phi_b \), for \( a, b \in S \). Let \( \Delta(a) = \langle u \rangle \) and \( \Delta(b) = \langle v \rangle \). For \( u \neq v \), say \( u < v \), then there exists a \( y \in Y \) such that \( u < y < v \). Then \( y \in \Delta(\phi_y) \cap \Delta(\phi_b) \) and \( \phi_a \neq \phi_b \), a contradiction. Hence \( u = v \). Now suppose that, for \( x \in \Delta(a) = \Delta(b) \), \( xa \neq xb \). If \( xa < xb \), then there exists an element \( y \in Y \) such that \( xa < y < xb \). Then \( y \in \Delta(\phi_a) \), and so there exists a \( z \in X \) such that \( zb = y \). Thus \( z < x \) and \( z \in Y \). Since \( z < x \), we have \( za < xa < y = zb \). Thus \( z\phi_a < z\phi_b \), and so \( \phi_a \neq \phi_b \), a contradiction. Similarly, if \( xb < xa \), we obtain a contradiction. Hence \( xa = xb \), for
all \( x \in \Delta(a) = \Delta(b) \), and so \( a = b \). Thus, if \( S \subseteq T_X \), we have the lemma.

If \( S \) is a group of order preserving permutations then the result is clear.

**Corollary 2.9.** Let \( X \) be a totally ordered set, \(|X| > 2\) and \( S \) be a subsemigroup of \( J_X \). Let \( S \) be \( \alpha \)-primitive. If \( S \) is a group of order preserving permutations or if \( S \) is actually contained in \( T_X \), then for any nontrivial transitivity class \( Y \) of \( X \) the induced representation \( \phi \) of \( S \) on \( Y \) is an isomorphism of \( S \) into \( J_Y \). Moreover, \( S\phi \) is transitive and \( \alpha \)-primitive on \( Y \).

**Proof.** Let \( Y \) be a nontrivial transitivity class of \( X \). From Theorem 2.6 and Corollary 2.7, either \( Y = X = N' \), or \( Z \) and \( S \) is transitive on \( X \), or \( Y \) is dense in \( X \). Of course, if \( S \) is a group of order preserving permutations of \( X \), then we cannot have \( X = N' \). However, if \( X = N' \) (and \( S \subseteq T_X \)) or \( X = Z \), then \( X \) is the only transitivity class and the induced representation is just the identity isomorphism. Thus, the result holds in this case.

Suppose now that \( X \neq N' \) or \( Z \) and that \( Y \) is any nontrivial transitivity class. By Lemma 2.8, the induced representation \( \phi \) of \( S \) on \( Y \) is an isomorphism of \( S \) into \( J_Y \). First suppose that \( S \) is a group of order preserving permutations. Then, since \( S \) is \( \alpha \)-primitive, \( X \) cannot have either a maximum or minimum element. Hence, since \( Y \) is dense in \( X \), we must have \( Y = X \). Let \( \phi \) be the induced representation of \( S \) on \( Y \). Then \( \phi \) followed by the natural embedding of \( J_Y \) into \( J_Y = J_X \) is just the natural embedding of \( S \) into \( J_X \). Hence by Lemma 2.5, \( S\phi \) is \( \alpha \)-primitive on \( Y \) if and only if \( S\phi \) is \( \alpha \)-primitive on \( Y \), that is, if and only if \( S \) is \( \alpha \)-primitive on \( X \) which holds, since \( S \) is \( \alpha \)-primitive on \( X \).

However, if \( S \subseteq T_X \) then the isomorphism \( \phi \) of \( S \) into \( J_Y \) followed by the natural embedding of \( J_Y \) into \( J_Y = J_X \), need not yield the natural embedding of \( S \) into \( J_X \), since the domains might not correspond. Therefore, to show that \( S\phi \) is \( \alpha \)-primitive on \( Y \), we proceed by contradiction.

Suppose that \( S\phi \) is not \( \alpha \)-primitive on \( Y \). Let \( \rho \) be a nontrivial convex congruence on \( Y \). Define \( \sigma \) on \( X \) by

\[
(x, y) \in \sigma \Leftrightarrow \text{either (i) } x = y; \quad \text{or (ii) there exist } u, v \in Y \text{ with } u \leq x, y \leq v \quad \text{and } (u, v) \in \rho.
\]

Then \( \sigma \) is clearly a convex equivalence relation on \( X \). However, it is possible for \( \sigma \) to fail to be a congruence. Now define the relation \( \tau \) on \( X \) by

\[
(x, y) \in \tau \Leftrightarrow \text{either (i) } (x, y) \in \sigma; \quad \text{or (ii) } x < y, y = \sup x\sigma \quad \text{and} \quad y \notin Y; \quad \text{or (iii) } y < x, x = \sup y\sigma \quad \text{and} \quad x \notin Y.
\]
Clearly, if \( y \notin Y \) and \( y = \sup x_\sigma \), then \( y_\sigma = \{y\} \). Hence, the relation \( \tau \) is a convex equivalence relation on \( X \) and we wish to show that \( \tau \) is a congruence.

Let \((x, y) \in \tau\) and \(x, y \in \Delta(a)\), \(a \in S\). Without loss of generality, let us assume that \( x < y \).

First suppose that \((x, y) \in \sigma\). Then, there exist \(u, v \in Y\) such that \(u < x < y < v\) and \((u, v) \in \rho\). If \(y \in Y\), then \((u, y) \in \rho\), \(ua < xa < ya\) and \((ua, ya) \in \rho\). Hence \((xa, ya) \in \sigma \subseteq \tau\). Therefore, suppose that \(y \notin Y\). Then \(yu \notin Y\). If there exists a \(z \in Y\) with \((uu, z) \in \rho\) and \(ya < z\), then \(ua < xa < ya < z\), and so \((xa, ya) \in \sigma \subseteq \tau\). So, now suppose that \(z \in Y\), \((ua, z) \in \rho\) implies that \(z < ya\). For any \(w \in Y\) such that \(xa < w < ya\), we have \(w - w'a\) for some \(w' \in X\) such that \(u < x < w' < y < v\). Then \((u, w') \in \rho\), \((ua, w) \in \rho\), \((ua, w) \in \sigma\) and \((ua, xa) \in \sigma\). Thus \(ya = \sup(ua)\sigma = \sup(ua)\sigma = \sup(xa)\sigma\). Since \(yu \notin Y\), \((xa, ya) \in \tau\).

Now suppose that \((x, y) \notin \sigma\). Then \(y \notin Y\) and \(y = \sup x_\sigma\). Hence \(ya \notin Y\). Since \(y = \sup x_\sigma\) either \(ya = \sup(xa)\sigma\), or \(ya \in (xa)\sigma\). In either case \((xa, ya) \in \tau\).

By a very similar argument to the one above, we can show that \((xa, ya) \in \tau\) implies that \((x, y) \in \tau\).

Hence \(\tau\) is a convex congruence on \(\overline{X}\). Since \(\rho = \tau_{|X \times Y}\) and \(\rho\) is nontrivial; \(\tau\) is also nontrivial. Thus \(S\) is not \(o\)-primitive on \(X\). But by Lemma 2.5, this contradicts the assumption that \(S\) is \(o\)-primitive on \(X\). Hence \(S\rho = \phi\) is \(o\)-primitive on \(Y\).

In certain circumstances, the converse of Theorem 2.6 will hold. In particular, when the following condition is satisfied:

(A) If \(x, y, z \in X\) and there exists an \(a \in S\) such that \(x \in \Delta(a)\) and \(xa = y\), then there exists an element \(b \in S\) such that \(x, z \in \Delta(b)\) and \(xb = y\).

If \(S\) is a group of order preserving permutations of \(X\), then this condition is satisfied.

**Theorem 2.10.** Let \(X\) be a totally ordered set and \(S\) be an inverse subsemigroup of \(J_X\). If either (i) \(X = \mathbb{N}^+\) or \(Z\) and \(S\) is transitive or (ii) every transitivity class of \(X\) is dense in \(X\) and \(S\) satisfies condition \(A\), then \(S\) is \(o\)-primitive.

**Proof.** Clearly the result is true in case (i). So suppose that (ii) holds. Let \(\rho\) be a nontrivial convex congruence on \(\overline{X}\). Let \(u, v, w \in X\) be such that \(u < v\), \((u, v) \in \rho\) and \((u, w) \notin \rho\). Suppose that \(v < w\), the alternative case \(w < u\) being treated similarly.

There exists an \(a \in S\) such that \(w \in \Delta(a)\) and \(u < w_\sigma < v\). Since \((ua, wa) \notin \rho\), we must have \((ua)\rho < (wa)\rho = up\). Let \(y = \sup(ua)\rho\). Then, for some \(b \in S\), \(y \in \Delta(b)\) and \(u < yb < v\). Thus \((yb, v) \in \rho\). Let \(z \in X\) be
such that \( u = zb \). Then \( z < y \) and \( (zb, yb) = (u, yb) \in \rho \). Hence \((z, y) \in \rho \) and, since \( y = \sup(u\alpha) \), \( y \in (ua)\rho \). Since \( S \) satisfies condition (A), there exists an element \( c \in S \) such that \((yb)c = y \) and \( v \in A(c) \). Then \( yb < v \) implies that \( y = ybc < vc \), and so \( vc \notin yp = (ua)\rho \). Thus \((yb)c, vc) = (y, vc) \notin \rho \), while \((yb, v) \in \rho \). Thus, we have a contradiction, and \( S \) must be \( o \)-primitive on \( X \) and so on \( X \).

**Note 1.** If, in Theorems 2.6 and 2.10, \( S \) is a group of order preserving permutations of \( X \), then these results constitute a slight refinement of a result due to Holland (Theorem 2, [3]) for lattice ordered groups of order preserving permutations of a totally ordered set. In [3], Holland assumes that \( S \) is transitive.

**Note 2.** In Theorem 2.10, we have required that \( S \) be an inverse semigroup solely to ensure that \( X \) is a disjoint union of transitivity classes. If the hypothesis that \( S \) is an inverse semigroup is replaced by the hypothesis that \( X \) is a disjoint union of transitivity classes, then the amended theorem will also be true.

If, in Theorem 2.10, we do not assume that \( S \) satisfies conditions (A), then the theorem need not be true as the following example illustrates.

**Example.** Let \( R \) denote the set of real numbers under their natural ordering. Let \( S = \{ \alpha \in T_R : \text{for } x \in A(\alpha), x \text{ not the maximum element of } A(\alpha), x_0 \in Z \text{ if and only if } x \in Z \} \). Then \( S \) is an inverse subsemigroup of \( T_R \). Moreover, \( S \) is transitive on \( R \) and \( R \) is equal to its Dedekind completion. However, \( S \) does not satisfy condition (A). If \( m \) is any integer, then there exists no element \( \alpha \in S \) such that \( m + 1 \in A(\alpha) \) and \( m\alpha = m + \frac{1}{2} \). Because, if \( m + 1 \in A(\alpha) \), then \( m \) is not the maximum element of \( A(\alpha) \), and so \( m\alpha \) is an integer. Thus, \( S \) satisfies the conditions of Theorem 2.10 apart from condition (A). Finally, the relation \( \rho \) defined on \( R \) by

\[
(x, y) \in \rho \quad m < x, y \leq m + 1 \quad \text{for some } m \in Z,
\]

is a nontrivial convex congruence on \( R \).

### III. Application to \( \Theta(S) \)

In this section we wish to deduce, from the results of the previous section, results regarding \( \Theta(S) \) for certain inverse semigroups \( S \).

First, we recall (cf. [2]) that two elements of a semigroup \( S \) are said to be \( \mathcal{L} - (\mathcal{R}-) \) equivalent if they generate the same principal left (right) ideal of \( S \). We write \( \mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} \), and call \( S \) bisimple if it has only
one \( \mathcal{D} \)-class. A necessary and sufficient condition for two idempotents \( e, f \) in an inverse semigroup \( S \) to be \( \mathcal{D} \)-equivalent is that there exist an element \( a \in S \) such that \( aa^{-1} = e \) and \( a^{-1}a = f \).

We call a congruence \( \rho \) on an inverse semigroup \( S \) idempotent separating, if no two distinct idempotents of \( S \) are \( \rho \)-equivalent. Howie [5] has shown that there is a maximum idempotent separating congruence on an inverse semigroup.

We need the following representation of an inverse semigroup due to Munn [7].

**Lemma 3.1.** Let \( S \) be an inverse semigroup and \( E_S = E \). Define a mapping \( \theta : S \to T_E \) by the rule that \( a\theta = \theta_a \) where

1. \( \Delta(\theta_a) = Eaa^{-1} \);
2. for \( e \in \Delta(\theta_a) \), \( e\theta_a = a^{-1}ea \).

Then \( \theta \) is a homomorphism of \( S \) into \( T_E \) inducing the maximum idempotent separating congruence on \( S \). Moreover, \( S\theta \) is transitive on \( E \) if and only if \( S \) is bisimple.

If the maximum idempotent separating congruence on an inverse semigroup \( S \) is the identity congruence, then \( S \) is called fundamental (cf. [6]). Thus, if \( S \) is fundamental, the representation \( \theta \) of Lemma 3.1 is an isomorphism.

We call an inverse subsemigroup \( S \) of \( T_X \) full if \( S \) contains all the idempotents of \( T_X \). It then follows from [6], Theorem 2.6, that a full inverse subsemigroup of \( T_X \) is fundamental. In Lemma 3.1, \( S\theta \) is a full inverse subsemigroup of \( T_E \).

We also require the following special case of Theorem 6.1 of [10], relating \( C(X) \) and \( \Theta(S) \);

**Lemma 3.2.** Let \( X \) be a totally ordered set and \( S \) be a full inverse subsemigroup of \( T_X \). Then \( C(X) \) and \( \Theta(S) \) are lattice isomorphic.

**Theorem 3.3.** Let \( S \) be an inverse semigroup such that \( E = E_S \) is totally ordered with respect to the natural partial ordering. Then \( \Theta(S) \) satisfies the infinite distributive law \( D \). If \( S \) is also bisimple, then \( \Theta(S) \) is totally ordered.

**Proof.** Let \( \mu \) denote the maximum idempotent separating congruence on \( S \). Then, by Lemma 3.1, \( S/\mu \) is isomorphic with a full inverse subsemigroup of \( T_E \). By Proposition 2.3, \( C(E) \) satisfies the infinite distributive law \( D \). Now \( \Theta(S) = \Theta(S/\mu) \), and so, by Lemma 3.2, \( \Theta(S) \) satisfies the infinite distributive law \( D \).

If, in addition \( S \) is a bisimple inverse semigroup, then \( S/\mu \) acts transitively
on $E$, and so, by Theorem 2.1, $C(E)$ is totally ordered. Hence $\Theta(S) = \Theta(S/\mu)$ is totally ordered.

If $S$ is a transitive inverse subsemigroup of $J_X$, then by Theorem 2.1, $C(X)$ is totally ordered. However, this need not imply that $\Theta(S)$ is totally ordered as Example 2 in Section 4 illustrates.

We shall require the following result, Proposition 6.3 of [10], in the next theorem.

**Lemma 3.4.** Let $X$ be a totally ordered set and $S$ be a full inverse subsemigroup of $J_X$. Let $T = S \cap T_X$. Then $C_T(X) = C_T(X)$.

If $S$ is an inverse semigroup such that $E_S$ is totally ordered, then each $D$-class of $S$ is a bisimple inverse subsemigroup $S$ by [11]. If $S$ is a bisimple inverse semigroup such that $E_S$ is order isomorphic with $\mathbb{Z} (N')$, then we say that $S$ is a bisimple $\mathbb{Z}$-semigroup (bisimple $\omega$-semigroup). Equivalently, let $\mu$ be the maximum idempotent separating congruence on $S$, if $S/\mu$ is isomorphic with $T_Z(T_N')$, then $S$ is a bisimple $\mathbb{Z}$-semigroup (bisimple $\omega$-semigroup).

**Theorem 3.5.** Let $S$ be an inverse semigroup such that $|E_S| > 2$, $E_S$ is totally ordered and $|\Theta(S)| = 2$. Then either $E_S = N'$ (and $S$ is a bisimple $\omega$-semigroup) or $E_S = \mathbb{Z}$ (and $S$ is a bisimple $\mathbb{Z}$-semigroup) or each class of $D$-equivalent idempotents is dense in $E_S$. Moreover, for each $D$-class $D$ of $S$, $D$ is a bisimple inverse semigroup such that $E_D$ is totally ordered and $|\Theta(D)| = 2$. If $S$ is fundamental, then so also is each $D$-class $D$.

**Proof.** Let $\mu$ be the maximum idempotent separating congruence on $S$ and $E = E_S$. Then $S/\mu$ is (isomorphic to) a full inverse subsemigroup of $T_E$. Since $|\Theta(S)| = 2$, we have $|\Theta(S/\mu)| = 2$ and, by Lemma 3.2, $|C(E)| = 2$. Hence, by Theorem 2.10, either $E = N'$ or $\mathbb{Z}$ and $S/\mu$ is transitive, or every transitivity class in $E$ is dense in $E$.

If $E = N'$, then $S/\mu$ is a transitive inverse subsemigroup of $T_{N'}$. Thus $S/\mu = T_{N'}$ and $S$ is a bisimple $\omega$-semigroup. Similarly, if $E = \mathbb{Z}$, then $S$ is a bisimple $\mathbb{Z}$-semigroup.

Now suppose that $E \neq N'$ or $\mathbb{Z}$ and that every transitivity class is dense in $E$. Let $e, f \in E$ be elements in the same transitivity class. Then, in the notation of Lemma 3.1, there exists an $a \in S$, such that $e \in \Delta(\theta_a)$, and $e \theta_a = f$. Now $e \in \Delta(\theta_e)$ and $e \theta_e = e$. Hence $e \in \Delta(\theta_e \theta_a) = \Delta(\theta_{ea})$. Hence $e \leq (ea)(ea)^{-1} = eaa^{-1}e = eaa^{-1} \leq e$; that is, $(ea)(ea)^{-1} = e$. Also, $f = e \theta_{ea} = (ea)^{-1} e (ea) = a^{-1}ea = (ea)^{-1} (ea)$. Thus, if $b = ea$, then $bb^{-1} = e$ and $b^{-1}b = f$. Hence $(e, f) \in D$.

Conversely, if $(e, f) \in D$ then, for some $a \in S$, $aa^{-1} = e$ and $a^{-1}a = f$. Then $e \theta_a = f$ and $e$ and $f$ belong to the same transitivity class. Thus, the
transitivity classes of $E$ are just the classes of $\mathcal{D}$-equivalent idempotents in $E$. Thus, any class of $\mathcal{D}$-equivalent idempotents is dense in $E$.

To establish the second statement, only the assertion that $|\Theta(D)| = 2$, remains to be shown.

If $S$ is a bisimple $\omega$-semigroup or a bisimple $Z$-semigroup, then the assertion is clear. (For details of this and other results on bisimple $\omega$-semigroups and bisimple $Z$-semigroups see [8] and [12].) Suppose that neither of these possibilities holds, and that $D$ is a $\mathcal{D}$-class with $E_D = Y$. Then $Y$ is a transitivity class in the representation of $S/\mu$ on $E$, and so by Corollary 2.9, the induced representation $\phi$ of $S/\mu$ on $Y$ is faithful. Clearly $(S/\mu)\phi \cap T_Y = (D/\mu)\phi$ and $D\phi$ is full in $T_Y$. Thus $\Theta(D) = \Theta(D/\mu) = \Theta((D/\mu)\phi)$ is isomorphic with $C(D/\mu)\phi(Y)$. However, by Lemma 3.4,

$$C(S/\mu)\phi(Y) = C(D/\mu)\phi(Y).$$

So it only remains to be shown that $C(S/\mu)\phi(Y)$ is of cardinality 2. But this follows from Corollary 2.9. Hence $|\Theta(D)| = 2$.

Now, suppose that $S$ is fundamental. If $S$ is a bisimple $\omega$-semigroup or a bisimple $Z$-semigroup, then $S$ has only one $\mathcal{D}$-class and, by assumption, it is fundamental. So let us assume that each $\mathcal{D}$-class $D$ of $S$ is such that $E_D$ is dense in $E_S$ and let $D$ be an arbitrary $\mathcal{D}$-class. Since $\mu$ is just the identity congruence, we have, by Corollary 2.9, that the induced representation $\phi$ of $S$ on $E_D$ (induced from the representation of $S$ from Lemma 3.1 on $E_S$) is faithful and clearly maps $D$ onto a full inverse subsemigroup of $T_{E_D}$. Hence $D$ is fundamental.

In connection with Theorem 3.5, Munn has shown independently (unpublished) that if $S$ is an inverse semigroup such that $E_S$ is totally ordered, $|E_S| > 2$, and $S$ is congruence free, that is, admits only the identity and universal congruences, then $E_S$ is dense in itself. Now if $S$ is congruence free and $|E_S| > 2$, then necessarily $|\Theta(S)| = 2$. Also, since the additive group of integers is a homomorphic image of any bisimple $\omega$-semigroup or bisimple $Z$-semigroup, such a semigroup cannot be congruence free. Hence we could deduce Munn’s result from Theorem 3.5.

If in Theorem 3.5, $S$ is neither a bisimple $Z$-semigroup, nor a bisimple $\omega$-semigroup then, even although $E_D$ is dense in $E_S$, for each $\mathcal{D}$-class $D$, the $\mathcal{D}$-classes need not be isomorphic semigroups. Let $\mathbb{Q}(R)$ denote the set of rational (real) numbers under their natural ordering. Let $S = \{x \in T_R : \text{for } x \in \mathcal{D}(\alpha), x \alpha \in \mathcal{Q} \text{ if and only if } x \in \mathcal{Q}\}$. Then $R$ has two transitivity classes, namely $\mathcal{Q}$ and $R\setminus\mathcal{Q}$. Now $E_S$ is isomorphic to $R$ and, for $e, f \in E_S$ such that $\mathcal{D}(e) = \langle x \rangle$ and $\mathcal{D}(f) = \langle y \rangle$, it follows that $(e, f) \in \mathcal{D}$ if and only if $x$ and $y$ belong to the same transitivity class. Thus $S$ has two $\mathcal{D}$-classes $C$ and $D$ such that $E_C$ is isomorphic to $\mathcal{Q}$ and $E_D$ is isomorphic to $R\setminus\mathcal{Q}$. Hence $C$ and $D$ are certainly not isomorphic.
The converse of the first part of Theorem 3.5 does not hold, as it stands. In other words, from the fact that $S$ is an inverse semigroup such that each class of $\mathcal{D}$-equivalent idempotents of $S$ is dense in $E_S$, it does not follow that $|\Theta(S)| = 2$. Let $X = R \setminus \mathbb{Z}$, where $R$ is the set of real numbers and $\mathbb{Z}$ is the set of integers. Let $S = T_X$. Then $X$ is dense in $X$ and $S$ is transitive on $X$. However, the relation $\rho$ defined on $X$ by $(x, y) \in \rho \iff m < x, y < m + 1$, for some $m \in \mathbb{Z}$, is a nontrivial convex congruence on $X$. Hence $|\Theta(X)| > 2$ and so $|\Theta(S)| > 2$. (See also the example at the end of Section 2.)

Thus it appears essential, for the converse, to consider the representation of $S$ on $E_S$, via Lemma 3.1 and Lemma 2.4. As a corollary of Theorem 2.10, one can then obtain a rather inelegant converse to Theorem 3.5, which we decline to state.

We conclude this section by identifying a certain sublattice of $\Lambda(S)$, where $S$ is an inverse semigroup such that $E_S$ is totally ordered.

**Proposition 3.6.** Let $S$ be an inverse semigroup such that $E_S$ is totally ordered. Then the set $\Sigma(S)$ consisting of the minimum congruences from each $\theta$-class is a sublattice of $\Lambda(S)$. Moreover $\Sigma(S)$ is isomorphic to $\Theta(S)$ and so is a distributive lattice.

**Proof.** Let $\sigma_1, \sigma_2 \in \Sigma(S)$ and $\sigma_3$ be the minimum element in the $\theta$-class containing $\sigma_1 \vee \sigma_2$. Then, by the choice of $\sigma_3$, $\sigma_3 \subseteq \sigma_1 \vee \sigma_2$. But $\sigma_1, \sigma_2 \subseteq \sigma_3$ implies that $\sigma_1 \vee \sigma_2 \subseteq \sigma_3$. Thus $\sigma_1 \vee \sigma_2 = \sigma_3 \in \Sigma(S)$.

Now, let $\sigma_3$ be the minimum congruence in the $\theta$-class containing $\sigma_1 \wedge \sigma_2$. Then $\sigma_3 \subseteq \sigma_1 \wedge \sigma_2$, and we want the converse inclusion. Let $(a, b) \in \sigma_1 \wedge \sigma_2$. By Lemma 1.2, we have $(aa^{-1}, bb^{-1}) \in \sigma_1 \wedge \sigma_2$ and so $(aa^{-1}, bb^{-1}) \in \sigma_3$. Also, by Lemma 1.2, there exist idempotents $e, f$ such that $(e, aa^{-1}) \in \sigma_1$, $ea = eb$, $(f, aa^{-1}) \in \sigma_2$ and $fa = fb$. Without loss of generality, we may assume that $e, f \leq aa^{-1}, bb^{-1}$. Since $E_S$ is totally ordered, $e$ and $f$ are comparable. So suppose that $e \leq f$. Then $e \leq f \leq aa^{-1}, bb^{-1}$. Hence $(f, aa^{-1}) \in \sigma_1 \wedge \sigma_2$, and so $(f, aa^{-1}) \in \sigma_3$. Moreover, $fa = fb$. Thus, by Lemma 1.2, $(a, b) \in \sigma_3$ and $\sigma_1 \wedge \sigma_2 = \sigma_3 \in \Sigma(S)$. Hence $\Sigma(S)$ is a sublattice of $\Lambda(S)$.

Since $\Sigma(S)$ contains exactly one element from each $\theta$-class, $\Sigma(S)$ is isomorphic to $\Theta(S)$ and hence is a distributive lattice by Theorem 3.3.

For an example to illustrate that $\Sigma(S)$ need not be a sublattice of $\Lambda(S)$ for an arbitrary inverse semigroup $S$, see [9], Section 6, Example 2.

**IV. Examples**

This section is devoted to examples. One of these demonstrates the existence of a bisimple inverse semigroup $S$ such that $\Theta(S)$ is an arbitrary finite chain and each $\theta$-class contains only a single element. Munn, in his
discussion on congruence free inverse semigroups in [6], has provided such examples where $|\Theta(S)| = 2$.

We need the following result from [10], by combining Propositions 5.1, 5.2, and Theorem 6.1.

**Proposition 4.1.** Let $X$ be a totally ordered set and $S$ be a full inverse subsemigroup of $T_X$. Then $\Theta(S)$ is isomorphic with $C(X)$. Let $\rho \in C(X)$. Define $\xi = \xi_\rho$ on $S$ by

$$(a, b) \in \xi \iff (i) \ U(a) = U(b);$$

and (ii) $x \in \Delta(a)$, $y \in \Delta(b)$ and $(x, y) \in \rho$

implies that $(xa, yb) \in \rho$,

where, for $\alpha \in T_X$, $U(\alpha) = \{xp : xp \cap \Delta(\alpha) \neq \emptyset\}$. Define $\eta = \eta_\rho$ on $S$ by

$$(a, b) \in \eta \iff (i) \ U(a) = U(b),$$

and (ii) If $xp \in U(a) = U(b)$, then there exists a $y \in xp \cap \Delta(a) \cap \Delta(b)$ such that $za = zb$, for all $z \leq y, z \in X$.

Then $\xi$ and $\eta$ are, respectively, the maximum and minimum congruences on $S$ in the $\Theta$-class corresponding to $\rho$.

Let $n$ be a positive integer. We now construct a bisimple inverse semigroup $S$ such that $|\Theta(S)| = n + 1$, $E_S$ is totally ordered and each $\Theta$-class contains only one element.

For an inverse semigroup $T$ with idempotents $E$, we write $E_\omega = \{a \in T : e \ll a$, in the natural partial ordering of $T$, for some $e \in E\} = \{a \in T : ea^{-1} = e$, for some $e \in E\}$.

**Example 1.** Let $X = R \times R \times \cdots \times R$ where $R_i = R$ (the set of real numbers), $i = 1, \ldots, n$.

Order $X$ lexicographically by defining

$$(x_1, \ldots, x_n) > (y_1, \ldots, y_n) \iff x_i > y_i,$$

where $i$ is the largest integer for which $x_i \neq y_i$. Let $E_i$ denote the set of idempotents of $T_{R_i}$.

Let $S$ denote the set of elements $\alpha$ of $T_X$ such that, if $\Delta(\alpha) = \langle (y_1, \ldots, y_n) \rangle$ and $(x_1, \ldots, x_n) \in \Delta(\alpha)$ then

$$(x_1, \ldots, x_n)\alpha = (x_1\alpha(x_1, \ldots, x_n), x_2\alpha(x_2, \ldots, x_n), \ldots, x_n\alpha(x_n)).$$
where

(i) \( \alpha_n \in E_{n}\omega; \)

(ii) \( \alpha(x_{i+1},...,x_n) \in E_{i}\omega \) with domain \( \langle y_i \rangle \), if \( x_{i+1} = y_{i+1},...,x_n = y_n \);

and \( \alpha(x_{i+1},...,x_n) \) is the identity permutation of \( R_i \), otherwise.

Then \( S \) is a full transitive inverse subsemigroup of \( T_X \) and, hence \( E_S \) is totally ordered and \( S \) is bisimple.

For each \( i = 1, 2, ..., n \) define the relation \( \rho_i \) on \( X \) by

\[
((x_1, ..., x_n), (x'_1, ..., x'_n)) \in \rho_i \iff x_i = x'_i, x_{i+1} = x'_{i+1}, ..., x_n = x'_n.
\]

Then each \( \rho_i \) is easily seen to be a convex congruence on \( X \) and, moreover, if \( \rho_{n+1} \) is the universal congruence, then \( C(X) = \{\rho_1, ..., \rho_{n+1}\} \) where \( \rho_1 \subset \rho_2 \subset \cdots \rho_n \subset \rho_{n+1} \). Thus, \( C(X) \) is an \( n + 1 \) element chain with \( \rho_1 \) the identity congruence and \( \rho_{n+1} \) the universal congruence.

Now let \( \rho_i, 1 \leq i \leq n + 1 \), induce the congruences \( \xi_i \) and \( \eta_i \) as in Proposition 4.1. We wish to show that \( \xi_i = \eta_i \).

Let \( (a, b) \in \xi_i \). Let \( u \rho_i \cap \Delta(a) \neq \phi \). Then, \( u \rho_i \cap \Delta(b) \neq \phi \). Let \( u = (u_1, ..., u_n) \), \( \Delta(a) = \langle (x_1, ..., x_n) \rangle \) and \( \Delta(b) = \langle (y_1, ..., y_n) \rangle \). Choose \( z_{i-1} \in R_{i-1} \) such that

(a) \( z_{i-1} \leq x_{i-1}, y_{i-1} \);

(b) \( y \alpha(u_1, ..., u_n) = y, \) for all \( y \in R_{i-1}, y \leq z_{i-1} \);

(c) \( y \beta(u_1, ..., u_n) = y, \) for all \( y \in R_{i-1}, y \leq z_{i-1} \).

Choose \( z_1, ..., z_{i-2} \) arbitrarily in \( R_1, ..., R_{i-2} \), respectively. Let \( z = (z_1, ..., z_{i-1}, u_i, ..., u_n) \). Then \( (z, u) \in \rho_i \) and, for \( z' \leq z, z'a = z'b \). Thus \( (a, b) \in \eta_i \) and \( \xi_i = \eta_i \). Thus the \( \theta \)-classes corresponding to \( \rho_2, ..., \rho_{n+1} \) have only one element. Since \( S \) is full in \( T_X \), \( S \) is fundamental and so the \( \theta \)-class corresponding to \( \rho_1 \) has only the one element. Thus \( S \) has the desired properties.

Note that in the above example each \( R_i \) could be replaced by any totally ordered field, with possibly distinct fields replacing distinct \( R_i \).

Using the techniques of [4], examples similar to the above can be constructed to yield a more general totally ordered set for \( \Theta(S) \).

We now give an example to show that it does not follow from the fact that \( S \) is a transitive inverse subsemigroup of \( J_X \) that \( \Theta(S) \) is totally ordered. (cf. Theorems 2.1 and 3.3).

**Example 2.** Let \( X = Z \times Z \), where \( (m, n) > (r, s) \) if and only if \( m > r \) or \( m = r \) and \( n > s \). Let \( S = \{x \in J_X : \Delta(x) \) is a proper ideal of \( X \} \). For \( x \in X \), let \( e_x \) denote the idempotent of \( S \) with \( \Delta(e_x) = \langle x \rangle \). For \( m \in Z \), let \( e_m \) denote the idempotent of \( S \) with \( \Delta(e_m) = \{(r, s) : r \leq m \} \). Then
$E_S = \{e_x : x \in X\} \cup \{e_m : m \in Z\}$, and $S$ is transitive on $X$. Define the relations $\sigma$ and $\tau$ on $E_S$ by

$$(e_u, e_v) \in \sigma \iff \text{either (i) } u = (m, n), \quad \quad \text{for some } m \in Z;$$

or (ii) $u = (m, n), \quad \quad \text{for some } m \in Z;$

or (iii) $u = m, \quad \quad \text{for some } m \in Z;$

and

$$(e_u, e_v) \in \tau \iff \text{either (i) } u = (m, n), \quad \quad \text{for some } m \in Z;$$

or (ii) $u = (m, n), \quad \quad \text{for some } m \in Z;$

or (iii) $u = m - 1, \quad \quad \text{for some } m \in Z.$

Then $\sigma$ and $\tau$ are noncomparable normal equivalences on $E_S$. Thus, $\mathcal{O}(S)$ is not totally ordered.

We finally give an example to show that the semilattice homomorphism of Lemma 2.5, need not be a lattice homomorphism.

**Example 3.** Let $X$ be as in Example 2, and $S$ be the group of order preserving permutations of $X$. Then $X = X \cup \{x_m : m \in Z\}$, where $x_m > (r, s)$ if and only if $r < m$. Define the relations $\pi, \rho$ on $X$ by

$$(u, v) \in \pi \iff \text{either (i) } u = (m, n), \quad \quad \text{for some } m \in Z;$$

or (ii) $u = (m, n), \quad \quad \text{for some } m \in Z;$

or (iii) $u = x_m, \quad \quad \text{for some } m \in Z;$

and

$$(u, v) \in \rho \iff \text{either (i) } u = (m, n), \quad \quad \text{for some } m \in Z;$$

or (ii) $u = (m, n), \quad \quad \text{for some } m \in Z;$

or (iii) $u = x_{m-1}, \quad \quad \text{for some } m \in Z.$

Then $\pi, \rho \in C(X)$ and $\pi \lor \rho = \omega_S$. Let $\lambda$ be the relation defined on $X$ by $(u, v) \in \lambda \iff u = (m, n), \quad v = (m, r)$, for some $m \in Z$. Then $\lambda \in C(X)$, and $\pi \alpha = \rho \alpha = \lambda$. Thus, $\pi \alpha \lor \rho \alpha = \lambda \neq \omega_X = \omega_{\mathcal{O}\alpha} = (\pi \lor \rho)\alpha$. Hence, $\alpha$ is not a lattice homomorphism.

**References**