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The evaluation subgroup of a fibre inclusion

Gregory Lupton [a], Samuel Bruce Smith [b,*]

[a] *Department of Mathematics, Cleveland State University, Cleveland, OH 44115, USA*
[b] *Department of Mathematics, Saint Joseph's University, Philadelphia, PA 19131, USA*

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Abstract

Let $\xi : X \xrightarrow{j} E \xrightarrow{p} B$ be a fibration of simply connected CW complexes of finite type with classifying map $h : B \to \mathrm{B\,aut}_1(X)$. We study the evaluation subgroup $G_n(E, X; j)$ of the fibre inclusion as an invariant of the fibre-homotopy type of ξ. For spherical fibrations, we show the evaluation subgroup may be expressed as an extension of the Gottlieb group of the fibre sphere provided the classifying map h induces the trivial map on homotopy groups. We extend this result after rationalization: We show that the decomposition $G_*(E, X; j) \otimes \mathbb{Q} = (G_*(X) \otimes \mathbb{Q}) \oplus (\pi_*(B) \otimes \mathbb{Q})$ is equivalent to the condition $(h_\sharp)_\mathbb{Q} = 0$.

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1. Introduction

The nth Gottlieb group $G_n(X)$ of a space X is the subgroup of $\pi_n(X)$ consisting of homotopy classes of maps $g : S^n \to X$ such that the map $(g \mid 1_X) : S^n \vee X \to X$, defined on the wedge, extends to some map $F : S^n \times X \to X$ of the product. Alternately, $G_n(X)$ is the image of the map induced on homotopy groups by the evaluation map $\omega : \mathrm{aut}_1(X) \to X$, where $\mathrm{aut}_1(X)$ is the space of continuous functions homotopic to the identity of X. The definition may be generalized by replacing the identity by an arbitrary based map $f : X \to Y$: the *nth evaluation subgroup* $G_n(Y, X; f)$ *of the map* f is the subgroup of $\pi_n(Y)$ represented by maps $g : S^n \to Y$ such that $(g \mid f) : S^n \vee X \to Y$ extends to some map $F : S^n \times X \to Y$. The generalization provides some functorality in that the map on homotopy groups induced by f restricts to a map $f_\sharp : G_n(X) \to G_n(Y, X; f)$. On the other hand, it is well known that the Gottlieb group fails to be a functor since, generally, a map $f : X \to Y$ does not yield a homomorphism $f_\# : G_n(X) \to G_n(Y)$.

The Gottlieb groups $G_n(X)$ play a well-known role in the homotopy theory of fibrations with fibre a CW complex X of finite type. Notably, the result of Gottlieb [7, Theorem 2] identifies $G_*(X)$ as the image of the connecting homomorphism in the long exact homotopy sequence of the universal fibration with fibre X (see diagram (3) below). In this paper, we investigate the role of the more general evaluation subgroups for the homotopy theory of fibrations.

* Corresponding author.
E-mail addresses: g.lupton@csuohio.edu (G. Lupton), smith@sju.edu (S.B. Smith).

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Let $\xi : X \xrightarrow{j} E \xrightarrow{p} B$ be a fibration of simply connected CW complexes of finite type. We consider the homomorphism $j_\sharp : G_n(X) \to G_n(E, X; j)$ induced by the fibre inclusion of ξ. By the universal property of the Gottlieb group mentioned above, restricting the homomorphisms in the long exact homotopy sequence of ξ yields a sequence of groups and homomorphisms

$$\cdots \longrightarrow \pi_{n+1}(B) \xrightarrow{\partial} G_n(X) \xrightarrow{j_\sharp} G_n(E, X; j) \xrightarrow{p_\sharp} \pi_n(B) \longrightarrow \cdots. \tag{1}$$

We call (1) the *Gottlieb sequence of the fibration* ξ. In general, it is not exact. Our overriding purpose in this paper is to show that exactness properties of the Gottlieb sequence represent an interesting measure of the relative triviality of the fibration.

Successive compositions of maps in the Gottlieb sequence are zero, since they are restrictions of those in the long exact homotopy sequence of ξ. Thus we may consider homology groups at each type of term. We focus on the generalized evaluation subgroup terms and define the nth *Gottlieb homology group of* ξ to be the subquotient

$$GH_n(\xi) = \frac{\ker\{p_\sharp : G_n(E, X; j) \to \pi_n(B)\}}{\operatorname{im}\{j_\sharp : G_n(X) \to G_n(E, X; j)\}}$$

of $G_n(E, X; j) \subseteq \pi_n(E)$. As we shall see, the Gottlieb homology detects the nontriviality of the fibration in the sense of fibre-homotopy type. We say the fibration ξ is *Gottlieb trivial in degree n* if (1) breaks into a short exact sequence

$$0 \longrightarrow G_n(X) \xrightarrow{j_\sharp} G_n(E, X; j) \xrightarrow{p_\sharp} \pi_n(B) \longrightarrow 0 \tag{2}$$

in degree n, and *Gottlieb trivial* if this occurs in all degrees $n \geqslant 2$.

Recall that a fibration ξ as above is classified by a map $h : B \to \mathrm{B}\,\mathrm{aut}_1(X)$, where $\mathrm{B}\,\mathrm{aut}_1(X)$ is the classifying space of the monoid $\mathrm{aut}_1(X)$ in the sense of Dold–Lashof [3] (see [1,19,2]). Our main results relate the notion of Gottlieb triviality to the vanishing of the homomorphism induced on homotopy groups by the classifying map of the fibration. In Section 2, we make some observations in ordinary homotopy theory that relate the behaviour of the Gottlieb sequence with that of the classifying map. We then prove that a spherical fibration is Gottlieb trivial whenever $h_\sharp = 0$ (Theorem 2.5).

In Section 4, we obtain a complete result in this vein within the framework of rational homotopy theory. We prove that, after tensoring with the rationals, the Gottlieb sequence of a fibration with fibre an arbitrary finite complex splits as in (2) if and only if the homomorphism induced on rational homotopy groups by the classifying map is trivial (Theorem 4.2). This result depends upon a description of the map $(h_\sharp)_{\mathbb{Q}}$ in terms of derivations of the Sullivan model of the fibre. We give this description in Section 3. The key step here comes from a close study of the holonomy action of a fibration. This description of $(h_\sharp)_{\mathbb{Q}}$ (Theorem 3.2) fits the classifying map into the framework developed in [15], whereby chain complexes of (generalized) derivations of Sullivan models are used to describe the rationalized evaluation subgroups.

Remark 1.1. The Gottlieb sequence of a fibration $\xi : F \xrightarrow{j} E \xrightarrow{p} B$ is a special case of the G-sequence of a map as developed and studied by Lee and Woo (cf. [14]). We have elected to focus on the Gottlieb sequence of a fibration since the construction is direct and avoids consideration of the relative term which occurs in the general G-sequence. However, the results of this paper may generally be rephrased in terms of the G-sequence and the ω-homology of a map. In particular, our definition of the Gottlieb homology corresponds to that of the ω-homology group $H_*^{\omega b}(E, X; j)$ [14].

2. First results in ordinary homotopy theory

Here and throughout, all spaces will be assumed to be based CW complexes of finite type. A fibration $\xi : X \xrightarrow{j} E \xrightarrow{p} B$ will be a Hurewicz fibration [18, p. 66]. The map j will usually denote the inclusion of the fibre $X = p^{-1}(b_0)$. However, by [8, Proposition 1.4], the evaluation subgroups are homotopy invariants of a map and so we may, when necessary, assume j is only a fixed equivalence from X to the actual fibre over the basepoint. We will write $\partial : \Omega B \to X$ for the connecting map in the Puppe sequence for ξ [22, Theorem III.6.22]. After the identification $\pi_n(\Omega B) \cong \pi_{n+1}(B)$, ∂ induces the connecting homomorphism $\partial_\sharp : \pi_{n+1}(B) \to \pi_n(X)$ in the long exact homotopy sequence for ξ. We let $\partial_\infty : \Omega\mathrm{B}\,\mathrm{aut}_1(X) \to X$ and $\partial_U : \Omega\mathrm{B}\,\mathrm{aut}_1(X) \to \mathrm{aut}_1(X)$ denote the connecting maps in the universal X-fibration $X \to UX \to \mathrm{B}\,\mathrm{aut}_1(X)$ and the universal principal $\mathrm{aut}_1(X)$-fibration $\mathrm{aut}_1(X) \to U\,\mathrm{aut}_1(X) \to \mathrm{B}\,\mathrm{aut}_1(X)$, respectively. By results of Gottlieb [8, §4] and Dror–Zabrodsky [4, Proposition 4.1], we then have the following homotopy commutative diagram

$$\begin{tikzcd} \Omega B \arrow[r, "\Omega h"] \arrow[dr, "\partial"'] & \Omega \mathrm{B}\,\mathrm{aut}_1(X) \arrow[r, "\partial_U", "\simeq"'] \arrow[d, "\partial_\infty"] & \mathrm{aut}_1(X) \arrow[dl, "\omega"] \\ & X & \end{tikzcd} \tag{3}$$

The Gottlieb sequence is an invariant of the fibre-homotopy type of ξ. Precisely, say two group homomorphisms $\phi: G_1 \to G_2$ and $\psi: H_1 \to H_2$ are *equivalent* if there are isomorphisms $\theta_i: G_i \to H_i$ such that $\psi \circ \theta_1 = \theta_2 \circ \phi$. This notion of equivalence extends in an obvious way to any sequence or commutative diagram of groups and homomorphisms. Suppose $\xi_1: X_1 \xrightarrow{j_1} E_1 \xrightarrow{p_1} B$ is fibre-homotopy equivalent to $\xi_2: X_2 \xrightarrow{j_2} E_2 \xrightarrow{p_2} B$ [18, p. 100]. Using [8, Proposition 1.4] again, we obtain that the equivalence between long exact homotopy sequences induced by the fibre-homotopy equivalence preserves the evaluation subgroups and thus restricts to an equivalence of Gottlieb sequences.

We next observe that Gottlieb triviality fits properly between two familiar notions of the relative triviality of a fibration. Say $\xi: X \xrightarrow{j} E \xrightarrow{p} B$ is *fibre-homotopically trivial* if ξ is fibre-homotopy equivalent to the product $\pi: X \xrightarrow{i_2} B \times X \xrightarrow{p_1} B$. Say ξ is *weak-homotopically trivial* if $\partial_\# = 0: \pi_{n+1}(B) \to \pi_n(X)$ for all n.

Theorem 2.1. *For any fibration we have the following implications*:

fibre-homotopically trivial $\Longrightarrow$ Gottlieb trivial $\Longrightarrow$ weak-homotopically trivial.

Furthermore, each of the reverse implications fails.

Proof. An easy direct argument shows that a trivial fibration is Gottlieb trivial (cf. [23, Corollary 13]). The first implication follows from this and the fibre-homotopy invariance mentioned above. For the second implication, note that $p_\sharp$ surjective implies $\partial_\# = 0$. A separating example for the first implication is given by Example 2.2. Separating examples for the second are given by Examples 2.6 and 4.4. □

Example 2.2. Let $\xi: G \xrightarrow{j} E \xrightarrow{p} B$ be a principal bundle with structure group G. It is well known that the Gottlieb groups of G coincide with the homotopy groups of G. This identity extends to the evaluation subgroups $G_n(E, G; j)$: For let $\alpha: S^n \to E$ represent an arbitrary homotopy class. Following Steenrod [21, §8.7], consider the principal map $P: G \times E \to E$ induced by the action of G on the fibres of $p: E \to B$. Define $F: S^n \times G \to E$ by $F(s, g) = P(g, \alpha(s))$. It is easy to check F extends $(\alpha \mid j): S^n \vee G \to E$ to the product and the claim follows. Thus for principal bundles the Gottlieb sequence coincides with the long exact homotopy sequence and we have Gottlieb trivial is equivalent to weak-homotopically trivial. The above analysis holds as well for principal H-fibrations as in [3]. We thus obtain Gottlieb trivial fibrations which are not fibre-homotopy trivial by considering, for example, the (nontrivial) stages in the Postnikov decomposition of a space.

Given a fibration $\xi: X \xrightarrow{j} E \xrightarrow{p} B$ with classifying map $h: B \to \mathrm{B}\,\mathrm{aut}_1(X)$ recall that ξ is fibre-homotopically trivial if and only if h is null-homotopic. The weaker condition $h_\sharp = 0$ is thus an approximation to ξ being fibre-homotopically trivial. In what follows we compare this condition to Gottlieb triviality. We begin with the following general fact.

Theorem 2.3. *Let $\xi: X \xrightarrow{j} E \xrightarrow{p} B$ be a fibration of simply connected, finite type CW complexes with classifying map $h: B \to \mathrm{B}\,\mathrm{aut}_1(X)$. Then we have*

$$p_\sharp\big(G_n(E, X; j)\big) \supseteq \ker\big\{h_\sharp: \pi_n(B) \to \pi_n\big(\mathrm{B}\,\mathrm{aut}_1(X)\big)\big\}.$$

Proof. Let $\alpha : S^n \to B$ represent a homotopy class in $\pi_n(B)$ with $h_\sharp(\alpha) = 0$. Let $X \xrightarrow{j^*} E^* \xrightarrow{p^*} S^n$ denote the pullback of $X \xrightarrow{j} E \xrightarrow{p} B$ by $\alpha : S^n \to B$ and $\alpha^* : E^* \to E$ the fibre map covering α. By hypothesis, the composition $h \circ \alpha : S^n \to \mathrm{B\,aut}_1(X)$ is homotopically trivial. This means $X \xrightarrow{j^*} E^* \xrightarrow{p^*} S^n$ is fibre-homotopically trivial. We thus have a fibre-homotopy equivalence $H : S^n \times X \to E^*$ giving a commutative diagram

$$\begin{tikzcd}
 & X \arrow[dl, "i_2"'] \arrow[d, "j^*"] \arrow[dr, "j"] & \\
S^n \times X \arrow[r, "H"] \arrow[dr, "\pi_1"'] & E^* \arrow[r, "\alpha^*"] \arrow[d, "p^*"] & E \arrow[d, "p"] \\
 & S^n \arrow[r, "\alpha"] \arrow[ur, dotted, "\tilde{\alpha}"] \arrow[ul, bend left, "i_1"] & B
\end{tikzcd}$$

Define a lifting $\tilde{\alpha} : S^n \to E$ by $\tilde{\alpha} = \alpha^* \circ H \circ i_1$. To see $\tilde{\alpha}$ represents a class in $G_n(E, X; j)$ define $G : S^n \times X \to E$ by $G = \alpha^* \circ H$ and observe that G extends $(\tilde{\alpha} \mid j) : S^n \vee X \to E$ to the product. $\square$

As a consequence, we obtain:

Theorem 2.4. *Let $\xi : X \xrightarrow{j} E \xrightarrow{p} B$ be a fibration of simply connected, finite type CW complexes with classifying map $h : B \to \mathrm{B\,aut}_1(X)$. Suppose $h_\sharp = 0 : \pi_n(B) \to \pi_n(\mathrm{B\,aut}_1(X))$ for some $n \geqslant 2$. Then $j_\# : G_{n-1}(X) \to G_{n-1}(E, X; j)$ is injective and $p_\# : G_n(E, X; j) \to \pi_n(B)$ is surjective.*

Proof. Our hypothesis and the factorization $\partial = \partial_\infty \circ \Omega h$ given by (3) imply $j_\sharp : \pi_{n-1}(X) \to \pi_{n-1}(E)$ is injective and so the restriction of $j_\sharp$ to $G_n(X)$ is also. Surjectivity is direct from Theorem 2.3. $\square$

Focusing on the case of a spherical fibration, we obtain the following result:

Theorem 2.5. *Let $\xi : S^k \xrightarrow{j} E \xrightarrow{p} B$ be a fibration with classifying map $h : B \to \mathrm{B\,aut}_1(S^k)$. Suppose $h_\sharp = 0 : \pi_i(B) \to \pi_i(\mathrm{B\,aut}_1(S^k))$ for $i = n, n+1$ and $n+k$. Then the fibration ξ is Gottlieb trivial in degree n, that is, we have a short exact sequence*

$$0 \longrightarrow G_n\big(S^k\big) \xrightarrow{j_\sharp} G_n\big(E, S^k; j\big) \xrightarrow{p_\sharp} \pi_n(B) \longrightarrow 0.$$

Proof. By Theorem 2.4, it suffices to show $GH_n(\xi) = 0$. Let $\beta : S^n \to E$ represent a homotopy class in $\ker\{p_\sharp : G_n(E, S^k; j) \to \pi_n(B)\}$. Because β represents an element of $G_n(E, S^k; j)$, we have a map $G : S^n \times S^k \to E$ that extends $(\beta \mid j) : S^n \vee S^k \to E$. Since $p \circ \beta \sim * : S^n \to B$, and since ξ is a fibration, we may choose a class $\alpha : S^n \to S^k$ such that $j \circ \alpha \sim \beta$. We need to show that $\alpha \in G_n(S^k)$, that is, we must produce a map $F : S^n \times S^k \to S^k$ that extends $(\alpha \mid 1) : S^n \vee S^k \to S^k$. Let $\eta : S^{n+k-1} \to S^n \vee S^k$ denote the Whitehead product map $\eta = [\iota_n, \iota_k]$, which gives the cofibre sequence $S^{n+k-1} \xrightarrow{\eta} S^n \vee S^k \to S^n \times S^k$. We have a commutative diagram

$$\begin{tikzcd}
S^{n+k-1} \arrow[d, "\eta"] & \\
S^n \vee S^k \arrow[r, "(\alpha|1)"] \arrow[d, "J"'] & S^k \arrow[d, "j"] \\
S^n \times S^k \arrow[r, "G"'] \arrow[ur, dotted, "F"] & E
\end{tikzcd}$$

in which $j \circ (\alpha \mid 1) \circ \eta \sim G \circ J \circ \eta \sim *$. As observed in the proof of Theorem 2.4, we have $j_\# : \pi_{n+k-1}(S^k) \to \pi_{n+k-1}(E)$ injective. It follows that $(\alpha \mid 1) \circ \eta \sim * : S^{n+k-1} \to S^k$ and hence that there exists an extension of $(\alpha \mid 1)$ to a map $F : S^n \times S^k \to S^k$ as desired. $\square$

The following example is primarily intended to separate Gottlieb trivial from weak-homotopically trivial. It also illustrates that, at least under rather restricted circumstances, converses of Theorems 2.3, 2.4 and 2.5 may hold.

Example 2.6. We claim that there is a fibration $\xi : S^3 \xrightarrow{j} E \xrightarrow{p} S^3$ that is weak-homotopically trivial but not Gottlieb trivial. Specifically, $p_\# : G_3(E, S^3; j) \to \pi_3(S^3)$ will not be surjective in our example. First observe, that any fibration $\xi : S^n \xrightarrow{j} E \xrightarrow{p} S^n$ must be weak-homotopically trivial (at least). Indeed, it must have a section $\sigma : S^n \to E$. This follows, since the connecting homomorphism $\partial_\# : \pi_n(S^n) \to \pi_{n-1}(S^n) = 0$ is necessarily trivial, and so $p_\# : \pi_n(E) \to \pi_n(S^n)$ is surjective. Choose σ to be any element of $\pi_n(E)$ for which $p_\#(\sigma) = 1 \in \pi_n(S^n)$.

Next, any fibration $\xi : S^n \xrightarrow{j} E \xrightarrow{p} S^n$ satisfies

$$p_\sharp\bigl(G_n\bigl(E, S^n; j\bigr)\bigr) \subseteq \ker\bigl\{h_\sharp : \pi_n\bigl(S^n\bigr) \to \pi_n\bigl(\mathrm{B\,aut}_1\bigl(S^n\bigr)\bigr)\bigr\}.$$

Together with Theorem 2.3, this shows that $p_\sharp(G_n(E, S^n; j)) = \ker\{h_\sharp\}$ in this case. For suppose that we have $\alpha = p_\#(\beta)$ for $\alpha \in \pi_n(S^n)$ and $\beta \in G_n(E, S^n; j)$. Then there exists some $G : S^n \times S^n \to S^n$ that extends $(\beta \mid j) : S^n \vee S^n \to S^n$. We now argue that, without loss of generality, we may assume that $p \circ G = \alpha \circ p_1 : S^n \times S^n \to S^n$, where $p_1 : S^n \times S^n \to S^n$ denotes projection onto the first factor. The cofibration sequence $S^{2n-1} \to S^n \vee S^n \to S^n \times S^n$ gives rise to a diagram of Puppe sequences

$$\begin{tikzcd}
{[S^{2n}, E]} \arrow[r] \arrow[d, "p_*"] & {[S^n \times S^n, E]} \arrow[r] \arrow[d, "p_*"] & {[S^n \vee S^n, E]} \arrow[d, "p_*"] \\
{[S^{2n}, S^n]} \arrow[r] \arrow[u, bend left, "\sigma_*"] & {[S^n \times S^n, S^n]} \arrow[r] & {[S^n \vee S^n, S^n]}
\end{tikzcd}$$

in which the left-hand terms act on the middle terms in the usual way. Since $p \circ G$ and $\alpha \circ p_1$ both map to the same element in the lower right set, we have $\alpha \circ p_1 = (p \circ G)^\gamma$ for some $\gamma \in \pi_{2n}(S^n)$. Since we have the section σ, we may write $\gamma = p_*\sigma_*(\gamma)$ and so we have $\alpha \circ p_1 = (p \circ G)^{p_*\sigma_*(\gamma)} = p_*(G^{\sigma_*(\gamma)})$. That is, we may replace G by $G^{\sigma_*(\gamma)}$ to obtain a map that extends $(\beta \mid j) : S^n \vee S^n \to S^n$ and also projects under p to $\alpha \circ p_1$. Finally, consider the pullback of the fibration ξ over α. This gives a fibration $\xi^* : S^n \xrightarrow{j^*} E^* \xrightarrow{p^*} S^n$ with classifying map $h \circ \alpha : S^n \to \mathrm{B\,aut}_1(S^n)$. The maps $G : S^n \times S^n \to E$ and $p_1 : S^n \times S^n \to S^n$ that satisfy $p \circ G = \alpha \circ p_1$ define a map $f : S^n \times S^n \to E^*$ into the pullback. This map gives a commutative diagram

$$\begin{tikzcd}
S^n \arrow[r, "1"] \arrow[d, "i_2"'] & S^n \arrow[d, "j^*"] \\
S^n \times S^n \arrow[r, "f"] \arrow[d, "p_1"'] & E^* \arrow[d, "p^*"] \\
S^n \arrow[r, "1"] & S^n
\end{tikzcd}$$

Since this displays the induced fibration ξ^* as fibre-homotopically trivial, it follows that its classifying map $h \circ \alpha$ is null-homotopic.

To complete our example, it remains to identify a specific instance in which $h_\# : \pi_n(S^n) \to \pi_n(\mathrm{B\,aut}_1(S^n))$ may be chosen nonzero. For this, take $n = 3$. We have $\pi_3(\mathrm{B\,aut}_1(S^3)) \cong \pi_2(\Omega\mathrm{B\,aut}_1(S^3)) \cong \pi_2(\mathrm{map}(S^3, S^3; 1))$. Since S^3 is an H-space, the evaluation fibration $\mathrm{map}_*(S^3, S^3; 1) \to \mathrm{map}(S^3, S^3; 1) \to S^3$ admits a section and it follows that $\pi_2(\mathrm{map}(S^3, S^3; 1)) \cong \pi_2(\mathrm{map}_*(S^3, S^3; 1))$. Using again that S^3 is an H-space, we have a well-known homotopy equivalence of components $\mathrm{map}_*(S^3, S^3; 1) \simeq \mathrm{map}_*(S^3, S^3; 0)$ and it now follows by standard methods that $\pi_2(\mathrm{map}_*(S^3, S^3; 0)) \cong \pi_5(S^3) \cong \mathbb{Z}_2$. In summary, we have computed that $\pi_3(\mathrm{B\,aut}_1(S^3)) \cong \mathbb{Z}_2$. Choose $h : S^3 \to \mathrm{B\,aut}_1(S^3)$ to represent the nontrivial element. This is the classifying map of a weak-homotopically trivial fibration $S^3 \to E \to S^3$ that is not Gottlieb trivial, as claimed.

By [13, Corollary 2.2], $G_n(S^2) \cong \pi_n(S^3)$ for all n. We complement this result with the following:

Example 2.7. Let $\eta_2 : S^3 \to S^2$ denote the Hopf map. We claim $G_n(S^2, S^3; \eta_2) = \pi_n(S^2)$ for all n. Write $k : S^2 \to BS^1 = K(\mathbb{Z}, 2)$ for the classifying map for η_2 viewed as a principal S^1-fibration. Converting k to a fibration, we obtain an S^3-fibration $\xi : S^3 \xrightarrow{\eta_2} S^2 \xrightarrow{k} K(\mathbb{Z}, 2)$. By Theorem 2.5, ξ is Gottlieb trivial in degrees $n > 2$. We check directly that ξ is Gottlieb trivial in degree 2 as well. For note that $G_2(S^3) = 0$, while the Whitehead identity $[\iota_2, \eta_2] = 0$ implies $G_2(S^2, S^3; \eta_2) = \pi_2(S^2)$ and so $k_\sharp : G_2(S^2, S^3; j) \to \pi_2(K(\mathbb{Z}, 2))$ is an isomorphism.

3. Derivations of Sullivan models, the holonomy action and the classifying map

In this section, we describe the map induced on rational homotopy groups by a classifying map for a fibration, in terms of certain chain complexes of derivations of Sullivan models. We first introduce some notation for working in Sullivan's differential graded (DG) algebra framework for rational homotopy theory, for which our general reference is [6].

By a *DG algebra* we mean a pair A, d where A is a connected, commutative graded algebra over $\mathbb{Q}$. The differential d increases degree by one. We write $\varepsilon : A \to \mathbb{Q}$ for the augmentation and A^+ for the augmentation ideal. When appropriate, we will view $\mathbb{Q}$ as the DG algebra concentrated in degree 0 with trivial differential and ε as a DG algebra map. A nilpotent, finite type CW complex X admits a Sullivan minimal model $\mathcal{M}_X, d_X$ which is a minimal DG algebra. Writing $\mathcal{M}_X = \Lambda V$ for some graded vector space V, we recall that if X is a *simple space* (that is, the fundamental group of X is abelian and acts trivially on the homotopy groups of X) then the rational homotopy groups of X are recovered by V. Specifically, given graded spaces V and W, let $\mathrm{Hom}_n(V, W)$ denote the space of linear maps between the graded spaces V and W reducing degrees by n. We then have Sullivan's isomorphism $\pi_n(X) \otimes \mathbb{Q} \cong \mathrm{Hom}_n(V, \mathbb{Q})$ [6, Theorem 15.11]. A map of spaces $f : X \to Y$ has a minimal model which is a map of DG algebras $\mathcal{M}_f : \mathcal{M}_Y \to \mathcal{M}_X$.

Let A, d_A and B, d_B be DG algebras and $\phi : A \to B$ a DG algebra map. We say $\theta \in \mathrm{Hom}_n(A, B)$ is a *ϕ-derivation* of degree n if $\theta(xy) = \theta(x)\phi(y) - (-1)^{n|x|}\phi(x)\theta(y)$. Let $\mathrm{Der}_n(A, B; \phi)$ denote the vector space of ϕ-derivations of degree n, for $n \geqslant 0$. Define a linear map $D_\phi : \mathrm{Der}_n(A, B; \phi) \to \mathrm{Der}_{n-1}(A, B; \phi)$ by $D_\phi(\theta) = d_B \circ \theta - (-1)^{|\theta|}\theta \circ d_A$. Then $\mathrm{Der}_*(A, B; \phi), D_\phi$ is a chain complex. We write $H_n(\mathrm{Der}(A, B; \phi))$ for the homology in degree n.

Given a map $f : X \to Y$, let $\mathrm{map}(X, Y; f)$ denote the path component of f in the space of (unbased) continuous functions from X to Y. When X and Y are simply connected with X a finite complex, we have $\pi_n(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q} \cong H_n(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f))$ for all $n \geqslant 2$ [15, Theorem 2.1]. We recall this identification and describe a minor extension here.

Given $F : B \to \mathrm{map}(X, Y; f)$ with B a simple CW complex of finite type, we describe the map induced by F on rational homotopy groups. Let $\mathcal{F} : B \times X \to Y$ denote the adjoint map so that $\mathcal{F} \circ i_2 \sim f$ where $i_2 : X \to B \times X$ is the inclusion. Write $\mathcal{M}_B = \Lambda W$ and $\mathcal{M}_X = \Lambda V$. Then $\mathcal{M}_\mathcal{F} : \mathcal{M}_Y \to \mathcal{M}_{B\times X} = \Lambda(W \oplus V)$. Fix a homogeneous basis $W = \mathbb{Q}(w_1, w_2, \ldots)$. Given $\chi \in \mathcal{M}_Y$, we may then write $\mathcal{M}_\mathcal{F}(\chi) = \mathcal{M}_f(\chi) + \sum_j w_j\psi_j(\chi) + A_\mathcal{F}(\chi)$ where $\psi_j \in \mathrm{Hom}_{|w_j|}(\mathcal{M}_Y, \mathcal{M}_X)$ and $A_\mathcal{F}(\chi)$ is in the ideal of $\Lambda(W \oplus V)$ generated by the decomposables of ΛW. A standard check shows that each ψ_j is an $\mathcal{M}_f$-derivation and a cycle. We define

$$\Psi_F : \mathrm{Hom}_*(W, \mathbb{Q}) \to H_*\big(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)\big)$$

by setting $\Psi_F(w_j^*) = \langle\psi_j\rangle$ and extending by linearity. Here $w_j^* \in \mathrm{Hom}_{|w_j|}(W, \mathbb{Q})$ denotes the dual of the basis element w_j.

Theorem 3.1. *Let $f : X \to Y$ be a map between simply connected CW complexes of finite type with X finite. Let $F : B \to \mathrm{map}(X, Y; f)$ be a given map with B a connected simple CW complex of finite type. Then, with notation as above, we have commutative diagrams*

$$\begin{tikzcd}
\pi_n(B) \otimes \mathbb{Q} \arrow[r, "(F_\#)_\mathbb{Q}"] \arrow[d, "\cong"', "\Phi_B"] & \pi_n(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q} \arrow[d, "\Phi_f"', "\cong"] \\
\mathrm{Hom}_n(W, \mathbb{Q}) \arrow[r, "\Psi_F"] & H_n(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f))
\end{tikzcd}$$

for all $n \geqslant 2$. In the case $X = Y$ and $f = 1$ the result holds for $n = 1$ as well.

Proof. The map Φ_f is defined in the proof of [15, Theorem 2.1] by following the procedure above with $B = S^n$ but replacing the minimal model of S^n with its rational cohomology. The map Φ_B corresponds to the case $B = S^n$, $X = *$

and $Y = B$ is readily seen to be Sullivan's isomorphism, as recalled above. Compatibility with Ψ_F is thus direct from definitions. For $n \geqslant 2$ the fact that Φ_f is an isomorphism is [15, Theorem 2.1]. When $n = 1$, the map Φ_f extends to a well-defined set map and is a surjection [16, Theorem 2.1c], but will not generally be a homomorphism in degree 1. When $Y = X$ and $f = 1$, however, the multiplication in $\pi_1(\mathrm{map}(X, X; 1))$ is induced by the multiplication in $\mathrm{map}(X, X; 1)$, which is an H-space under composition of maps. Thus, the adjoint of the product of two classes $\alpha, \beta : S^1 \to \mathrm{map}(X, X; 1)$ is given by

$$S^1 \times X \xrightarrow{\Delta \times 1} S^1 \times S^1 \times X \xrightarrow{\Gamma} X$$

where $\Gamma(z_1, z_2, x) = (\alpha(z_1) \circ \beta(z_2))(x)$. Since $1 : X \to X$ is a two-sided identity in $\mathrm{map}(X, X; 1)$, we have $\Gamma \circ (i_1, 1) = A$ and $\Gamma \circ (i_2, 1) = B$, where A and B denote the adjoints to α and β, respectively, and $i_1, i_2 : S^1 \to S^1 \times S^1$ are the inclusions. It follows easily from this that Φ_f is a homomorphism in degree 1. Injectivity then follows by the argument in [15, Theorem 2.1] which only requires Φ_f a homomorphism to extend to $n = 1$. □

Now fix a fibration $\xi : X \xrightarrow{j} E \xrightarrow{p} B$ of simply connected finite type CW complexes with X a finite complex and with classifying map $h : B \to \mathrm{B\,aut}_1(X)$. We are interested in describing h at the level of rational homotopy groups and so we may consider $\Omega h : \Omega B \to \Omega \mathrm{B\,aut}_1(X)$. Using the equivalence $\partial_U : \Omega \mathrm{B\,aut}_1(X) \to \mathrm{aut}_1(X) = \mathrm{map}(X, X; 1)$ we obtain a map

$$H = \partial_U \circ \Omega h : \Omega B \to \mathrm{map}(X, X; 1) \tag{4}$$

which fits the setting of Theorem 3.1. Recall that the Koszul–Sullivan model of the fibration $\xi : X \xrightarrow{j} E \xrightarrow{p} B$ is a short exact sequence

$$\Lambda W, d_B \xrightarrow{P} \Lambda(W \oplus V), d_E \xrightarrow{J} \Lambda V, d_X \tag{5}$$

of DG algebras. The differential d_E satisfies $d_E(w) = d_B(w)$ for $w \in W$ and $d_E(v) - d_X(v) \in (\Lambda W)^+ \cdot \Lambda(W \oplus V)$ [6, Proposition 15.5]. The map P is the inclusion and the map J satisfies $J(w) = \varepsilon(w)$ and $J(v) = v$. The DG algebras $\Lambda W, d_B$ and $\Lambda V, d_X$ are Sullivan minimal models for B and X, respectively. The DG algebra $\Lambda(W \oplus V), d_E$ is a Sullivan model for the total space E but is not, in general, a minimal DG algebra. Given $\chi \in \Lambda V$ we may then write $d_E(\chi) = d_X(\chi) + \sum_j w_j \theta_j(\chi) + B_E(\chi)$ where $B_E(\chi)$ is in the ideal of $\Lambda(W \oplus V)$ generated by the decomposables in ΛW. Again we check directly that $\theta_j \in \mathrm{Der}_{|w_j|-1}(\Lambda V, \Lambda V; 1)$ and is a cycle. Define

$$\Theta_\xi : \mathrm{Hom}_n(W, \mathbb{Q}) \to H_{n-1}\big(\mathrm{Der}(\Lambda V, \Lambda V; 1)\big)$$

by setting $\Theta_\xi(w_j^*) = \langle \theta_j \rangle$ and extending by linearity.

Theorem 3.2. *Let $\xi : X \xrightarrow{j} E \xrightarrow{p} B$ be a fibration of simply connected CW complexes of finite type with X finite and classifying map $h : B \to \mathrm{B\,aut}_1(X)$. Then, with notation as above, we have a commutative diagram*

$$\begin{tikzcd} \pi_n(B) \otimes \mathbb{Q} \arrow[r, "(h_\#)_\mathbb{Q}"] \arrow[d, "\Phi_B", "\cong"'] & \pi_n(\mathrm{B\,aut}_1(X)) \otimes \mathbb{Q} \arrow[d, "\Phi'_{1_X}"', "\cong"] \\ \mathrm{Hom}_n(W, \mathbb{Q}) \arrow[r, "\Theta_\xi"] & H_{n-1}(\mathrm{Der}(\Lambda V, \Lambda V; 1)) \end{tikzcd}$$

for $n \geqslant 2$.

Proof. The map Φ'_{1_X} is obtained by precomposing Φ_{1_X} from Theorem 3.1 with the identification $\pi_n(\mathrm{B\,aut}_1(X)) \otimes \mathbb{Q} \cong \pi_{n-1}(\mathrm{map}(X, X; 1)) \otimes \mathbb{Q}$. Let $\overline{W}$ denote the desuspension of W so that $\pi_*(\Omega B) \otimes \mathbb{Q} \cong \mathrm{Hom}_*(\overline{W}, \mathbb{Q})$. Define $\overline{\Theta}_\xi : \mathrm{Hom}_*(\overline{W}, \mathbb{Q}) \to H_*(\mathrm{Der}(\Lambda V, \Lambda V; 1))$ by $\overline{\Theta}_\xi(\overline{w}_j^*) = \Theta_\xi(w_j^*)$. We show $\Psi_H = \overline{\Theta}_\xi$; the result follows.

We first argue that the adjoint $\mathcal{H}: \Omega B \times X \to X$ of the map H in (4) is homotopic to the *holonomy action* $\mathcal{H}_\xi = s_0 \circ i_0$ of ξ as defined by the following diagram:

$$\begin{tikzcd}
\Omega B \times X \arrow[d, "i_0"] \arrow[dr, "i"] \arrow[dd, bend right=60, "\mathcal{H}_\xi"'] & & \\
PB \times_B E \arrow[r, "i'"] \arrow[d, shift right, "s_0"'] & MB \times_B E \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] \arrow[dr, "q_1"] & MB \arrow[d, "q_0"] \\
X \arrow[u, shift right, "j_0"'] \arrow[r, "j"] & E \arrow[r, "p"] & B
\end{tikzcd}$$

Here, with notation as in [6, §2], MB is the space of Moore paths on B with q_0, q_1 evaluation at 0 and the length of the path, respectively, PB the subspace of paths which end at the basepoint of B and the other maps are the obvious inclusions and projections. The inclusion j_0 is a homotopy equivalence [6, Proposition 2.5(ii)] and we have denoted a homotopy inverse by s_0. Let $\mathcal{H}_\infty : \Omega \mathrm{B}\,\mathrm{aut}_1(X) \times X \to X$ denote the holonomy action of the universal X-fibration. Then $\mathcal{H}_\xi \sim \mathcal{H}_\infty \circ (\Omega h \times 1_X)$ by the naturality of the holonomy action with respect to pull-backs [10, Proposition 11.4]. Taking adjoints, we obtain $H_\xi \sim H_\infty \circ \Omega h$. Finally, $H_\infty \sim \partial_U : \Omega \mathrm{B}\,\mathrm{aut}_1(X) \to \mathrm{aut}_1(X)$ by Stasheff's formulation of the classification theorem in terms of parallel transports [20]: the maps ∂_U and H_∞ are transports giving the same principal $\mathrm{aut}_1(X)$-fibration (namely, the universal one) and hence are homotopic.

On [6, p. 419], the authors obtain a Sullivan model for the diagram $\Omega B \times X \xrightarrow{i_0} PB \times_B E \xleftarrow{j_0} X$ occurring in the definition of the holonomy action $\mathcal{H}_\xi$. In our notation, this is a diagram of DG algebras

$$\Lambda(\overline{W} \oplus V), \bar{d}_X \xleftarrow{\ I\ } \Lambda(W \oplus \overline{W} \oplus V), \bar{d}_E \xrightarrow{\ J\ } \Lambda V, d_X.$$

The differential $\bar{d}_X$ is given by $\bar{d}_X(v) = d_X(v)$ while $\bar{d}_X(\overline{W}) = 0$. The differential $\bar{d}_E$ satisfies $\bar{d}_E(x) = d_E(x)$ for $x \in W \oplus V$. The differential $\bar{d}_E$ on $\overline{w} \in \overline{W}$ is determined by the (push-out) construction involved. Tracing this through, we see $\bar{d}_E(\overline{w}) - w \in \mathcal{J}$ where $\mathcal{J}$ is the ideal of $\Lambda(W \oplus \overline{W} \oplus V)$ generated by decomposables in $\Lambda(W \oplus \overline{W})$. The maps I and J are projections. A DG algebra map $S : \Lambda V, d_X \to \Lambda(W \oplus \overline{W} \oplus V), \bar{d}_E$ with $S \circ J = 1_{\Lambda V}$ exists by the standard lifting lemma for minimal models (cf. [6, Lemma 12.4]). The map S is a Sullivan model for the equivalence s_0 and $I \circ S$ is one for $\mathcal{H}$. Given $\chi \in \Lambda V$ write $S(\chi) = \chi + \sum_j \overline{w}_j \bar{\theta}_j(\chi) + \sum_j w_j \varphi_j(\chi) + C_S(\chi)$ for $C_S(\chi) \in \mathcal{J}$. Using the fact that S is a map of algebras, we obtain that $\bar{\theta}_j \in \mathrm{Der}_{|w_j|-1}(\Lambda V, \Lambda V; 1)$ and $\varphi_j \in \mathrm{Der}_{|w_j|}(\Lambda V, \Lambda V; 1)$. Using the fact that S is a chain map and $\bar{d}_E(\mathcal{J}) \subseteq \mathcal{J}$, we obtain $D_{1_X}(\bar{\theta}_j) = 0$ while $D_{1_X}(\varphi_j) = \theta_j - \bar{\theta}_j$. Thus $\Psi_H(\overline{w}_j^*) = \langle \bar{\theta}_j \rangle = \langle \theta_j \rangle = \overline{\Theta}_\xi(\overline{w}_j^*)$. $\square$

Remark 3.3. The *holonomy representation* of ξ is the action of the homotopy Lie algebra $\pi_*(\Omega B) \otimes \mathbb{Q}$ on $H_*(X; \mathbb{Q})$ induced by the holonomy action $H_\xi : \Omega B \times X \to X$ (cf. [6, p. 415]). Define the "induced derivation" map $I : H_*(\mathrm{Der}(\Lambda V, \Lambda V; 1)) \to \mathrm{Der}_*(H^*(X; \mathbb{Q}), H^*(X; \mathbb{Q}); 1)$ by setting $I(\langle \theta \rangle)(\langle \chi \rangle) = \langle \theta(\chi) \rangle$ for θ a derivation cycle and χ a cycle of $\Lambda V, d_X$. It is easy to check I is well-defined. By Theorem 3.2 and [6, Theorem 31.3], we see $I \circ \Theta_\xi : \mathrm{Hom}_*(W, \mathbb{Q}) \to \mathrm{Der}_*(H^*(X; \mathbb{Q}), H^*(X; \mathbb{Q}); 1)$ is dual to the holonomy representation (up to sign).

4. The rationalized Gottlieb sequence

The Gottlieb sequence is a P-local invariant of the fibre-homotopy type of fibrations of simply connected finite type CW complexes, provided the fibre is a finite complex. This fact follows directly from isomorphisms $G_n(X) \otimes \mathbb{Z}_P \cong G_n(X_P)$ [12] and $G_n(E, X; j) \otimes \mathbb{Z}_P \cong G_n(E_P, X_P; j_P)$ [17]. We define the P-local Gottlieb homology $GH_n(\xi; \mathbb{Z}_P)$ of a fibration ξ by setting $GH_n(\xi; \mathbb{Z}_P) = GH_n(\xi_P)$. We will focus exclusively on the rational case here. We say a fibration ξ is *rationally Gottlieb trivial* if the sequence (1) splits into short exact sequences in each degree after tensoring with $\mathbb{Q}$.

Consider, again, a fibration $\xi : X \xrightarrow{j} E \xrightarrow{p} B$ with X a finite complex and classifying map $h : B \to \mathrm{B}\,\mathrm{aut}_1(X)$. We wish to compare the condition that ξ is rationally Gottlieb trivial, with that of the vanishing of $(h_\sharp)_\mathbb{Q}$. Since both of these conditions imply ξ is *rationally weak-homotopically trivial*, that is, that the rationalized connecting homomorphism $(\partial_\sharp)_\mathbb{Q} = 0$, we may assume this condition holds for ξ. In this case, the rationalization of the Gottlieb sequence for ξ breaks up into three-term sequences at each degree $n \geqslant 2$ of the form

$$0 \longrightarrow G_n(X) \otimes \mathbb{Q} \xrightarrow{(j_\sharp)_\mathbb{Q}} G_n(E, X; j) \otimes \mathbb{Q} \xrightarrow{(p_\sharp)_\mathbb{Q}} \pi_n(B) \otimes \mathbb{Q} \longrightarrow 0 \tag{6}$$

Moreover, the model of the total space of the fibration given in (5) for ξ is a minimal DG algebra: $\mathcal{M}_E, d_E = \Lambda(W \oplus V), d_E$ [9, Proposition 4.12]. As above, we have $d_E(w) = d_B(w)$ while, for $\chi \in \Lambda V$, $d_E(\chi) = d_X(\chi) + \sum_j w_j\theta_j(\chi) + B_E(\chi)$ where $B_E(\chi)$ is in the ideal $\mathcal{I}$ of $\Lambda(W \oplus V)$ generated by decomposables in ΛW. The maps $J : \Lambda(W \oplus V), d_E \to \Lambda V, d_X$ and $P : \Lambda W, d_B \to \Lambda(W \oplus V), d_E$ are Sullivan models for j and p, respectively. They induce *linearization* maps $Q(J) : W \oplus V \to V$ and $Q(P) : W \to W \oplus V$ (see [6, p. 171]) which in turn induce maps $Q(J)^* : \mathrm{Hom}_n(V, \mathbb{Q}) \to \mathrm{Hom}_n(W \oplus V, \mathbb{Q})$ and $Q(P)^* : \mathrm{Hom}_n(W \oplus V, \mathbb{Q}) \to \mathrm{Hom}_n(W, \mathbb{Q})$ by taking duals. These maps correspond, under the identifications Φ_X, Φ_E, Φ_B of Theorem 3.1, to the maps induced on rational homotopy groups by j and p.

Composition with $\varepsilon : \Lambda V \to \mathbb{Q}$ induces a chain map $\varepsilon_* : \mathrm{Der}_n(\Lambda(W \oplus V), \Lambda V; J) \to \mathrm{Der}_n(\Lambda(W \oplus V), \mathbb{Q}; \varepsilon)$. The minimality of $\mathcal{M}_E, d_E$ implies $H_n(\mathrm{Der}(\Lambda(W \oplus V), \mathbb{Q}; \varepsilon)) \cong \mathrm{Hom}_n(W \oplus V, \mathbb{Q})$ and so we obtain a map

$$H(\varepsilon_*) : H_n\big(\mathrm{Der}\big(\Lambda(W \oplus V), \Lambda V; J\big)\big) \to \mathrm{Hom}_n\big(\Lambda(W \oplus V), \mathbb{Q}\big).$$

Define the *nth rationalized evaluation subgroup of J* by

$$G_n\big(\Lambda(W \oplus V), \Lambda V; J\big) = \mathrm{im}\big(H(\varepsilon_*)\big)$$

for $n \geqslant 2$. Thus $w^* \in \mathrm{Hom}_n(W, \mathbb{Q})$ is in the subspace $G_n(\Lambda(W \oplus V), \Lambda V; J)$ if and only if w^* extends to a J-derivation θ of degree n with $D_J(\theta) = 0$. We write $G_n(\Lambda V) = G_n(\Lambda V, \Lambda V; 1)$ and call this the *nth rationalized Gottlieb group* of $\Lambda V, d_X$. We then obtain a sequence

$$0 \longrightarrow G_n(\Lambda V) \xrightarrow{Q(J)^*} G_n\big(\Lambda(W \oplus V), \Lambda V; J\big) \xrightarrow{Q(P)^*} \mathrm{Hom}_n(W, \mathbb{Q}) \longrightarrow 0 \tag{7}$$

for each $n \geqslant 2$. The following is a direct consequence of [15, Corollary 2.2].

Theorem 4.1. *Let $\xi : X \xrightarrow{j} E \xrightarrow{p} B$ be a fibration of simply connected, finite type CW complexes with X finite and $(\partial_\sharp)_\mathbb{Q} = 0$. Then, for each $n \geqslant 2$, the sequence* (6) *is equivalent to the sequence* (7).

Using these identifications, we extend the result above for spherical fibrations (Theorem 2.5) to the following:

Theorem 4.2. *Let $\xi : X \xrightarrow{j} E \xrightarrow{p} B$ be a fibration of simply connected, finite type CW complexes with X finite and classifying map $h : B \to \mathrm{B\,aut}_1(X)$. The following are equivalent*:

(1) $(h_\sharp)_\mathbb{Q} = 0 : \pi_n(B) \otimes \mathbb{Q} \to \pi_n(\mathrm{B\,aut}_1(X)) \otimes \mathbb{Q}$ *for each $n \geqslant 2$,*
(2) *ξ is rationally Gottlieb trivial. That is, we have short exact sequences*

$$0 \longrightarrow G_n(X) \otimes \mathbb{Q} \xrightarrow{(j_\sharp)_\mathbb{Q}} G_n(E, X; j) \otimes \mathbb{Q} \xrightarrow{(p_\sharp)_\mathbb{Q}} \pi_n(B) \otimes \mathbb{Q} \longrightarrow 0$$

for each $n \geqslant 2$,
(3) $(p_\sharp)_\mathbb{Q} : G_n(E, X; j) \otimes \mathbb{Q} \to \pi_n(B) \otimes \mathbb{Q}$ *is surjective for all $n \geqslant 2$.*

Proof. We begin with the implication (1) $\Rightarrow$ (2). By virtue of Theorem 2.4, we need only show exactness at the middle term, that is, that $GH_*(\xi; \mathbb{Q}) = 0$. For this, suppose that $\psi \in \mathrm{Der}_n(\Lambda(W \oplus V), \Lambda V; J)$ is a D_J-cycle for which there exists $v \in V$ with $\psi(v) = 1$. (This corresponds to an element $x \in G_n(E, X; j) \otimes \mathbb{Q}$ with $(p_\sharp)_\mathbb{Q}(x) = 0$.) We may also assume that $\psi(w_j) = 0$ for $|w_j| \leqslant n$. Write D for the differential in $\mathrm{Der}_*(\Lambda V, \Lambda V; 1)$. We wish to find a D-cycle $\theta \in \mathrm{Der}_n(\Lambda V, \Lambda V; 1)$ that satisfies $\theta(v) = 1$. (This corresponds to the element $x \in G_n(E, X; j) \otimes \mathbb{Q}$ being in the image of $G_n(X) \otimes \mathbb{Q}$ under $(j_\sharp)_\mathbb{Q}$.) As a first approximation to such a θ, consider the derivation $\psi_X \in \mathrm{Der}_n(\Lambda V, \Lambda V; 1)$ obtained by simply restricting ψ to ΛV. Since $D_J(\psi) = 0$, for $\chi \in \Lambda V$ we have

$$\begin{aligned}0 &= D_J(\psi)(\chi)\\ &= d_X\psi(\chi) - (-1)^n\psi d_E(\chi)\\ &= d_X\psi(\chi) - (-1)^n\psi d_X(\chi) - (-1)^n\psi\left(\sum_j w_j\theta_j(\chi) + B_E(\chi)\right).\end{aligned}$$

Since ψ is a J-derivation, and $J(W) = 0$, we have $\psi(w_j\theta_j(\chi)) = \psi(w_j)\theta_j(\chi)$ and $\psi(B_E(\chi)) = 0$. This yields the identity

$$0 = D(\psi_X)(\chi) - (-1)^n\sum_j \psi(w_j)\theta_j(\chi). \tag{8}$$

The sum on the right-hand side is thus an obstruction to ψ_X being a D-cycle, as we would wish. With our hypothesis on the classifying map, we may overcome this obstruction as follows. Since $(h_\sharp)_{\mathbb{Q}} = 0$, Theorem 3.2 implies that each $\theta_j \in \mathrm{Der}_{|w_j|-1}(\Lambda V, \Lambda V; 1)$ is a D-boundary: $\theta_j = D(\varphi_j)$ for some $\varphi_j \in \mathrm{Der}_{|w_j|}(\Lambda V, \Lambda V; 1)$. For each j, define derivations $\widehat{\theta_j} \in \mathrm{Der}_{n-1}(\Lambda V, \Lambda V; 1)$ and $\widehat{\varphi_j} \in \mathrm{Der}_n(\Lambda V, \Lambda V; 1)$ by setting

$$\widehat{\theta_j}(\chi) = \psi(w_j)\theta_j(\chi) \quad \text{and} \quad \widehat{\varphi_j}(\chi) = \psi(w_j)\varphi_j(\chi)$$

for $\chi \in \Lambda V$. Since ψ is a J-derivation and a D_J-cycle, and since $J(W) = 0$, we see easily that $d_X(\psi(w_j)) = 0$. Using this, a straightforward computation leads to the identity

$$D(\widehat{\varphi_j}) = (-1)^{|w_j|-n}\widehat{\theta_j}. \tag{9}$$

Now set $\theta = \psi_X - \sum_j(-1)^{|w_j|}\widehat{\varphi_j} \in \mathrm{Der}_n(\Lambda V, \Lambda V; 1)$. Since $\psi(w_j)$ is never scalar, we have $\theta(v) = 1$. The identities (8) and (9) now give $D(\theta) = 0$, as required.

The implication (2) $\Rightarrow$ (3) is immediate. We prove (3) $\Rightarrow$ (1). By Theorem 3.2, it suffices to show that each of the derivations $\theta_j \in \mathrm{Der}_{|w_j|}(\Lambda V, \Lambda V; 1)$ are D-boundaries. Let $\{w_j\}_{j\in J}$ be a well-ordered homogeneous basis of W. Because $(p_\sharp)_{\mathbb{Q}}$ is surjective, each $w_j^* \in \mathrm{Hom}_{|w_j|}(W, \mathbb{Q})$ extends to a J-derivation $\eta_j \in \mathrm{Der}_{|w_j|}(\Lambda(W \oplus V), \Lambda V; J)$ with $D_J(\eta_j) = 0$ and $\eta_j(w_j) = 1$. Without loss of generality, we may suppose that $\eta_j(w_i) = 0$ for $i < j$ and that $\eta_j(w_i)$ is of positive degree (or zero) for $i > j$.

For each j, let $\eta_{j,X} \in \mathrm{Der}_{|w_j|}(\Lambda V, \Lambda V; 1)$ denote the derivation obtained by restricting η_j to ΛV. Then we have, for each j,

$$D\big((-1)^{|w_j|}\eta_{j,X}\big) = \theta_j + \eta_j(w_{j+1})\theta_{j+1} + \eta_j(w_{j+2})\theta_{j+2} + \cdots. \tag{10}$$

This is simply a re-written version of (8), above. Now choose a particular θ_r. We will show that we may add a suitable D-boundary to (10), so as to remove the terms from the right-hand side other than θ_r. For this, we suppose inductively that, for each $k \geqslant 1$, we have a derivation $\widehat{\varphi_{k-1}} \in \mathrm{Der}_{|w_r|}(\Lambda V, \Lambda V; 1)$ such that

$$D\left((-1)^{|w_r|}\eta_{r,X} - \sum_{t=1}^{k-1}\widehat{\varphi_t}\right) = \theta_r + c_{r+k}\theta_{r+k} + c_{r+k+1}\theta_{r+k+1} + \cdots, \tag{11}$$

with each $c_{r+s} \in \Lambda V$, of degree $|c_{r+s}| = |w_{r+s}| - |w_r|$, a sum of terms in $\Lambda^+ V$ each of the form

$$\eta_r(w_{p_1})\eta_{p_1}(w_{p_2})\cdots\eta_{p_l}(w_{r+s}),$$

for $r < p_1 < p_2 < \cdots < p_l < r + s$.

For the inductive step, define $\widehat{\varphi_k} = (-1)^{|w_r|}c_{r+k}\eta_{r+k,X}$. Since each η_j is a J-derivation and a D_J-cycle, and since $J(W) = 0$, we have that $d_X(\eta_j(w_i)) = 0$ for $i > j$. It follows that $d_X(c_{r+s}) = 0$ for each s. Then

$$\begin{aligned}D(\widehat{\varphi_k}) &= (-1)^{|w_{r+k}|}c_{r+k}D(\eta_{r+k,X})\\ &= c_{r+k}\big(\theta_{r+k} + \eta_{r+k}(w_{r+k+1})\theta_{r+k+1} + \eta_{r+k}(w_{r+k+2})\theta_{r+k+2} + \cdots\big),\end{aligned}$$

from (10) with $j = r + k$, and hence we have

$$\begin{aligned}D\left((-1)^{|w_r|}\eta_{r,X} - \sum_{t=1}^{k}\widehat{\varphi_t}\right) &= \theta_r + \big(c_{r+k+1} - c_{r+k}\eta_{r+k}(w_{r+k+1})\big)\theta_{r+k+1}\\ &\quad + \big(c_{r+k+2} - c_{r+k}\eta_{r+k}(w_{r+k+2})\big)\theta_{r+k+2} + \cdots.\end{aligned}$$

The coefficients $c_{r+k+s} - c_{r+k}\eta_{r+k}(w_{r+k+s})$ of each θ_{r+k+s} in this expression are of the required form, and the inductive step is proven. Induction starts with $k=1$, where we have (10) for $j=r$ (take $\widehat{\varphi_0}=0$).

By induction, we may write

$$D\biggl((-1)^{|w_r|}\eta_{r,X} - \sum_{t\geqslant 1}\widehat{\varphi_t}\biggr) = \theta_r.$$

Notice that each $\widehat{\varphi_t}$ is a derivation, and that, for any given $\chi \in \Lambda V$, we have $\widehat{\varphi_t}(\chi)=0$ for all but a finite number of t. Indeed, since $\eta_{r+k,X}$, used in the definition of $\widehat{\varphi_k}$, decreases degree by $|w_{r+k}|$, we will have $\widehat{\varphi_t}(\chi)=0$ for all t with $|w_{r+t}| \geqslant |\chi|$. The infinite sum is "locally finite", therefore, and defines a derivation. This proves that each θ_r is a D-boundary in $\mathrm{Der}_{|w_r|}(\Lambda V, \Lambda V; 1)$, as required. □

Theorem 4.2 provides a link between the rationalized Gottlieb sequence and well-known results on rational L-S category. Let $\mathrm{cat}_0(X)$ denote the rational L-S category of the space X, that is, $\mathrm{cat}_0(X) = \mathrm{cat}(X_{\mathbb{Q}})$ where $X_{\mathbb{Q}}$ is the rationalization of X and $\mathrm{cat}(X)$ denotes the ordinary L-S category of X.

Corollary 4.3. *Let $\xi : X \xrightarrow{j} E \xrightarrow{p} B$ be a fibration of simply connected, finite type CW complexes with X finite. If ξ satisfies one of the three equivalent conditions of Theorem* 4.2*, then*

(1) $\dim(G_{\mathrm{odd}}(E,X;j)\otimes\mathbb{Q}) \leqslant \mathrm{cat}_0(X) + \dim(\pi_{\mathrm{odd}}(B)\otimes\mathbb{Q})$,
(2) $G_{\mathrm{even}}(E,X;j)\otimes\mathbb{Q} \cong \pi_{\mathrm{even}}(B)\otimes\mathbb{Q}$.

If, further, B has finite-dimensional rational homotopy then

(3) $\mathrm{cat}_0(E) \geqslant \mathrm{cat}_0(X) + \mathrm{cat}_0(B)$.

Proof. The first two results are direct consequence of [5, Theorem III] and condition (2) of Theorem 4.2. For the third result, observe that if $(h_\sharp)_{\mathbb{Q}} = 0$, then by Remark 3.3 the holonomy representation of ξ is rationally trivial. The result in this case follows from [11, Theorem 2]. □

With rational techniques now at our disposal, we may rapidly add to the store of illustrative examples begun in the integral setting above. The following provides a further example of a weak-homotopically trivial fibration that is not Gottlieb trivial.

Example 4.4. Consider a fibration of the form $\xi : \mathbb{C}P^2 \xrightarrow{j} E \xrightarrow{p} S^4$. Any such fibration admits a section and, in particular, is weak-homotopically trivial. This follows by reasoning as in Example 2.6, using the fact that $\pi_3(\mathbb{C}P^2)=0$. Now consider classifying maps for such a ξ. These are elements of $\pi_4(\mathrm{B\,aut}_1(\mathbb{C}P^2)) \cong \pi_3(\mathrm{map}(\mathbb{C}P^2, \mathbb{C}P^2; 1))$. Let $\Lambda V, d_X = \Lambda(v_2, v_5)$, with $d_X(v_2)=0$ and $d_X(v_5)=v_2^3$, denote the minimal model of $\mathbb{C}P^2$. Then the derivation $\theta \in \mathrm{Der}_3(\Lambda V, \Lambda V; 1)$, defined by $\theta(v_2)=0$ and $\theta(v_5)=v_2$, gives a nonzero class in $H_3(\mathrm{Der}(\Lambda V, \Lambda V; 1))$. Hence, $\pi_3(\mathrm{map}(\mathbb{C}P^2, \mathbb{C}P^2; 1))\otimes\mathbb{Q}$ is nonzero and so $\pi_4(\mathrm{B\,aut}_1(\mathbb{C}P^2))$ contains elements of infinite order. Choose $h : S^4 \to \mathrm{B\,aut}_1(\mathbb{C}P^2)$ to be any map that represents a class of infinite order. Since $(h_\#)_{\mathbb{Q}} \neq 0$, by Theorem 4.2 we have $(p_\#)_{\mathbb{Q}} : G_*(E, \mathbb{C}P^2; j)\otimes\mathbb{Q} \to \pi_*(S^4)\otimes\mathbb{Q}$ is not surjective. We conclude that $p_\# : G_*(E, \mathbb{C}P^2; j) \to \pi_*(S^4)$ cannot be surjective.

It is perhaps interesting to note that the minimal model of ξ is determined by the requirement that its classifying map be nontrivial rationally. Write $\mathcal{M}_{S^4} = \Lambda(w_4, w_7)$ with $d_B(w_4)=0$ and $d_B(w_7)=w_4^2$. Then any fibration with base S^4 and fibre $\mathbb{C}P^2$ has minimal model of the form $\Lambda(w_4, w_7), d_B \to \Lambda(w_4, w_7, v_2, v_5), d_E \to \Lambda(v_2, v_5), d_X$. The only possible "twisting" of the differential d_E, for degree reasons, is of the form $d_E(v_5) = v_2^3 + c w_4 v_2$, for $c \in \mathbb{Q}$. If $c=0$, then the fibration is rationally trivial; we have $(p_\#)_{\mathbb{Q}} : G_*(E, \mathbb{C}P^2; j)\otimes\mathbb{Q} \to \pi_*(S^4)\otimes\mathbb{Q}$ surjective, as is easily checked by the methods of this section; and hence by Theorem 4.2 its classifying map is rationally trivial. If we assume this is not the case, then the minimal model of the fibration is determined, up to isomorphism, by the differential $d_E(v_5) = v_2^3 + w_4 v_2$.

Our final example illustrates that the dimension of the Gottlieb homology and thus that of the rationalized evaluation subgroup of a fibre inclusion can be arbitrarily large even when the dimension of $G_*(X)\otimes\mathbb{Q}$ and $\pi_*(B)\otimes\mathbb{Q}$ are small.

Example 4.5. Let $n, k > 0$ be any odd integers. We construct a rational fibration $\xi_{\mathbb{Q}} : X_{\mathbb{Q}} \xrightarrow{j} E_{\mathbb{Q}} \xrightarrow{p} (S^{kn})_{\mathbb{Q}}$ where X has the rational homotopy type of a finite complex with $\dim(G_*(X)\otimes\mathbb{Q}) = 1$ such that $\dim(GH_k(\xi_{\mathbb{Q}})) = n$. Let $V = \mathbb{Q}(v_1, \ldots, v_{n+1}, u)$ where each v_j is of degree k and $|u| = k(n+1)-1$. Define d_X on ΛV by setting $d_X(v_i) = 0$ and $d_X(u) = v_1 v_2 \cdots v_{n+1}$. Note that $\Lambda V, d_X$ is an elliptic model and so may be realized as a finite complex X. Define a K.S. model

$$\Lambda(w_{kn}), 0 \xrightarrow{P} \Lambda\big(\mathbb{Q}(w_{kn}) \oplus V\big), d_E \xrightarrow{J} \Lambda V, d_X$$

by setting $d_E(u) = v_1 v_2 \cdots v_{n+1} + w_{kn} v_{n+1}$ with $d_E(w_{kn}) = d_E(v_i) = 0$ for $i = 1, \ldots, n$. We show each $v_i^* \in \mathrm{Hom}_k(V, \mathbb{Q})$ extends to a J-derivation cycle for $i = 1, \ldots, n$. Define $\psi_i \in \mathrm{Der}(\Lambda(\mathbb{Q}(w_{kn}) \oplus V), \Lambda V; J)$ by setting $\psi_i(v_i) = 1$, $\psi_i(v_j) = \psi_i(u) = 0$ for $j \neq i$ and $\psi_i(w_{kn}) = -v_1 v_2 \cdots \widehat{v_i} \cdots v_n$. It is easy to check $D_J(\psi_i) = 0$ as needed. Visibly $G_j(X)\otimes\mathbb{Q} = 0$ for $j \neq k(n+1)-1$ while $G_{k(n+1)-1}(X)\otimes\mathbb{Q} \cong \mathbb{Q}$. Thus $\dim(GH_k(\xi_{\mathbb{Q}})) = n$.

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