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On multidegrees of tame and wild automorphisms of $\mathbb{C}^3$

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ABSTRACT

In this note we show that the set $\mathrm{mdeg}(\mathrm{Aut}(\mathbb{C}^3))\setminus\mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$ is not empty, where mdeg denotes multidegree. Moreover we show that this set has infinitely many elements. Since for Nagata's famous example N of a wild automorphism, $\mathrm{mdeg}\,N = (5, 3, 1) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$, and since for other known examples of wild automorphisms the multidegree is of the form $(1, d_2, d_3)$ (after permutation if necessary), we give the very first example of a wild automorphism F of $\mathbb{C}^3$ with $\mathrm{mdeg}\,F \notin \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$.

We also show that, if d_1, d_2 are odd numbers such that $\gcd(d_1, d_2) = 1$, then $(d_1, d_2, d_3) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$ if and only if $d_3 \in d_1\mathbb{N} + d_2\mathbb{N}$. This a crucial fact that we use in the proof of the main result.

1. Introduction

Let us recall that a *tame automorphism* is, by definition, a composition of linear automorphisms and triangular automorphisms, where a *triangular automorphism* is a mapping of the form

$$T : \mathbb{C}^n \ni \left\{ \begin{array}{l} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right\} \mapsto \left\{ \begin{array}{l} x_1 \\ x_2 + f_2(x_1) \\ \vdots \\ x_n + f_n(x_1, \dots, x_{n-1}) \end{array} \right\} \in \mathbb{C}^n.$$

Recall also that an automorphism is called *wild* if it is not tame.

The *multidegree* of any polynomial mapping $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$, denoted as $\mathrm{mdeg}\,F$, is the sequence $(\deg F_1, \dots, \deg F_n)$. We will denote by $\mathrm{Tame}(\mathbb{C}^n)$ the group of all tame automorphisms of $\mathbb{C}^n$, and by mdeg the mapping from the set of all polynomial endomorphisms of $\mathbb{C}^n$ into the set $\mathbb{N}^n$. In [3] it was proved that $(3, 4, 5), (3, 5, 7), (4, 5, 7), (4, 5, 11) \notin \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$. Next in [5] it was proved that $(3, d_2, d_3) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$, for $3 \le d_2 \le d_3$, if and only if $3|d_2$ or $d_3 \in 3\mathbb{N} + d_2\mathbb{N}$, and in [4] it was shown that for $d_3 \ge d_2 > d_1 \ge 3$, where d_1 and d_2 are prime numbers, $(d_1, d_2, d_3) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$ if and only if $d_3 \in d_1\mathbb{N} + d_2\mathbb{N}$. In this paper we give a generalization of this result (Theorem 2.1 below), and using this fact we show the following theorem.

Theorem 1.1. *The set* $\mathrm{mdeg}(\mathrm{Aut}(\mathbb{C}^3))\setminus\mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$ *is infinite.*

Notice that the existence of wild automorphisms does not imply the above result. For example Nagata's famous example is wild, but its multidegree is (after permutation) $(1, 3, 5) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$, because for instance the map $\mathbb{C}^3 \ni (x, y, z) \mapsto (x, y + x^3, z + x^5) \in \mathbb{C}^3$ is a tame automorphism.

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2. Tame automorphisms of $\mathbb{C}^3$ with multidegree (d_1, d_2, d_3) with $\gcd(d_1, d_2) = 1$ and odd d_1, d_2

In the proof of Theorem 1.1 we will use the following generalization of the result of [4].

Theorem 2.1. *Let $d_3 \geq d_2 > d_1 \geq 3$ be positive integers. If d_1 and d_2 are odd numbers such that $\gcd(d_1, d_2) = 1$, then $(d_1, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if $d_3 \in d_1\mathbb{N} + d_2\mathbb{N}$, i.e. if and only if d_3 is a linear combination of d_1 and d_2 with coefficients in $\mathbb{N}$.*

In the proof of this theorem (which is an appropriate adaptation of the proof of the result of [4]) we will need the following results that we include here for the convenience of the reader.

Theorem 2.2 (*see e.g. [1]*). *If d_1, d_2 are positive integers such that $\gcd(d_1, d_2) = 1$, then for every integer $k \geq (d_1 - 1)(d_2 - 1)$ there are $k_1, k_2 \in \mathbb{N}$ such that*

$$k = k_1 d_1 + k_2 d_2.$$

Moreover $(d_1 - 1)(d_2 - 1) - 1 \notin d_1\mathbb{N} + d_2\mathbb{N}$.

Proposition 2.3 (*[3], Proposition 2.2*). *If for a sequence of integers $1 \leq d_1 \leq \cdots \leq d_n$ there is $i \in \{1, \ldots, n\}$ such that*

$$d_i = \sum_{j=1}^{i-1} k_j d_j \quad \text{with } k_j \in \mathbb{N},$$

then there exists a tame automorphism F of $\mathbb{C}^n$ with $\operatorname{mdeg} F = (d_1, \ldots, d_n)$.

Proposition 2.4 (*[5], Proposition 2.4*). *Suppose that $f, g \in \mathbb{C}[X_1, \ldots, X_n]$ are algebraically independent and such that $\overline{f} \notin \mathbb{C}[\overline{g}], \overline{g} \notin \mathbb{C}[\overline{f}]$ ($\overline{h}$ means the highest homogeneous part of h). Assume that $\deg f < \deg g$, put*

$$p = \frac{\deg f}{\gcd(\deg f, \deg g)},$$

and suppose that $G(x, y) \in \mathbb{C}[x, y]$ with $\deg_y G(x, y) = pq + r$, $0 \leq r < p$. Then

$$\deg G(f, g) \geq q\,(p \deg g - \deg g - \deg f + \deg[f, g]) + r \deg g.$$

In the above proposition $[f, g]$ means the Poisson bracket of f and g defined as the following formal sum:

$$\sum_{1 \leq i < j \leq n} \left(\frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} - \frac{\partial f}{\partial X_j} \frac{\partial g}{\partial X_i} \right) [X_i, X_j]$$

and

$$\deg [f, g] = \max_{1 \leq i < j \leq n} \deg \left\{ \left(\frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} - \frac{\partial f}{\partial X_j} \frac{\partial g}{\partial X_i} \right) [X_i, X_j] \right\},$$

where by definition $\deg [X_i, X_j] = 2$ for $i \neq j$ and $\deg 0 = -\infty$.

From the definition of the Poisson bracket we have

$$\deg [f, g] \leq \deg f + \deg g$$

and by Proposition 1.2.9 of [2],

$$\deg[f, g] = 2 + \max_{1 \leq i < j \leq n} \deg \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right)$$

if f, g are algebraically independent, and $[f, g] = 0$ if f, g are algebraically dependent.

The last result that we will need is the following theorem.

Theorem 2.5 (*[6], Theorem 3*). *Let $F = (F_1, F_2, F_3)$ be a tame automorphism of $\mathbb{C}^3$. If $\deg F_1 + \deg F_2 + \deg F_3 > 3$ (in other words if F is not a linear automorphism), then F admits either an elementary reduction or a reduction of types I–IV (see [6], Definitions 2–4).*

Let us recall that an automorphism $F = (F_1, F_2, F_3)$ *admits an elementary reduction* if there exists a polynomial $g \in \mathbb{C}[x, y]$ and a permutation σ of the set $\{1, 2, 3\}$ such that $\deg(F_{\sigma(1)} - g(F_{\sigma(2)}, F_{\sigma(3)})) < \deg F_{\sigma(1)}$; in other words, if there exists an elementary automorphism $\tau : \mathbb{C}^3 \to \mathbb{C}^3$ such that $\operatorname{mdeg}(\tau \circ F) < \operatorname{mdeg} F$, where $(d_1, \ldots, d_n) < (k_1, \ldots, k_n)$ means that $d_l \leq k_l$ for all $l \in \{1, \ldots, n\}$ and $d_i < k_i$ for at least one $i \in \{1, \ldots, n\}$. Recall also that a mapping $\tau = (\tau_1, \ldots, \tau_n) : \mathbb{C}^n \to \mathbb{C}^n$ is called an *elementary automorphism* if there exists $i \in \{1, \ldots, n\}$ such that

$$\tau_j(x_1, \ldots, x_n) = \begin{cases} x_j & \text{for } j \neq i, \\ x_i + g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) & \text{for } j = i. \end{cases}$$

Proof (of Theorem 2.1). Assume that $F = (F_1, F_2, F_3)$ is an automorphism of $\mathbb{C}^3$ such that $\operatorname{mdeg} F = (d_1, d_2, d_3)$. Assume also that $d_3 \notin d_1\mathbb{N} + d_2\mathbb{N}$. By Theorem 2.2 we have

$$d_3 < (d_1 - 1)(d_2 - 1). \tag{1}$$

First of all we show that this hypothetical automorphism F does not admit reductions of types I–IV.

By the definitions of those reductions (see [6], Definitions 2–4), if $F = (F_1, F_2, F_3)$ admits such a reduction, then $2 | \deg F_i$ for some $i \in \{1, 2, 3\}$. Thus if d_3 is odd, then F does not admit a reduction of types I–IV. Assume that $d_3 = 2n$ for some positive integer n.

If F admits a reduction of type I or II, then by the definition (see [6], Definitions 2 and 3) we have $d_1 = sn$ or $d_2 = sn$ for some odd $s \geq 3$. Since $d_1, d_2 \leq d_3 = 2n < sn$, we obtain a contradiction.

If F admits a reduction of type III or IV, then by the definition (see [6], Definition 4) we have either

$$n < d_1 \leq \frac{3}{2}n, \quad d_2 = 3n,$$

or

$$d_1 = \frac{3}{2}n, \quad \frac{5}{2}n < d_2 \leq 3n.$$

Since $d_1, d_2 \leq d_3 = 2n < \frac{5}{2}n, 3n$, we obtain a contradiction. Thus we have proved that our hypothetical automorphism F does not admit a reduction of types I–IV.

Now we will show that it also does not admit an elementary reduction.

Assume, to the contrary, that

$$(F_1, F_2, F_3 - g(F_1, F_2)),$$

where $g \in \mathbb{C}[x, y]$, is an elementary reduction of (F_1, F_2, F_3). Then $\deg g(F_1, F_2) = \deg F_3 = d_3$. But, by Proposition 2.4, we have

$$\deg g(F_1, F_2) \geq q(d_1 d_2 - d_1 - d_2 + \deg[F_1, F_2]) + r d_2,$$

where $\deg_y g(x, y) = q d_1 + r$ with $0 \leq r < d_1$. Since F_1, F_2 are algebraically independent, $\deg[F_1, F_2] \geq 2$ and so

$$d_1 d_2 - d_1 - d_2 + \deg[F_1, F_2] \geq d_1 d_2 - d_1 - d_2 + 2 > (d_1 - 1)(d_2 - 1).$$

This and (1) imply that $q = 0$, and that

$$g(x, y) = \sum_{i=0}^{d_1 - 1} g_i(x) y^i.$$

Since $\operatorname{lcm}(d_1, d_2) = d_1 d_2$, the sets

$$d_1\mathbb{N}, d_2 + d_1\mathbb{N}, \ldots, (d_1 - 1)d_2 + d_1\mathbb{N}$$

are pairwise disjoint. This yields

$$d_3 = \deg\left(\sum_{i=0}^{d_1-1} g_i(F_1) F_2^i\right) = \max_{i=0,\ldots,d_1-1} (\deg F_1 \deg g_i + i \deg F_2).$$

Thus

$$d_3 \in \bigcup_{r=0}^{d_1-1} (r d_2 + d_1\mathbb{N}) \subset d_1\mathbb{N} + d_2\mathbb{N},$$

contrary to assumption.

Now, assume that

$$(F_1, F_2 - g(F_1, F_3), F_3),$$

where $g \in \mathbb{C}[x, y]$, is an elementary reduction of (F_1, F_2, F_3). Since $d_3 \notin d_1\mathbb{N} + d_2\mathbb{N}$, we get $d_1 \nmid d_3$. This implies that

$$p = \frac{d_1}{\gcd(d_1, d_3)} > 1.$$

Since d_1 is odd, we also have $p \neq 2$. Thus by Proposition 2.4,

$$\deg g(F_1, F_3) \geq q(p d_3 - d_3 - d_1 + \deg[F_1, F_3]) + r d_3,$$

where $\deg_y g(x, y) = qp + r$ with $0 \leq r < p$. Since $p \geq 3$, we find that $p d_3 - d_3 - d_1 + \deg[F_1, F_3] \geq 2d_3 - d_1 + 2 > d_3$. Since we want to have $\deg g(F_1, F_3) = d_2$, it follows that $q = r = 0$, and so $g(x, y) = g(x)$. This means that $d_2 = \deg g(F_1, F_3) = \deg g(F_1)$. But this contradicts $d_2 \notin d_1\mathbb{N}$ (remember that $\gcd(d_1, d_2) = 1$).

Finally, if we assume that $(F_1 - g(F_2, F_3), F_2, F_3)$ is an elementary reduction of (F_1, F_2, F_3), then in the same way as in the previous case we obtain a contradiction. □

3. Proof of the theorem

Let $N : \mathbb{C}^3 \ni (x, y, z) \mapsto (x + 2y(y^2 + zx) - z(y^2 + zx)^2, y - z(y^2 + zx), z) \in \mathbb{C}^3$ be Nagata's example and suppose that $T : \mathbb{C}^3 \ni (x, y, z) \mapsto (z, y, x) \in \mathbb{C}^3$. We start with the following lemma.

Lemma 3.1. *For all $n \in \mathbb{N}$ we have* $\operatorname{mdeg}(T \circ N)^n = (4n - 3, 4n - 1, 4n + 1)$.

Proof. We have $T \circ N(x, y, z) = (z, y - z(y^2 + zx), x + 2y(y^2 + zx) - z(y^2 + zx)^2)$, so the above equality is true for $n = 1$. Suppose that $(f_n, g_n, h_n) = (T \circ N)^n$ for $f_n, g_n, h_n \in \mathbb{C}[X, Y, Z]$. One can see that $g_1^2 + h_1 f_1 = Y^2 + ZX$, and by induction that $g_n^2 + h_n f_n = Y^2 + ZX$ for any $n \in \mathbb{N} \setminus \{0\}$. Thus

$$\begin{aligned}(f_{n+1}, g_{n+1}, h_{n+1}) &= (T \circ N)(f_n, g_n, h_n)\\ &= \left(h_n, g_n - h_n\left(g_n^2 + h_n f_n\right), f_n + 2h_n\left(g_n^2 + h_n f_n\right) - h_n\left(g_n^2 + h_n f_n\right)^2\right)\\ &= \left(h_n, g_n - h_n\left(Y^2 + ZX\right), f_n + 2h_n\left(Y^2 + ZX\right) - h_n\left(Y^2 + ZX\right)^2\right).\end{aligned}$$

So if we assume that $\operatorname{mdeg}(f_n, g_n, h_n) = (4n - 3, 4n - 1, 4n + 1)$, we obtain $\operatorname{mdeg}(f_{n+1}, g_{n+1}, h_{n+1}) = (4n + 1, (4n + 1) + 2, (4n + 1) + 2 \cdot 2) = (4(n + 1) - 3, 4(n + 1) - 1, 4(n + 1) + 1)$. □

By the above lemma and Theorem 2.1 we obtain the following theorem.

Theorem 3.2. *For every $n \in \mathbb{N}$ the automorphism $(T \circ N)^n$ is wild.*

Proof. For $n = 1$ this is the result of Shestakov and Umirbaev [6,7]. So we can assume that $n \geq 2$. The numbers $4n - 3$, $4n - 1$ are odd and $\gcd(4n - 3, 4n - 1) = \gcd(4n - 3, 2) = 1$. Since $4n - 3 > 2$, we have $4n + 1 \notin (4n - 3)\mathbb{N} + (4n - 1)\mathbb{N}$. Then by Theorem 2.1, $(4n - 3, 4n - 1, 4n + 1) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ for $n > 1$. This proves that $(T \circ N)^n$ is not a tame automorphism. □

Let us notice that in the proof of the above theorem we have also proved that

$$\{(4n - 3, 4n - 1, 4n + 1) : n \in \mathbb{N}, n \geq 2\} \subset \operatorname{mdeg}(\operatorname{Aut}(\mathbb{C}^3)) \setminus \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3)).$$

This proves Theorem 1.1.

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