On the vertex-arboricity of planar graphs

André Raspaud, Weifan Wang

LaBRI UMR CNRS 5800, Université Bordeaux I, 33405 Talence Cedex, France

Available online 24 January 2008

Abstract

The vertex-arboricity $a(G)$ of a graph $G$ is the minimum number of subsets into which the set of vertices of $G$ can be partitioned so that each subset induces a forest. It is well-known that $a(G) \leq 3$ for any planar graph $G$. In this paper we prove that $a(G) \leq 2$ whenever $G$ is planar and either $G$ has no 4-cycles or any two triangles of $G$ are at distance at least 3.

1. Introduction

All graphs considered in this paper are finite simple graphs. A plane graph is a particular drawing of a planar graph on the Euclidean plane. For a plane graph $G$, let $V(G)$, $E(G)$, $F(G)$, $|G|$, $G^*$, $\Delta(G)$, and $\delta(G)$ denote, respectively, its vertex set, edge set, face set, order, dual, maximum degree, and minimum degree. A linear forest is a forest in which every connected component is a path. The vertex-arboricity $a(G)$ (linear vertex-arboricity $la(G)$, respectively) of a graph $G$ is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a forest (a linear forest, respectively).

The vertex-arboricity of a graph was first introduced by Chartrand, Kronk, and Wall [8], named by point-arboricity. Among other things, they proved that the vertex-arboricity of planar graphs is at most 3. Chartrand and Kronk [9] provided a planar graph of the vertex-arboricity 3. Poh [22] strengthened this result by showing that the linear vertex-arboricity of planar graphs is at most 3.

The following theorem, which will be cited later, characterizes completely maximal plane graphs with the vertex-arboricity 2.

---

E-mail addresses: raspaud@labri.fr (A. Raspaud), wwf@zjnu.cn (W. Wang).

On leave of absence from the Department of Mathematics, Zhejiang Normal University, Jinhua 321004, PR China.
Theorem 1 ([24]). Let $G$ be a maximal plane graph of order at least 4. Then $a(G) = 2$ if and only if $G^*$ is Hamiltonian.

As an extension of Theorem 1, Hakimi and Schmeichel [16] proved that a plane graph $G$ has $a(G) = 2$ if and only if $G^*$ contains a connected Eulerian spanning subgraph. It was known [14] that determining the vertex-arboricity of a graph is NP-hard. Hakimi and Schmeichel [16] showed that determining whether $a(G) \leq 2$ is NP-complete for maximal planar graphs $G$. The reader is referred to [5–7,10,12,17,20,23,26] for other results about the vertex-arboricity of graphs.

The purpose of this paper is to give some sufficient conditions for a planar graph to have the vertex-arboricity at most 2. In short, we prove the following theorems:

**Theorem 2.** For each $k \in \{3, 4, 5, 6\}$, every planar graph $G$ without $k$-cycles has $a(G) \leq 2$.

**Theorem 3.** Every planar graph $G$ without triangles at distance less than 2 has $a(G) \leq 2$.

2. Preliminaries

Let $G$ be a plane graph. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of $f$ and write $f = [u_1u_2\cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices of $b(f)$ in the clockwise order. Sometimes, we write $V(f) = V(b(f))$. For $x \in V(G) \cup F(G)$, let $d_G(x)$, or simply $d(x)$, denote the degree of $x$ in $G$. A vertex (or face) of degree $k$ is called a $k$-vertex (or $k$-face). If $k \leq 4$, $x$ is called a minor vertex (or face), and likewise a major vertex (or face). We say that $f$ is an $(m_1, m_2, \ldots, m_n)$-face if $d(u_i) = m_i$ for $i = 1, 2, \ldots, n$. For $S \subseteq V(G) \cup E(G)$, let $G[S]$ denote the subgraph of $G$ induced by $S$. A graph $G$ is called $k$-degenerate if every subgraph $H$ of $G$ contains a vertex of degree at most $k$.

Now we introduce an equivalent definition to the vertex-arboricity in terms of the coloring version. An acyclic $k$-coloring of a graph $G$ is a mapping $\phi$ from the vertex set $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that each color class induces an acyclic subgraph, i.e., a forest. The vertex-arboricity $a(G)$ of $G$ is the smallest integer $k$ such that $G$ has an acyclic $k$-coloring.

Analogously to the Brooks’ Theorem on the vertex coloring, Kronk and Mitchem [21] proved the following result:

**Theorem 4 ([21]).** Let $G$ be a simple connected graph. If $G$ is neither a cycle nor a clique of odd order, then $a(G) \leq \lceil \Delta(G)/2 \rceil$.

Theorem 4 implies that planar graphs $G$ of maximum degree 4 have $a(G) \leq 2$.

**Lemma 5.** If $G$ is a $k$-degenerate graph, then $a(G) \leq \lceil (k + 1)/2 \rceil$.

**Lemma 5** can be established by using induction on the order of graphs. It is well-known that every planar graph is 5-degenerate and that every planar graph without 3-cycles is 3-degenerate. It is shown in [27] that every planar graph without 5-cycles is 3-degenerate and in [13] that every planar graph without 6-cycles is 3-degenerate. Note that an icosidodecahedron, i.e., the line graph of a dodecahedron, is a 4-regular planar graph without 4-cycles. Hence the lack of 4-cycles does not imply the 3-degeneracy of a planar graph. Choudum [11] constructed, for each $k \geq 7$, 4-regular 3-connected planar graphs with no $k$-cycles.

These facts together with Lemma 5 establish the following result:

**Theorem 6.** If $G$ is a planar graph without 3-cycles, or without 5-cycles, or without 6-cycles, then $a(G) \leq 2$. 
Lemma 7. Suppose that a graph $G$ is the union of two graphs $G_1$ and $G_2$ with $|V(G_1) \cap V(G_2)| \leq 1$. Then $a(G) \leq \max\{a(G_1), a(G_2)\}$.

3. Planar graphs without 4-cycles

In this section, we focus on the vertex-arboricity of planar graphs without cycles of length 4, starting with the study of their structural properties.

Let $G$ be a plane graph with $\delta(G) = 4$ and without 4-cycles. For a vertex $v \in V(G)$, we use $F_3(v)$ to denote the set of 3-faces incident with $v$. For a face $f \in F(G)$, let $m(f)$ denote the number of 3-faces adjacent to $f$. We say that two faces (or cycles) are adjacent or intersecting if they share a common edge or a common vertex respectively. Suppose that $v$ is a 4-vertex and $v_1, v_2, v_3, v_4$ are the neighbors of $v$ in the clockwise order. Let $f_i$ denote the face incident with the vertex $v$ with $v v_1, v v_{i+1}$ as boundary edges, where $i = 1, 2, 3, 4$ and the summation in the indices are taken modulo 4. We say that $f_1$ is a source of $f_3$, and $f_3$ is a sink of $f_1$, if $d(f_2) = d(f_4) = 3$, $d(f_3) = 5$, $d(f_1) \geq 5$, $d(v_3) = d(v_4) = 4$, and $d(v_i) \geq 5$ for $i = 1, 2$. A 5-face $f$ is said to be weak if $f$ is adjacent to exactly four 3-faces and is incident with five 4-vertices. Let $s(f)$ denote the number of weak 5-faces adjacent to a face $f$.

Lemma 8. Let $G$ be a 2-connected plane graph with $\delta(G) = 4$ and without 4-cycles. Then $G$ contains a 5-face $[x_1x_2 \cdots x_5]$ adjacent to four 3-faces $[x_1y_1x_2]$, $[x_2y_2x_3]$, $[x_3y_3x_4]$, and $[x_4y_4x_5]$ so that $d(y_2) = d(x_i) = 4$ for all $i = 1, 2, \ldots, 5$ (see Fig. 1).

Proof. Assume that the lemma is false. Let $G$ be a counterexample. Then $G$ is a 2-connected plane graph with $\delta(G) = 4$, without 4-cycles, and not having a 5-face that satisfies the requirement of the lemma. Since $G$ is 2-connected, the boundary of each face of $G$ forms a simple cycle. Since $G$ contains no 4-cycles, $G$ has neither 4-faces nor two adjacent 3-faces. These basic facts will be used frequently in the following proof without further notice.

Using Euler’s formula, $|G| - |E(G)| + |F(G)| = 2$ and $\sum\{d(v) \mid v \in V(G)\} = \sum\{d(f) \mid f \in F(G)\} = 2|E(G)|$, we can derive the following identity.

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -8. \quad (1)$$

Let $w$ denote a weight function defined by $w(x) = d(x) - 4$ for each $x \in V(G) \cup F(G)$. So the total sum of weights is equal to $-8$. We shall design some discharging rules and redistribute weights according to them. Once the discharging is finished, a new weight function $w'$ is produced. However, the total sum of weights is kept fixed when the discharging is in progress. On the other hand, we will show that $w'(x) \geq 0$ for all $x \in V(G) \cup F(G)$. This leads to an obvious contradiction.
The following are the discharging rules. For \( x, y \in V(G) \cup F(G) \), we use \( \tau(x \to y) \) to denote the amount of weight transferred from \( x \) to \( y \).

(R1) Every face \( f \) of degree at least 6 sends \( \frac{1}{2} \) to each adjacent 3-face, \( \frac{1}{2} \) to each adjacent weak 5-face, and 2/15 to each sink.

(R2) Let \( v \) be a vertex of degree at least 6 and \( f \) be a major face incident with \( v \). If \( f \) is adjacent to two 3-faces in \( F_3(v) \), then we let \( \tau(v \to f) = \frac{2}{3} \). If \( f \) is adjacent to exactly one 3-face in \( F_3(v) \), then we let \( \tau(v \to f) = \frac{1}{3} \).

(R3) Let \( v \) be a 5-vertex. Suppose that \( f_1, f_2, \ldots, f_5 \) are the incident faces of \( v \) in the clockwise order.

If \( |F_3(v)| = 1 \), say \( d(f_1) = 3 \), we let \( \tau(v \to f_2) = \tau(v \to f_3) = \frac{1}{3} \).

If \( |F_3(v)| = 2 \), say \( d(f_1) = d(f_2) = 3 \), we first let \( \tau(v \to f_2) = \frac{2}{3} \). Afterwards, if there exists exactly one face \( f^* \in \{f_4, f_5\} \) such that \( d(f^*) = 5 \), \( m(f^*) = 4 \) and \( f^* \) is incident with one 5-vertex and four 4-vertices, then we let \( \tau(v \to f^*) = \frac{1}{3} \). Otherwise, we let \( \tau(v \to f_4) = \tau(v \to f_5) = \frac{1}{6} \).

(R4) Every 5-face \( f \) sends \( \frac{1}{2} \) to each adjacent 3-face and 2/15 to each sink.

Let \( \beta(f) \) denote the resulting weight of a 5-face \( f \) after the discharging procedure is performed according to (R1)–(R4).

(R5) Let \( f \) be a 5-face adjacent to at most two 3-faces. If \( \beta(f) > 0 \) and \( s(f) \geq 1 \), then \( f \) sends \( \beta(f)/s(f) \) to each weak 5-face adjacent to \( f \).

Let \( w' \) denote the final weight function after (R1) to (R5) are carried out in the graph \( G \).

Suppose \( v \in V(G) \). Since \( \delta(G) = 4 \), we know that \( d(v) \geq 4 \). If \( d(v) = 4 \), then \( w'(v) = w(v) = 0 \). If \( d(v) = 5 \), then \( w(v) = 1 \). Since \( G \) contains no two adjacent 3-faces, \( |F_3(v)| \leq 2 \). It follows from the rule (R3) that \( w'(v) \geq 0 \). Assume that \( d(v) \geq 6 \). For \( k = 1, 2 \), we use \( t_k \) to denote the number of faces incident with \( v \) each of which is adjacent to exactly \( k \) 3-face(s) in \( F_3(v) \). It is easy to show that \( t_1 + 2t_2 \leq d(v) \), and hence \( w'(v) = w(v) - \frac{1}{3}t_1 - \frac{2}{3}t_2 = d(v) - 4 - \frac{1}{3}(t_1 + 2t_2) \geq d(v) - 4 - \frac{1}{3}d(v) = \frac{2}{3}d(v) - 6 \geq 0 \).

Suppose \( f \in F(G) \). Then \( d(f) \neq 4 \). We have to consider the different possible values of \( d(f) \).

1. If \( d(f) = 3 \), then each of the faces adjacent to \( f \) is of degree at least 5. By (R1) and (R4),
   \[ w'(f) = -1 + 3 \cdot \frac{1}{3} = 0. \]
   Assume that \( d(f) \geq 5 \). It is easy to see that \( m(f) + s(f) \leq d(f) \) and \( f \) has at most \( \frac{m(f)/2}{12} \) sinks.

2. If \( d(f) \geq 7 \), then \( w'(f) = w(f) - \frac{1}{3}m(f) + \frac{1}{12}s(f) + \frac{2}{15} \cdot \frac{1}{2}d(f) \geq d(f) - 4 - \frac{1}{3}m(f) + s(f) - \frac{1}{12}d(f) \geq d(f) - 4 - \frac{1}{3}d(f) = \frac{2}{3}d(f) - 4 \geq 0 \) by (R1).

3. Suppose that \( d(f) = 6 \). Then \( w(f) = 2 \). Let \( f = [x_1x_2 \cdots x_6] \) and let \( f_i \) denote the adjacent face of \( f \) with \( x_ix_{i+1} \) as their common boundary edge for \( i = 1, 2, \ldots, 6 \), where the indices are taken modulo 6. Suppose that \( f \) is adjacent to a weak 5-face, say \( f_1 \). Then, by definition, \( d(x_1) = d(x_2) = 4 \) and \( f_1 \) is adjacent to four 3-faces, i.e., all the adjacent faces, different from \( f \), of \( f_1 \) are of degree 3. If \( d(f_2) = 3 \), then either \( d(x_2) \geq 5 \) or \( x_2 \) is incident with two adjacent 3-faces, always producing a contradiction. Thus, it follows that \( d(f_2) > 3 \), and similarly \( d(f_3) > 3 \). This shows that if \( f \) is adjacent to a weak 5-face, then \( f \) is adjacent to at most three 3-faces.

- If \( m(f) \leq 4 \), then \( f \) has at most two sinks. Thus, by (R1), \( w'(f) \geq 2 - \frac{1}{3}m(f) - \frac{1}{2}s(f) - \frac{2}{2} = -2 \cdot \frac{1}{12}d(f) - \frac{1}{3}m(f) - \frac{1}{5}(6 - m(f)) - \frac{4}{15} = \frac{8}{15} - \frac{2}{15}m(f) \geq 0 \).
- If \( m(f) = 5 \), then \( s(f) = 0 \) by the above argument and \( w'(f) \geq 2 - 5 \cdot \frac{1}{3} - 2 \cdot \frac{2}{15} = \frac{1}{15} \) by (R1).
Assume that $m(f) = 6$. Then $s(f) = 0$. If $f$ is incident with a vertex $v$ of degree at least 5, then $\tau(v \to f) = \frac{2}{3}$ by (R2) and (R3), and $w'(f) \geq 2 + \frac{2}{3} - 6 \cdot \frac{1}{3} - 3 \cdot \frac{2}{15} = \frac{4}{15}$ accordingly. If all the vertices incident with $f$ are of degree 4, then it follows that $f$ has no sinks and therefore $w'(f) = 2 - 6 \cdot \frac{1}{3} = 0$.

4. Suppose that $d(f) = 5$. Then $w(f) = 1$. Let $x_1, x_2, \ldots, x_5$ be the boundary vertices of $f$ in the clockwise order. Let $f_i$ denote the adjacent face of $f$ with $x_i x_{i+1}$ as a boundary edge, $i = 1, 2, \ldots, 5$ (indices modulo 5).

- If $m(f) \leq 2$, $f$ has at most one sink. By (R4), $\beta(f) \geq 1 - 2 \cdot \frac{1}{3} - \frac{2}{15} = \frac{1}{3}$, implying that $w'(f) \geq 0$ by (R5).

- Assume that $m(f) = 3$. Notice that $f$ admits at most one sink. If $f$ has no sink, then $w'(f) = 1 - 3 \cdot \frac{1}{3} = 0$ by (R4) and (R5). Suppose that $f$ has one sink, say $d(f_1) = d(f_2) = 2$, and $f$ intersects with its sink at a vertex $x_2$. By definition, both $d(x_1)$ and $d(x_3)$ are of degree at least 5. Without loss of generality, we argue the two cases as follows:

  (i) $d(f_3) = 3$. Since $x_3$ is incident with two 3-faces, namely $f_2$ and $f_3$, $x_3$ gives $\frac{2}{3}$ to $f$ by (R2) or (R3). Thus $w'(f) \geq 1 + \frac{2}{3} - 3 \cdot \frac{1}{3} - \frac{2}{15} = \frac{8}{15}$.

  (ii) $d(f_4) = 3$. If either $d(x_1) \geq 6$ or $d(x_3) \geq 6$, then $f$ receives $\frac{1}{3}$ from $x_1$ or $x_3$ by (R2) and thus $w'(f) \geq 1 + \frac{1}{3} - 3 \cdot \frac{1}{3} - \frac{2}{15} = \frac{1}{5}$. Suppose that $d(x_1) = d(x_3) = 5$. If $x_1$ is incident with only one 3-face, i.e., $f_1$, then $\tau(x_1 \to f) = \frac{1}{3}$ by (R3), so that $w'(f) \geq 1 + \frac{1}{3} - 3 \cdot \frac{1}{3} - \frac{2}{15} = \frac{1}{5}$. So suppose that $|F_3(x_1)| = 2$. If either $d(f_5) \geq 6$ or $d(f_5) = 5$ and $m(f_5) \leq 3$, then $x_1$ sends at least $\frac{1}{5}$ to $f$ by (R3). If $d(f_5) = 5$ and $m(f_5) = 4$, then since $d(f_4) = 3$, we see that $d(x_5) \geq 5$. It follows that $f_5$ is incident with at least two vertices of degree at least 5. Thus (R3), we also have $\tau(x_1 \to f) \geq \frac{1}{5}$.

Now we always obtain $w'(f) \geq 1 + \frac{1}{6} - 3 \cdot \frac{1}{3} - \frac{2}{15} = \frac{4}{30}$.

- Assume that $m(f) = 4$, say $d(f_1) = d(f_2) = d(f_3) = d(f_4) = 3$. For $i \in \{1, 2, 3, 4\}$, let $y_i$ denote the third vertex on the boundary of $f_i$ distinct from $x_i$ and $x_{i+1}$. Note that $f$ has at most two sinks. If $f$ has a sink, then $f$ is incident with a vertex $u$ such that $\tau(u \to f) = \frac{2}{3}$ by (R2) or (R3). Hence $w'(f) \geq 1 + \frac{2}{3} - 4 \cdot \frac{1}{3} - 2 \cdot \frac{2}{15} = \frac{1}{15}$. Assume that $f$ has no sink. If one of $x_2, x_3, x_4$ is of degree at least 5, then $w'(f) \geq 1 + \frac{2}{3} - \frac{4}{3} = \frac{1}{3}$. If either $d(x_1) \geq 6$, or $d(x_3) \geq 6$, or $d(x_1) = d(x_3) = 5$, or $d(f_5) \geq 6$ and exactly one of $d(x_1)$ and $d(x_3)$ is equal to 5, then it is easy to check that $w'(f) \geq 1 + \frac{1}{3} - \frac{2}{3} = 0$. Now suppose that $d(f_5) = 5$, $d(x_1) = 5$ and $d(x_i) = 4$ for all $i \in \{2, 3, 4, 5\}$. We see that $f_5$ is adjacent to at most three 3-faces for otherwise it follows that $d(x_5) \geq 5$. By (R3), $\tau(x_1 \to f) = \frac{1}{5}$ and $w'(f) \geq 1 + \frac{1}{3} - \frac{4}{3} = 0$.

Finally, suppose that $d(x_1) = d(x_2) = \cdots = d(x_5) = 4$, that is $f$ is a weak 5-face. Since we assume the lemma to be false, we observe that $d(y_2) \geq 5$ and $d(y_3) \geq 5$. Thus, the face $f'$ having $y_2x_3, x_3y_3$ as two boundary edges is a source of $f$, so $\sigma(f' \to f) = \frac{7}{15}$ by (R1) or (R4). It suffices to show that $f_5$ gives $f$ at least $\frac{1}{5}$, thus $w'(f) \geq 1 + \frac{1}{5} + \frac{4}{15} = 0$. This is obvious by (R1) when $d(f_5) \geq 6$. Thus assume that $d(f_5) = 5$ and $f_5 = [x_5z_1z_2z_3x_1]$. Suppose that $g_1, g_2, g_3, g_4$ are the adjacent faces of $f_5$ different from $f$, where $f_5$ shares the edge $x_5z_1$ with $g_1, z_1z_2$ with $g_2, z_2z_3$ with $g_3$, and $z_3x_5$ with $g_4$. Since $d(x_1) = d(x_5) = 4$ and $G$ contains no adjacent 3-faces, we have $d(g_1) \geq 5$ and $d(g_4) \geq 5$. This implies that $f_5$ is adjacent to at most two 3-faces and so has at most one sink. If $m(f_5) = 0$, then $f_5$ has no sink and has at most five adjacent weak 5-faces. Thus $f_5$ gives $f$ at least $\frac{1}{5}$ by (R5). Assume that $m(f_5) = 1$, say $d(g_2) = 3$ and $d(g_3) \geq 5$. Then either $d(z_1) \geq 5$, or
$d(g_1) > 5$ or $g_1$ is adjacent to at most three 3-faces. We in both (all) cases derive that $g_1$ is not a weak 5-face in these two cases. Consequently, $\tau(f_5 \to f) \geq (1 - \frac{1}{3})/3 = \frac{2}{9} > \frac{1}{3}$ by (R5). Assume that $m(f_5) = 2$, i.e., both $g_2$ and $g_3$ are 3-faces. With a similar argument, $g_1$ and $g_4$ are not weak 5-faces. If $d(z_2) \geq 5$, then $\tau(z_2 \to f_5) = \frac{2}{3}$ by (R2) or (R3) and henceforth $\tau(f_5 \to f) \geq 1 + \frac{2}{3} - \frac{2}{3} = 1$ by (R5). Thus suppose that $d(z_2) = 4$. Now we have $\tau(f_5 \to f) \geq 1 - \frac{2}{3} - \frac{2}{15} = \frac{1}{5}$.

- Assume that $m(f) = 5$. If $f$ has a sink, then $f$ is incident with at least two vertices of degree at least 5. (R2) and (R3) guarantee that $w'(f) \geq 1 + 2 \cdot \frac{2}{3} - 5 \cdot \frac{1}{3} - 2 \cdot \frac{2}{15} = \frac{2}{3}$. Suppose $f$ has no sink. If $f$ is incident with a vertex of degree at least 5, then $w'(f) \geq 1 + \frac{2}{3} - 5 \cdot \frac{1}{3} = 0$

This completes the proof of the Lemma. □

In Figs. 1 and 2, by the heavy vertices we mean that they are of the same degree as in the original graph $G$.

**Theorem 9.** If $G$ is a plane graph without 4-cycles, then $a(G) \leq 2$.

**Proof.** We make use of induction on the order of $G$. If $|G| \leq 4$, then the theorem holds clearly. Suppose that $G$ is a plane graph with $|G| \geq 5$ and without 4-cycles. If $G$ contains a vertex $v$ of degree at most 3, we let $H = G - v$. Then $H$ is a plane graph without 4-cycles and $|H| = |G| - 1$. By the induction hypothesis, $H$ is acyclically 2-colorable. It is easy to show that any acyclic 2-coloring of $H$ can be extended into an acyclic 2-coloring of $G$.

Now suppose that $\delta(G) = 4$. By Lemma 7, we may assume that $G$ is 2-connected. By Lemma 8, $G$ contains a 5-face $[x_1x_2 \cdots x_5]$ adjacent to four 3-faces $[x_1y_1x_2]$, $[x_2y_2x_3]$, $[x_3y_3x_4]$, and $[x_4y_4x_5]$ such that $d(y_2) = d(x_i) = 4$ for all $i = 1, 2, \ldots, 5$. In $G$, we let $y_5$ denote the neighbor of $x_5$ different from $y_4$, $x_1$, and $x_4$. Let $y_6$ denote the neighbor of $x_1$ different from $y_1$, $x_2$, and $x_5$. Let $y'_2$, $y''_2$ denote the neighbors of $y_2$ different from $x_2$ and $x_3$. We set $H = G - y_2$. Then $H$ is a plane graph without 4-cycles and $|H| < |G|$. By the induction hypothesis, $H$ has an acyclic 2-coloring $\phi$ using the colors 1 and 2. If at least three of four neighbors of $y_2$ have the color 1 (or 2), then we assign 2 (or 1) to $y_2$. It is easy to see that such coloring does not produce a monochromatic cycle, thus $\phi$ is extended into an acyclic 2-coloring of $G$. So suppose that each color 1 or 2 occurs exactly twice in the neighborhood of $y_2$. We only need to consider the following two possibilities (up to symmetry):
Case 1. $\phi(y'_2) = \phi(y''_2) = 1$ and $\phi(x_2) = \phi(x_3) = 2$.

Since $x_1x_2y_1x_1$ is a 3-cycle in $H$, at most one of $x_1$ and $y_1$ is colored with 2. If some of them has the color 2, then we color $y_2$ with 2 and recolor $x_2$ with 1. So suppose that $\phi(x_1) = \phi(y_1) = 1$, and similarly $\phi(x_4) = \phi(y_3) = 1$. If $\phi(y_6) = 1$, we recolor $x_1$ with 2 and $x_2$ with 1, then color $y_2$ with 2. Suppose that $\phi(y_6) = 2$. If $\phi(x_5) = 1$, we again recolor $x_1$ with 2 and $x_2$ with 1, then color $y_2$ with 2. Then suppose that $\phi(x_5) = 2$. If $\phi(y_4) = 1$, we recolor $x_4$ with 2 and $x_3$ with 1, and color $y_2$ with 2. Suppose that $\phi(y_4) = 2$. If $\phi(y_5) = 2$, we color (or recolor) the vertices $y_2, x_3, x_4, x_5$ with the colors 2, 1, 2, 1, respectively. If $\phi(y_5) = 1$, we color (or recolor) the vertices $y_2, x_3, x_4, x_5, x_1$ with the colors 2, 1, 2, 1, 2, respectively.

It is easy to see that $\phi$ is extended to the graph $G$ in every possible case.

Case 2. $\phi(y'_2) = \phi(x_2) = 1$ and $\phi(y''_2) = \phi(x_3) = 2$.

We first erase the colors of the vertices $x_1, x_2, \ldots, x_5$. For $i = 1, 2$, let $S(i)$ denote the subset of vertices in $\{y_1, y_3, y_4, y_5, y_6\}$ which get the color $i$ in the coloring $\phi$. Without loss of generality, we suppose that $|S(1)| \leq |S(2)|$. Thus $0 \leq |S(1)| \leq 2$. At first, we assign the color 1 to $y_2$.

If $|S(1)| = 0$, we color $x_1, x_2, x_4, x_5$ with 1 and $x_3$ with 2.

Assume that $|S(1)| = 1$.

If $S(1) = \{y_1\}$, we color $x_3, x_4, x_5$ with 1 and $x_1, x_2$ with 2.

If $S(1) = \{y_3\}$, we color $x_1, x_2, x_5$ with 1 and $x_3, x_4$ with 2.

If $S(1) = \{y_4\}$, or $S(1) = \{y_5\}$, we color $x_1, x_2, x_4$ with 1 and $x_3, x_5$ with 2.

If $S(1) = \{y_6\}$, we color $x_2, x_4, x_5$ with 1 and $x_1, x_3$ with 2.

Assume that $|S(1)| = 2$.

If $S(1) = \{y_1, y_3\}$, we color $x_1, x_5$ with 1 and $x_2, x_3, x_4$ with 2.

If $S(1) = \{y_1, y_4\}$, or $S(1) = \{y_1, y_5\}$, or $S(1) = \{y_3, y_5\}$, we color $x_1, x_4$ with 1 and $x_2, x_3, x_5$ with 2.

If $S(1) = \{y_1, y_6\}$, we color $x_4, x_5$ with 1 and $x_1, x_2, x_3$ with 2.

If $S(1) = \{y_3, y_4\}$, we color $x_1, x_2$ with 1 and $x_3, x_4, x_5$ with 2.

If $S(1) = \{y_3, y_6\}$, we color $x_2, x_5$ with 1 and $x_1, x_3, x_4$ with 2.

If $S(1) = \{y_4, y_5\}$, we color $x_1, x_2, x_4$ with 1 and $x_3, x_5$ with 2.

If $S(1) = \{y_5, y_6\}$, we color $x_2, x_4, x_5$ with 1 and $x_1, x_3$ with 2.

If $S(1) = \{y_4, y_6\}$, we need to recolor $y_2$ with 2, afterwards color $x_2, x_3, x_5$ with 1 and $x_1, x_4$ with 2.

We have exhausted all the possible cases to extend $\phi$ to the whole graph $G$. The proof of Theorem 9 is complete. \qed

Now Theorem 2 follows immediately from Theorems 6 and 9.

4. Planar graphs with sparse triangles

Theorem 6 affirms that the vertex-arboricity of planar graphs without 3-cycles is at most 2. Actually, this result can be improved by relaxing the requirement for 3-cycles. In what follows, it is assumed that a triangle is synonymous with a 3-cycle. The distance $\text{dist}(T, T')$ between two triangles $T$ and $T'$ is defined as the value $\min\{\text{dist}(x, y) | x \in V(T) \text{ and } y \in V(T')\}$.

**Lemma 10.** Let $G$ be a 2-connected plane graph with $\delta(G) \geq 4$ and without triangles at distance less than 2. Then $G$ contains a 5-cycle $C = v_1v_2 \cdots v_5v_1$ with a chord $v_2v_5$ such that $d(v_i) = 4$ for all $i = 1, 2, \ldots, 5$ (see Fig. 2).
Proof. Assume the lemma is false. Let $G$ be a counterexample. Then $G$ is a 2-connected plane graph satisfying the following properties:

(a) $\delta(G) \geq 4$.
(b) For any two triangles $T_1$ and $T_2$, $\text{dist}(T_1, T_2) \geq 2$. In particular, every vertex $v$ is incident with at most a 3-face.
(c) There does not exist a $(4, 4, 4)$-face adjacent to a $(4, 4, 4, 4)$-face.

As in the proof of Lemma 8, we define the initial weight function $w(x) = d(x) - 4$ for all $x \in V(G) \cup F(G)$. From the formula (1), it follows that $\sum_{x \in V(G) \cup F(G)} w(x) = -8$. We design the new discharging rules (R1)–(R3) below and then carry out them on the graph $G$. Let $w'(x)$ denote the resultant weight function once the discharging procedure is complete. It suffices to show that $w'(x) \geq 0$ for all $x \in V(G) \cup F(G)$ to derive a contradiction.

A 3-face $f$ is called bad if it is incident with three 4-vertices and adjacent to three 4-faces. The face $f$ is said to be a pendant 3-face of a vertex $v$ if $\min\{\text{dist}(v, x) | x \in V(f)\} = 1$, i.e., the distance between $v$ and $f$ is exactly one.

This time, our discharging rules are as follows:

(R1) Every major face $f$ sends 1 to each adjacent 3-face.
(R2) Every major vertex $v$ which is incident with a 3-face $f$ sends 1 to this $f$.
(R3) Every major vertex $v$ which is not incident with any 3-face sends $\frac{1}{3}$ to each pendant bad 3-face.

Let $f \in F(G)$. If $d(f) = 4$, then $w'(f) = w(f) = 0$. If $d(f) \geq 5$, then it is easy to derive from (b) that $f$ is adjacent to at most $\lfloor d(f)/3 \rfloor$ 3-faces. Thus $w'(f) \geq d(f) - 4 - \lfloor d(f)/3 \rfloor = [2d(f)/3] - 4 \geq 0$ by (R1). Assume that $d(f) = 3$. Then $w(f) = -1$. If $f$ is either adjacent to a major face or incident with a major vertex, then $w'(f) \geq -1 + 1 = 0$ by (R1) or (R2). Otherwise, $f$ is a bad 3-face. Suppose that $f = [xyz]$, with $d(x) = d(y) = d(z) = 4$, is adjacent to three 4-faces $[xu_1u_2y]$, $[yv_1v_2z]$, and $[zt_1t_2x]$. On the one hand, we note that every element in $\{u_1, u_2, v_1, v_2, t_1, t_2\}$ takes $f$ as a pendant bad 3-face, and is not incident with any 3-face by (b). On the other hand, by (c), either $u_1$ or $u_2$ is a major vertex and hence gives $1/3$ to $f$ by (R3). The same argument applies to $v_i$'s or $t_j$'s. It turns out that $w'(f) \geq -1 + 3 \times \frac{1}{3} = 0$.

Let $v \in V(G)$. Then $d(v) \geq 4$ by (a). If $d(v) = 4$, then $w'(v) = w(v) = 0$. Assume that $d(v) \geq 5$. Then $w(v) \geq 1$. When $v$ is incident with a 3-face, $v$ has no pendant bad 3-faces by (b), so that $w'(v) \geq 1 - 1 = 0$. So suppose that $v$ is not incident with any 3-face. If $d(v) \geq 6$, then since $v$ has at most $d(v)$ pendant bad 3-faces, $w'(v) \geq d(v) - 4 - \frac{1}{3}d(v) = \frac{2}{3}d(v) - 4 \geq 0$ by (R3). Now assume that $d(v) = 5$, and so $w(v) = 1$. We claim that $v$ has at most three pendant bad 3-faces, and henceforth $w'(v) \geq 1 - 3 \times \frac{1}{3} = 0$.

Let $x, y, z, t, s$ denote the neighbors of $v$ arranged around $v$ in clockwise direction. Suppose to the contrary that $v$ has four pendant bad 3-faces, e.g., $f_1 = [xx_1x_2]$, $f_2 = [yy_1y_2]$, $f_3 = [zz_1z_2]$, and $f_4 = [tt_1t_2]$, where all the vertices in $V(f_1) \cup V(f_2) \cup V(f_3) \cup V(f_4)$ are of degree 4. Let $f_{xy}$, $f_{yz}$, and $f_{zt}$ denote the incident faces of $v$ with $vx, vy \in b(f_{xy})$, $vy, vz \in b(f_{yz})$, and $vz, vt \in b(f_{zt})$, respectively. For each $r \in \{x, y, z, t\}$, suppose that $r_3$ is the neighbor of $r$ different from $v, r_1, r_2$.

If neither $yy_3$ nor $zz_3$ lie on the boundary of $f_{yz}$, then it follows that $f_{yz}$ is adjacent to $f_2$ and $f_3$. Thus $d(f_{yz}) = 4$ by virtue of the definition of $f_2$ and $f_3$. However, it is immediate to derive that $\text{dist}(f_2, f_3) \leq 2$, contradicting (b). So suppose that at least one of the edges $yy_3$ and $zz_3$ belongs to the boundary of $f_{yz}$. If $yy_3 \in b(f_{yz})$, then, again, $f_2$ is adjacent to $f_{xy}$. It follows that $f_{xy}$ is a 4-face of the form $[yvxy_1]$ for some $1 \leq i \leq 2$, which implies that $\text{dist}(f_1, f_2) \leq 1$, also a contradiction. If $zz_3 \in b(f_{yz})$, we consider the face $f_{zt}$ to obtain a similar contradiction. This completes the proof of the lemma. \qed
Proof of Theorem 3. The proof is proceeded by induction on $|G|$. If $|G| \leq 4$, then the theorem holds trivially. Suppose that $G$ is a plane graph with $|G| \geq 5$ and without triangles at distance less than 2. If $\delta(G) \leq 3$, the proof is similar to that of Theorem 9. Thus, by Lemma 7, we may assume that $G$ is 2-connected and $\delta(G) \geq 4$. By Lemma 10, $G$ contains a 5-cycle $C = v_1 v_2 \cdots v_5 v_1$ with a chord $v_2 v_5$ such that $d(v_i) = 4$ for all $i = 1, 2, \ldots, 5$. For $1 \leq i \leq 5$, let $x_i$ (and $y_i$ when $i = 1, 3, 4$) denote the neighbors of $v_i$ which are not in the cycle $C$. We put $H = G - v_1$. Then $H$ is a plane graph without triangles at distance less than 2 and $|H| < |G|$. By the induction hypothesis, $H$ has an acyclic 2-coloring $\phi$ using the colors 1 and 2. If at least three of four neighbors of $v$ have the color 1 (or 2), then we color $v_1$ with 2 (or 1). Otherwise, each color 1 or 2 occurs exactly twice in the neighborhood of $v$. The argument is divided into the two cases below (up to symmetry):

Case 1. $\phi(x_1) = \phi(y_1) = 1$ and $\phi(v_2) = \phi(v_5) = 2$.

If at least one of $x_2$ and $v_3$ is colored with 2, then we color $v_1$ with 2 and recolor $v_2$ with 1. Suppose that $\phi(x_2) = \phi(v_3) = 1$. Similarly, $\phi(x_5) = \phi(v_4) = 1$. If at most one of $x_3$ and $y_3$ is colored with 2, we recolor $v_3$ with 2 and $v_2$ with 1, and then color $v_1$ with 2. Thus assume that $\phi(x_3) = \phi(y_3) = 2$. With a similar discussion, we may assume that $\phi(x_4) = \phi(y_4) = 2$. In this case, we again color $v_1$ with 2 and recolor $v_2$ with 1. It is easy to check that $\phi$ is extended to the whole graph $G$ in each possible case.

Case 2. $\phi(x_1) = \phi(v_2) = 1$ and $\phi(y_1) = \phi(v_5) = 2$.

First assume that $\phi(x_2) = \phi(x_5) = 1$. If $\phi(v_4) = 1$, we color $v_1$ with 2. Suppose that $\phi(v_4) = 2$. If $\phi(v_3) = 1$, we color $v_1$ with 1 and recolor $v_2$ with 2. Suppose that $\phi(v_3) = 2$. If at least one of $x_3$ and $y_3$ is colored with 2, we recolor $v_3$ with 1 and $v_2$ with 2, then color $v_1$ with 1. Suppose that $\phi(x_3) = \phi(y_3) = 1$. If at least one of $x_4$ and $y_4$ is colored with 2, we recolor $v_4$ with 1 and color $v_2$ with 2. If $\phi(x_4) = \phi(y_4) = 1$, we need only to color $v_1$ with 2.

If $\phi(x_2) = \phi(x_5) = 2$, we have a similar argument.

Next assume that $\phi(x_2) = 2$ and $\phi(x_5) = 1$. If $\phi(v_3) = 2$, we color $v_1$ with 1. If $\phi(v_4) = 1$, we color $v_1$ with 2. So suppose that $\phi(v_3) = 1$ and $\phi(v_4) = 2$. If $\phi(x_3) = \phi(y_3) = 2$, we color $v_1$ with 1. If $\phi(x_4) = \phi(y_4) = 1$, we color $v_1$ with 2. Hence assume that $1 \in \{\phi(x_3), \phi(y_3)\}$ and $2 \in \{\phi(x_4), \phi(y_4)\}$. We switch the colors of $v_3$ and $v_4$ and then color $v_1$ with 1.

Finally assume that $\phi(x_2) = 1$ and $\phi(x_5) = 2$. Switching the colors of $v_2$ and $v_5$, we reduce the problem to the previous case. This completes the proof of the theorem. □

We conclude this section by the following conjecture and question:

Conjecture 1. If $G$ is a planar graph without intersecting (or adjacent) triangles, then $a(G) \leq 2$.

Question 1. Is there a constant $c$ such that every planar graph $G$ without triangles at distance less than $c$ has $\delta(G) \leq 3$?

5. Smallest planar graphs with the vertex-arboricity 3

In this section, we give an easy observation about the fact that the vertex-arboricity of a planar graph is at most 2 when its order is sufficiently small. We need to cite the following result by Holton and Mckay [18]:

Theorem 11 ([18]). Every 3-connected cubic planar graph of order at most 36 is Hamiltonian.
Theorem 11 is best possible in the sense that there exist 3-connected cubic planar graphs of order 38 which are not Hamiltonian. Some such examples were constructed in [1,18]. Thus the smallest non-Hamiltonian 3-connected cubic planar graphs have 38 vertices.

Theorem 12. (1) If $G$ is a plane graph with $|G| \leq 20$, then $a(G) \leq 2$.
(2) There exists a plane graph $G$ with $|G| = 21$ such that $a(G) = 3$.

Proof. (1) Suppose that $G$ is a plane graph with $|G| \leq 20$. By adding some diagonals, we subdivide every face of degree at least 4 of $G$ into the union of 3-faces. Let $H$ denote the resultant graph. Then $H$ is a maximal plane graph with $G$ as a spanning graph. Note that $H$ is 3-connected, thus $H^*$ is a 3-connected cubic plane graph. By Euler’s formula $|H| + |F(H)| - |E(H)| = 2$ and the relation $|E(H)| = 3|H| - 6$, we get $|F(H)| = 2|H| - 4$. Thus, $|H^*| = |F(H)| = 2|H| - 4 = 2|G| - 4 \leq 36$. By Theorem 11, $H^*$ is Hamiltonian. Therefore, by Theorem 1, we derive that $a(G) \leq a(H) = 2$.

(2) A non-Hamiltonian 3-connected cubic planar graph $G$ on 38 vertices appears in Fig. 3 (see [4,19]). Its dual $G^*$, a maximal plane graph of order 21, is depicted in Fig. 4. Theorem 1 yields that $a(G^*) = 3$. This completes the proof of the theorem. □

Since $a(K_5) = 3$, the assumption that $G$ is plane in (1) of Theorem 12 is essential. Moreover, it is easy to note that $G^*$ in Fig. 4 is a 4-degenerate graph. It means that there exist 4-degenerate planar graphs of the vertex-arboricity 3.

Let $\mu$ denote the largest integer such that every planar graph $G$ without $k$-cycles, for $3 \leq k \leq \mu$, has $a(G) \leq 2$. Theorems 2 and 12 assert that $6 \leq \mu \leq 21$.

Question 2. What is the exact value of $\mu$?

Fig. 3. A non-Hamiltonian 3-connected cubic plane graph $G$ of order 38.

Fig. 4. A plane graph $G^*$ with $|G^*| = 21$ and $a(G^*) = 3$. 
6. Further research

Recall that every planar graph $G$ has $\alpha(G) \leq 3$, that is, $V(G)$ can be partitioned into $(V_1, V_2, V_3)$ such that each $V_i$ induces a forest. This result can be improved in the sense that one of $V_i$'s is an independent set of $G$. To show this, we need to use the following result due to Thomassen [25]:

**Theorem 13.** Every planar graph $G$ has a vertex partition $(V_1, V_2)$ such that $V_1$ induces a forest and $V_2$ induces a 2-degenerate graph.

**Lemma 14.** Every 2-degenerate graph $G$ has a vertex partition $(V_1, V_2)$ such that $V_1$ is an independent set and $V_2$ induces a forest.

**Proof.** The proof proceeds by induction on the order of $G$. If $|G| \leq 3$, the result is trivial. Let $G$ be a 2-degenerate graph with $|G| \geq 4$. Then $G$ contains a vertex $v$ of degree at most 2 by definition. Let $H = G - v$. Then $H$ is a 2-degenerate graph with $|H| < |G|$. By the induction hypothesis, $V(H)$ has a partition $(V'_1, V'_2)$ such that $V'_1$ induces an independent set and $V'_2$ induces a forest. In $G$, if some of the neighbors of $v$ belongs to $V'_1$, we let $V_1 = V'_1$ and $V_2 = V'_2 \cup \{v\}$; otherwise, we let $V_1 = V'_1 \cup \{v\}$ and $V_2 = V'_2$. It is easy to see that $(V_1, V_2)$ is a partition of $V(G)$ such that $V_1$ is an independent set and $V_2$ induces a forest. This completes the proof of the lemma. □

By Lemma 14 and Theorem 13, we have the following:

**Theorem 15.** Every planar graph $G$ has a vertex partition $(V_1, V_2, V_3)$ such that $V_1$ is an independent set and each of $V_2, V_3$ induces a forest.

Theorem 15 implies that every planar graph is 5-colorable.

**Conjecture 2.** Every planar graph $G$ has a vertex partition $(V_1, V_2, V_3)$ such that $V_1, V_2$ are independent sets and $V_3$ induces a forest.

It should be remarked that Conjecture 2, if true, implies the well-known Four-Color Theorem [2]. Finally, we like to conclude this paper by the following problem:

**Conjecture 3.** Every planar graph $G$ without 3-cycles has a vertex partition $(V_1, V_2)$ such that $V_1$ is an independent set and $V_2$ induces a forest.

Borodin and Glebov [3] showed that every planar graph $G$ of girth at least 5 has a vertex partition $(V_1, V_2)$ such that $V_1$ is an independent set and $V_2$ induces a forest. If Conjecture 3 were true, then it would imply the Grötzsch’s 3-Color Theorem on triangle-free planar graphs [15].

**Acknowledgment**

We deeply thank an anonymous referee whose many constructive comments greatly improved the presentation of the paper.

**References**