1. Introduction

The problem treated in this paper concerns the maximum likelihood estimation of a finite partially or completely ordered set of parameters of probability distributions. A special case of this problem, the maximum likelihood estimation of a finite ordered set of probabilities, has been treated in [2].

The problem will be formulated in section 2; in section 4 and 5 methods will be given by means of which the estimates may be found. For the proofs of the theorems we need some lemma's which will be proved in section 3; in section 6 the consistency of the estimates will be investigated and in section 7 some examples will be given.

2. The problem

Consider \( k \) independent random variables \( x_1, x_2, \ldots, x_k \) and \( n_i \) independent observations \( x_{i,1}, x_{i,2}, \ldots, x_{i,n_i} \) of \( x_i \) (\( i = 1, 2, \ldots, k \)). Assume that the distribution of \( x_i \) contains one unknown parameter \( \theta_i \) (\( i = 1, 2, \ldots, k \)) and that its distribution function is

\[
F_i(x_i|\theta_i) \triangleq P[x_i \leq x_i|\theta_i] \quad (i = 1, 2, \ldots, k).
\]

Two types of restrictions are imposed on the parameters \( \theta_1, \theta_2, \ldots, \theta_k \). First let \( I_i \) be a closed interval such that \( F_i(x_i|y) \) is a distribution function for each value of \( y \in I_i \) (\( i = 1, 2, \ldots, k \)). By means of the choice of \( I_i \), restrictions of the type \( c_i \leq \theta_i \leq d_i \) may be imposed. The second type of restrictions consists of a partial or complete ordering of the parameters, which may be described as follows. Let \( \alpha_{i,j} \) (\( i, j = 1, 2, \ldots, k \)) be numbers satisfying the conditions

\[
\begin{cases}
1. & \alpha_{i,j} = -\alpha_{i,j}, \\
2. & \alpha_{i,j} = 0 \text{ if the intersection } I_i \cap I_j \text{ contains at most one point,} \\
3. & \alpha_{i,j} = 0, +1 \text{ or } -1 \text{ in all other cases}
\end{cases}
\]

1) Report SP 52 of the Statistical Department of the Mathematical Centre, Amsterdam.

2) Random variables will be distinguished from numbers (e.g. from the values they take in an experiment) by printing their symbols in bold type.
and
\[ \alpha_{i,j} = 1 \text{ if } \alpha_{i,h} = \alpha_{h,i} = 1 \text{ for any } h. \]
The restrictions imposed on \( \theta_1, \theta_2, \ldots, \theta_k \) are then
\[ \begin{align*}
1. & \quad \alpha_{i,j}(\theta_i - \theta_j) \leq 0 \\
2. & \quad \theta_i \in I_i \quad (i, j = 1, 2, \ldots, k)
\end{align*} \]
and it will be supposed that the parameters \( \theta_1, \theta_2, \ldots, \theta_k \) are numbered in such a way that
\[ \alpha_{i,j} \geq 0 \text{ for each pair of values } (i, j). \]
No other restrictions on \( \theta_1, \theta_2, \ldots, \theta_k \) are admitted, such that all points \( y_1, y_2, \ldots, y_k \) of the Cartesian product
\[ G \overset{\text{def}}{=} \prod_{i=1}^{k} I_i, \]
satisfying
\[ \alpha_{i,j}(y_i - y_j) \leq 0 \quad (i, j = 1, 2, \ldots, k) \]
belong to the parameterspace, which thus is a convex subdomain of \( G \). This subdomain will be denoted by \( D \).

Let
\[ \begin{align*}
1. & \quad \alpha_{i,j} = 0 \text{ for } r_0 \text{ pairs of values } (i, j) \text{ with } i < j, \\
2. & \quad \alpha_{i,j} = 1 \text{ for } r_1 \text{ pairs of values } (i, j) \text{ with } i < j,
\end{align*} \]
then
\[ r_0 + r_1 = \binom{k}{2}. \]

Let further \( f_i(x_i \mid \theta_i) \) denote the density function of \( x_i \) if \( x_i \) possesses a continuous probability distribution and \( P[x_i = x_i \mid \theta_i] \) if \( x_i \) possesses a discrete probability distribution and let
\[ \begin{align*}
1. & \quad L_i = L_i(y_i) \overset{\text{def}}{=} \sum_{\gamma=1}^{n_i} \ln f_i(x_{i,\gamma} \mid y_i) \quad (i = 1, 2, \ldots, k), \\
2. & \quad L = L(y_1, y_2, \ldots, y_k) \overset{\text{def}}{=} \sum_{i=1}^{k} L_i(y_i).
\end{align*} \]

Then the maximum likelihood estimates of \( \theta_1, \theta_2, \ldots, \theta_k \) are the values of \( y_1, y_2, \ldots, y_k \) which maximize \( L \) in the domain \( D \). Unless explicitly stated otherwise \( L \) will only be considered in this domain \( D \); the maximum likelihood estimates will throughout this paper be denoted by \( t_1, t_2, \ldots, t_k \). Further the restrictions \( \theta_i \leq \theta_j \) (i.e. \( \alpha_{i,j} = 1 \)) satisfying
\[ \alpha_{i,h} \cdot \alpha_{h,j} = 0 \text{ for each } h \text{ between } i \text{ and } j \]
will be denoted by \( R_1, R_2, \ldots, R_s \). Each \( R_s \) thus corresponds with one pair \( (i, j) \); this pair will be denoted by \( (i_s, j_s) \).
Because of the transitivity relations (2.3) the system \( R_1, R_2, \ldots, R_s \) is equivalent to (2.4.1) and uniquely determined by (2.4.1). The restrictions \( R_1, R_2, \ldots, R_s \) will be called the essential restrictions.

Remark 1: H. D. Brunk [1] described a method by means of which the estimates of \( \theta_1, \theta_2, \ldots, \theta_k \) may be found if the distribution of \( x_i \) belongs to the "exponential family" \((i = 1, 2, \ldots, k)\) and if moreover \( I_i \) is the set of all values of \( y \) for which \( F_i(x_i \mid y) \) is a distribution function \((i = 1, 2, \ldots, k)\). His method however leads to more complicated computations than ours.

3. Lemma's

Definition: A function \( \varphi(y) \) of a variable \( y \) will be called strictly unimodal in an interval \( J \) if there exists a value \( y^* \in J \) such that

\[
\varphi(y) < \varphi(y') < \varphi(y^*)
\]

for each pair of values \((y, y') \in J \) with

\[
y < y' < y^*
\]

and for each pair of values \((y, y') \in J \) with

\[
y^* < y' < y.
\]

It follows at once from this definition that a strictly unimodal function \( \varphi(y) \) is bounded in every closed subdomain of \( J \) not containing \( y^* \).

Now let \( \varphi_\kappa(y_\kappa) \) be a strictly unimodal function of \( y_\kappa \) in the interval \( J_\kappa(\kappa = 1, 2, \ldots, K) \) and let further

\[
\Phi (y_1, y_2, \ldots, y_K) \overset{\text{def}}{=} \sum_{\kappa = 1}^{K} \varphi_\kappa(y_\kappa),
\]

then it will be clear that \( \Phi(y_1, y_2, \ldots, y_K) \) possesses a unique maximum in

\[
I \overset{\text{def}}{=} \prod_{\kappa = 1}^{K} J_\kappa
\]

in the point \( (y_1^*, y_2^*, \ldots, y_K^*) \), where \( \varphi_\kappa(y_\kappa^*) \) is the maximum of \( \varphi_\kappa \) in \( J_\kappa(\kappa = 1, 2, \ldots, K) \).

We now define a function \( V \) as follows.

Let \( y_1^0, y_2^0, \ldots, y_K^0 \) be a given point in \( I \) with \( y_\kappa^0 \neq y_\kappa^* \) for at least one value of \( \kappa \) and let

\[
\begin{align*}
Y_\kappa(\beta) &\overset{\text{def}}{=} (1 - \beta) y_\kappa^0 + \beta y_\kappa^* \quad (\kappa = 1, 2, \ldots, K), \\
0 &\leq \beta \leq 1.
\end{align*}
\]

Then \( \{Y_1(\beta), Y_2(\beta), \ldots, Y_K(\beta)\} \) is a point in \( I \) and \( V \) is defined by

\[
V(\beta) \overset{\text{def}}{=} \Phi \{ Y_1(\beta), Y_2(\beta), \ldots, Y_K(\beta) \}.
\]
Lemma I: \( V(\beta) \) is a monotone increasing function of \( \beta \) in the interval \( 0 \leq \beta \leq 1 \).

Proof: Consider a value of \( \alpha \) with
\[
y^*_{\alpha} = y^*_{\alpha}
\]
then
\[
Y(\beta) = y^*_{\alpha} \text{ for each } \beta \text{ with } 0 \leq \beta \leq 1.
\]
Thus in this case we have
\[
q_{\alpha}(y^*_{\alpha}) = q_{\alpha}(Y(\beta)) = q_{\alpha}(y^*_{\alpha}) \text{ for each } \beta \text{ with } 0 \leq \beta \leq 1.
\]
Now consider a value of \( \alpha \) with
\[
y^*_{\alpha} \neq y^*_{\alpha}
\]
then it follows from the fact that \( q_{\alpha}(y^*_{\alpha}) \) is, in the interval \( J_{\alpha} \), a strictly unimodal function of \( y^*_{\alpha} \) and attains its maximum in \( J_{\alpha} \) for \( y^*_{\alpha} = y^*_{\alpha} \), that
\[
q_{\alpha}(y^*_{\alpha}) < q_{\alpha}(Y(\beta_1)) < q_{\alpha}(Y(\beta_2)) < q_{\alpha}(y^*_{\alpha})
\]
for each pair of values \( (\beta_1, \beta_2) \) with \( 0 < \beta_1 < \beta_2 < 1 \).
From (3.4) and the fact that there exists at least one value of \( \alpha \) with (3.11) it follows then that
\[
V(0) < V(\beta_1) < V(\beta_2) < V(1)
\]
for each pair of values \( (\beta_1, \beta_2) \) with \( 0 < \beta_1 < \beta_2 < 1 \).

Lemma II: If \( C \) is a closed convex subdomain of \( \Gamma \), not containing the point \( (y^*_{1}, y^*_{2}, ..., y^*_{K}) \), then \( \Phi(y_1, y_2, ..., y_K) \) attains its maximum in \( C \) only in one or more points on its boundary.

Proof: Consider any inner point \( y^*_{1}, y^*_{2}, ..., y^*_{K} \) of \( C \) and let \( Y(\beta) \) be defined by (3.6) \( (\alpha = 1, 2, ..., K) \). Then, \( C \) being a closed convex domain not containing the point \( (y^*_{1}, y^*_{2}, ..., y^*_{K}) \) there exists a value of \( \beta \) in the interval \( 0 < \beta < 1 \), say \( \beta_0 \), such that \( \{Y(\beta_0), Y_2(\beta_0), ..., Y_K(\beta_0)\} \) is a border point of \( C \). Further it follows from Lemma I that
\[
\Phi(Y_1(\beta_0), Y_2(\beta_0), ..., Y_K(\beta_0)) > \Phi(y^*_{1}, y^*_{2}, ..., y^*_{K}).
\]
Thus for each inner point \( (y^*_{1}, y^*_{2}, ..., y^*_{K}) \) of \( C \) there exists a border point \( (Y_1, Y_2, ..., Y_K) \) of \( C \) with a larger value of \( \Phi \). Moreover \( \Phi \) is bounded in \( C \), because the point \( (y^*_{1}, y^*_{2}, ..., y^*_{K}) \) is not contained in \( C \). Thus \( \Phi \) has a maximum in \( C \), which can evidently only be attained in border points.

4. The maximum likelihood estimates of \( \theta_1, \theta_2, ..., \theta_K \)

Let \( M \) be a subset of the numbers \( 1, 2, ..., k \); let further
\[
I_M = \bigcap_{i \in M} I_i
\]
and if $I_M \neq 0$

(4.2) \[ L_M(z) \overset{\text{def}}{=} \sum_{i \in M} L_i(z) \quad z \in I_M. \]

Throughout this paper it will be supposed that the following condition is satisfied.

(4.3) **Condition:** For each $M$ with $I_M \neq 0$ the function $L_M(z)$ is strictly unimodal in the interval $I_M$.

Now let $M_v$ ($v = 1, 2, \ldots, N$) be subsets of the numbers $1, 2, \ldots, k$ with

\[
\begin{aligned}
1. & \quad \bigcup_{v=1}^{N} M_v = \{1, 2, \ldots, k\}, \\
2. & \quad M_{v_1} \cap M_{v_2} = 0 \text{ for each pair of values } v_1, v_2 = 1, 2, \ldots, N \quad \text{with } v_1 \neq v_2, \\
3. & \quad I_{M_v} \neq 0 \text{ for each } v = 1, 2, \ldots, N,
\end{aligned}
\]

where

(4.5) \[ I_{M_v} \overset{\text{def}}{=} \bigcap_{i \in M_v} I_i \quad (v = 1, 2, \ldots, N). \]

Let further (cf. (2.6))

(4.6) \[ G_N \overset{\text{def}}{=} \prod_{v=1}^{N} I_{M_v} \]

and

(4.7) \[ L_{M_v}(z_v) \overset{\text{def}}{=} \sum_{i \in M_v} L_i(z_v) \quad z_v \in I_{M_v} (v = 1, 2, \ldots, N). \]

Then for all points in $G_N L(y_1, y_2, \ldots, y_k)$ reduces to a function of $N$ variables $z_1, z_2, \ldots, z_N$; we denote this function by $L'(z_1, z_2, \ldots, z_N)$ and thus have

(4.8) \[ L'(z_1, z_2, \ldots, z_N) = \sum_{v=1}^{N} L_{M_v}(z_v), \]

which is according to (4.3), a sum of strictly unimodal functions.

**Theorem I:** $L$ possesses a unique maximum in $D$.

**Proof:** This theorem will be proved by induction.

Let $M_1, M_2, \ldots, M_N$ be an arbitrary set of subsets of the numbers $1, 2, \ldots, k$ satisfying (4.4) and let

(4.9) \[ D_{N,s} \overset{\text{def}}{=} D \cap G_N, \]

where $s$ denotes the number of essential restrictions defining $D$ and where $G_N$ is defined by (4.6). Then $D_{N,s}$ is convex and:

- for $N=k$ we have $I_{M_v} = I_v (v = 1, 2, \ldots, N)$, thus $G_k = G$ and $D_{k,s} = D$ for $s=0$ we have $D = G$ thus $D_{N,0} = G_N$. 

We shall say that the function \( L'(z_1, z_2, \ldots, z_N) \) can be monotonously traced to its maximum in \( D_{N,s} \) if

\[
\begin{align*}
1. & \quad L'(z_1, z_2, \ldots, z_N) \text{ possesses a unique maximum in } D_{N,s}, \\
2. & \quad \text{every point of } D_{N,s} \text{ can be connected with the point in } D_{N,s} \text{ where } L' \text{ assumes its maximum by means of a broken line, consisting of a finite number of segments, in } D_{N,s} \text{ such that } L' \text{ increases monotonously along this line. (Such a line will be called a trace.)}
\end{align*}
\]

(4.10)

For \( s = 0 \) \( L'(z_1, z_2, \ldots, z_N) \) has this property for every set \( M_1, M_2, \ldots, M_N \) satisfying (4.4) and every \( N \). This follows from the fact that \( L' \) is the sum of strictly unimodal functions and that \( D_{N,0} \) is the Cartesian product of the intervals \( I_{M_v} (v = 1, 2, \ldots, N) \) so that lemma I may be applied.

Let us now suppose that it has been proved that \( L' \) can be monotonously traced to its maximum for all values of \( s \leq s_0 \) for every set \( M_1, M_2, \ldots, M_N \) satisfying (4.4) and for every \( N \). We then prove that the same holds for \( s_0 + 1 \) essential restrictions.

Consider, for a given set \( M_1, M_2, \ldots, M_N \), satisfying (4.4), a domain \( D_{N, s_{n+1}} \) and the domain \( D_{N, s_n} \), which is obtained by omitting one of the essential restrictions defining \( D_{N, s_{n+1}} \). Let this be the restriction \( R_1 \):

\[
\theta_{ij} \leq \theta_{ij}.
\]

Then clearly

\[
D_{N, s_{n+1}} \subset D_{N, s_n}.
\]

(4.11)

Now \( L' \) has a unique maximum in \( D_{N, s_n} \), attained in (say) the point \( T \). We first consider the case that \( T \) is outside \( D_{N, s_{n+1}} \). Then an arbitrary point \( P \) of \( D_{N, s_{n+1}} \) with \( z_{i_1} < z_{i_2} \) can be connected with \( T \) by means of a trace in \( D_{N, s_n} \) and this trace must contain at least one border point of \( D_{N, s_{n+1}} \) with \( z_{i_2} = z_{i_1} \), because within \( D_{N, s_{n+1}} \) we have: \( z_{i_1} < z_{i_2} \) and outside \( D_{N, s_{n+1}} \).

The first of these points when following the trace will be denoted by \( U \); then \( L' \) assumes a larger value in \( U \) than in \( P \). Now \( U \) lies in a domain \( D_{N', s_{n'}} \), where \( N' = N - 1 \) and \( s_{n'} \leq s_{n} \) and \( L' \) can thus monotonously be traced from \( U \) to its unique maximum in \( D_{N', s_{n'}} \) by means of a trace within \( D_{N', s_{n'}} \). The trace from \( P \) to \( U \) in \( D_{N, s_{n+1}} \) and from \( U \) to the maximum of \( L' \) in \( D_{N} \) together form a trace from \( P \) to the maximum of \( L' \) in \( D_{N, s_{n+1}} \).

Consider next the case where \( T \) is a point of \( D_{N, s_{n+1}} \). Then \( L' \) attains a unique maximum in \( D_{N, s_{n+1}} \) in \( T \). If the maximum of \( L' \) in \( G_N \) is attained in this point \( T \) then, according to Lemma I, \( L' \) can be monotonously traced to its maximum from every point of \( D_{N, s_{n+1}} \) by means of a straight line, connecting this point with \( T \). If \( T \) is not the point where \( L' \) assumes its maximum in \( G_N \) then it follows from Lemma II that \( T \) is a border point of \( D_{N, s_{n+1}} \) where at least two \( z \) from \( z_1, z_2, \ldots, z_N \) corresponding to an essential restriction for \( D_{N, s_{n+1}} \) are equal. Let this pair be

\[
z_{i_1} = z_{i_2}.
\]

(4.12)
then we consider the domain $D_{N,s}$ which is obtained from $D_{N,s+1}$ by omitting the restriction $R_i$: $\theta_{i} < \theta_{j}$ from the essential restrictions defining $D_{N,s+1}$. The maximum of $L'$ in $D_{N,s}$ then exists and the point where it is attained is a point of $D_{N,s}$ with $z_{k} \geq z_{j}$. The rest of the proof for this case is then the same as for the first case considered. Thus $L'$ can be monotonously traced to its maximum in every $D_{N,s}$; thus, taking $N = k$, $L$ can be monotonously traced to its maximum in $D$.

Remark 2: For $s = 0$ and $N = k$ we have $D_{N,s} = G$. Thus $L$ attains a unique maximum in $G$ in a point which will be denoted by $v_1, v_2, ..., v_k$.

Theorem II: If $t'_1, t'_2, ..., t'_k$ are the values of $y_1, y_2, ..., y_k$ which maximize $L$ in $G$ and under the restrictions $R_1, ..., R_{k-1}, R_{k+1}, ..., R_s$ then
\[
\begin{align*}
(4.13) & \\
& \begin{cases} 
1. & t_i = t'_i \quad (i = 1, 2, ..., k) \quad \text{if} \quad t'_i \leq t'_k, \\
2. & t_i = t'_k \quad \text{if} \quad t'_i > t'_k.
\end{cases}
\end{align*}
\]

Proof: The $R_i$ have not been arranged in a special order, thus we may take without any loss of generality $\lambda = s$. First consider the case that $t'_i \leq t'_k$; then $t'_1, t'_2, ..., t'_k$ satisfy all restrictions $R_1, R_2, ..., R_s$; thus in this case we have
\[
(4.14)\quad t_i = t'_i \quad (i = 1, 2, ..., k).
\]

If $t'_i > t'_k$ then (4.13.2) may be proved as follows. The domain defined by the essential restrictions $R_1, R_2, ..., R_{s-1}$ will be denoted by $D'$. Then for each point $(y_1, y_2, ..., y_k)$ in $D$ with $y'_k < y'_k$, there exists a trace in $D'$ from the point $(y_1, y_2, ..., y_k)$ to the point $(t'_1, t'_2, ..., t'_k)$ and this trace contains a point $(y'_1, y'_2, ..., y'_k)$ with
\[
(4.15)\quad \begin{cases} 
1. & y'_k = y'_k, \\
2. & L(y'_1, y'_2, ..., y'_k) > L(y_1, y_2, ..., y_k).
\end{cases}
\]

Thus, if $t'_i > t'_k$, then $L(y_1, y_2, ..., y_k)$ attains its maximum in $D$ for $y_i = y'_i$; (4.13.2) then follows from the uniqueness of this maximum.

Remark 3:

If
\[
(4.16)\quad P[x_i = 1] = \theta_i, \quad P[x_i = 0] = 1 - \theta_i \quad (i = 1, ..., k)
\]
and
\[
(4.17)\quad a_i \overset{\text{def}}{=} \sum_{\gamma = 1}^{m} x_{i,\gamma}, \quad b_i \overset{\text{def}}{=} n_i - a_i \quad (i = 1, 2, ..., k)
\]
then
\[
(4.18)\quad L(y_1, y_2, ..., y_k) = \sum_{i=1}^{k} \{a_i \ln y_i + b_i \ln (1 - y_i)\}.
\]

In [2] it has been proved that, if $I_i$ is the interval $(0,1)$, this function $L$ satisfies the following condition.
(4.19) Condition: If \( (y_1, y_2, \ldots, y_k) \) and \( (y'_1, y'_2, \ldots, y'_k) \) are any two points in \( G \) with \( y_i \neq y'_i \) for at least one value of \( i \) and if
\[
Y_i(\beta) = (1 - \beta) y_i + \beta y'_i \quad (i = 1, 2, \ldots, k),
\]
then \( L\{Y_1(\beta), Y_2(\beta), \ldots, Y_k(\beta)\} \) is a strictly unimodal function of \( \beta \) in the interval \( 0 \leq \beta \leq 1 \).

This condition is stronger than condition (4.3) and the theorems I and II of this paper have been proved in [2] by using condition (4.19).

Further if condition (4.19) is satisfied then theorem I of this paper may be proved in a more simple way then we did in [2] as follows. Consider any two points \( (y_1, y_2, \ldots, y_k) \) and \( (y'_1, y'_2, \ldots, y'_k) \) in \( D \) with \( y_i \neq y'_i \) for at least one value of \( i \) and
\[
L(y_1, y_2, \ldots, y_k) = L(y'_1, y'_2, \ldots, y'_k).
\]
Then it follows from condition (4.19) that there exists a point \( (Y_1, Y_2, \ldots, Y_k) \) in \( D \) with
\[
L(Y_1, Y_2, \ldots, Y_k) > L(y_1, y_2, \ldots, y_k).
\]
Thus \( L \) possesses a unique maximum in \( D \).

The maximum likelihood estimates of \( \theta_1, \theta_2, \ldots, \theta_k \) may always be found by applying theorem II repeatedly. This follows from the fact that \( L'(z_1, z_2, \ldots, z_N) \) is a sum of strictly unimodal functions and that \( D_{N,s} \) is a convex subdomain of the Cartesian product of the intervals \( I_{M_r} \) \( (r = 1, 2, \ldots, N) \) for each set \( M_1, M_2, \ldots, M_N \) and each \( N \).

This leads however to a rather complicated procedure which may often be simplified by using one of the theorems of the following section.

5. Some special theorems

The theorems III–VI in this section may be proved in precisely the same way as the theorems II–V in [2].

Theorem III: If \( \alpha_{i,j}(v_i - v_j) \leq 0 \) for each pair of values \( (i, j) \) then
\[
t_i = v_i \quad (i = 1, 2, \ldots, k).
\]

Theorem IV: If \( l_1, l_2, \ldots, l_m \) is a set of values satisfying
\[
\alpha_{i,l_i} = \alpha_{i,l_1} = \ldots = \alpha_{i,l_m} = 0 \quad \text{for each } i \neq l_1, l_2, \ldots, l_m
\]
then the maximum likelihood estimates of \( \theta_{l_1}, \theta_{l_2}, \ldots, \theta_{l_m} \) are the values of \( y_{l_1}, y_{l_2}, \ldots, y_{l_m} \) which maximize \( L_{l_1} + L_{l_2} + \ldots + L_{l_m} \) in the domain
\[
D \bigg\{ \alpha_{i,j}(y_i - y_j) \leq 0 \bigg\} \quad (i, j = l_1, l_2, \ldots, l_m).
\]

Theorem V: If for some pair of values \( (i, j) \) with \( i < j \)
\[
\alpha_{i,j}(v_i - v_j) > 0
\]

(5.4)
and

\begin{align*}
1. \quad & \alpha_{i,h} = \alpha_{h,i} = 0 \text{ for each } h \text{ between } i \text{ and } j, \\
2. \quad & \alpha_{h,i} = \alpha_{h,j} \text{ for each } h < i, \\
3. \quad & \alpha_{i,h} = \alpha_{j,h} \text{ for each } h > j,
\end{align*}

then

\begin{equation}
\begin{aligned}
& t_i = t_j.
\end{aligned}
\end{equation}

Theorem VI: If \((i, j)\) is a pair of values satisfying

\begin{equation}
v_i \leq v_j
\end{equation}

and

\begin{align*}
1. \quad & \alpha_{i,j} = 0, \\
2. \quad & \alpha_{h,i} \leq \alpha_{h,j} \text{ for each } h < i, \\
3. \quad & \alpha_{i,h} \geq \alpha_{j,h} \text{ for each } h > j,
\end{align*}

then

\begin{equation}
\begin{aligned}
& t_i \leq t_j.
\end{aligned}
\end{equation}

Theorem VII: If \((i, j)\) is a pair of values with

\begin{equation}
\begin{aligned}
& \alpha_{i,j} = 0,
\end{aligned}
\end{equation}

if \(D'\) is the subdomain of \(D\) where \(y_i \leq y_j\) and if \((t'_1, t'_2, \ldots, t'_k)\) is the point where \(L\) assume its maximum in \(D'\) then

\begin{equation}
\begin{aligned}
1. \quad & t_1 = t'_1, t_2 = t'_2, \ldots, t_k = t'_k \text{ if } t'_i < t_j, \\
2. \quad & t_i \geq t_j \text{ if } t'_i = t'_j.
\end{aligned}
\end{equation}

Proof: The proof of this theorem differs from the one given for theorem VI in [2] only in the form of the trace from a point in \(D'\) to the maximum of \(L\) in \(D\). This trace which is a straight line in [2], need not be straight now (cf. the proof of theorem II of the present paper).

(To be continued)

REFERENCES