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Finite automata and ordinals

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Abstract

Several definitions of automata on words indexed by ordinals have been proposed previously. The first one was introduced by Büchi to prove the decidability of the monadic second order theory of denumerable ordinals. Wojciechowski studied the properties of these automata independently of the length of the input. The second definition, proposed by Choueka, works only on words of length less than ω^n. In this paper, we restrict the domain of Wojciechowski automata to the domain of Choueka's ones (that is, given $n < \omega$, we keep only α-sequences for $\alpha < \omega^{n+1}$ in the language defined by a Wojciechowski automaton) in order to prove the equivalence between Choueka automata and Wojciechowski automata. Then, we obtain the closure under complementation of the class of Wojciechowski's definable sets, and finally we give an algorithm for determinizing Wojciechowski automata.

1. Introduction

Finite automata on ω-sequences were first introduced by Büchi [1] to prove the decidability of the second order monadic logic of integers. A Büchi automaton looks like an ordinary one (that is, like a Kleene automaton), but in this case a word is said to be accepted iff the set of states that appears infinitely often in a run of the automaton on the word contains at least a final state.

Independently of Büchi, Muller [7] used automata on infinite words to study the behavior of asynchronous circuits. A word is accepted by an automaton iff the set of states that appears infinitely often in the run of the automaton on the word belongs to a table associated with the automata. Muller automata are deterministic.

McNaughton proved in [6] the equivalence between Muller automata and Büchi's ones acting on infinite words.

Büchi, in [2], generalized his idea to transfinite sequences. Automata acting on transfinite words have two maps for transitions: one for successor ordinals (this is the

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map used in usual automata) and a second one for limit ordinals (to get through infinity: the state reached for a limit ordinal depends only of states reached before). For ξ a limit ordinal, if φ is the sequence of states denoting the run of the automaton on a transfinite sequence, Büchi defined $\varphi(\xi)$ to depend uniquely on $\{s \in \mathscr{S} / \forall \beta < \xi \; \exists \gamma \; \xi > \gamma > \beta \; \varphi(\gamma) = s\}$. He used these automata to prove the decidability of the monadic second order theory of $[\alpha, <]$ for α a countable ordinal, but could not use them for uncountable ordinals (he proved the decidability of the monadic second order theory of $[\omega_2, <]$ using an other kind of automata).

Choueka [4] generalized automata on infinite sequences to transfinite sequences of length less than ω^n for a given $n < \omega$. The main difference between his idea and Büchi's idea [2] is the second map (call it f). For ξ a limit ordinal, $\xi = \beta + \omega^n$, $n < \omega$, $\beta = 0$ or $\beta \geqslant \omega^n$, p belongs to $f(\xi)$ iff there exists an infinity of $k < \omega$ such that $f(\beta + \omega^{n-1} \cdot k) = p$. Note that f is defined only for $n < \omega$, and that $f(\xi)$ depends only on values of $f(\alpha)$ for $\alpha < \xi$. Those automata are equivalent to regular expressions, looking like Büchi's ω-regular expressions, but with a free ω operator.

Wojciechowski [13] studied the behavior of Büchi's [2] automata, without a limit for the length of inputs. He gave in [14] regular expressions equivalent to his automata, being like Choueka's but with one more operator, $\#$, where $a^{\#}$ means the letter a repeated zero, a finite number, an infinite number or a transfinite number of times.

The main result of this paper is the equivalence between Choueka automata and Wojciechowski's ones (if of course the domain of Wojciechowski automata is restricted to Choueka's one). As the class of Choueka definable languages is closed under complementation (because Choueka's deterministic and nondeterministic automata are equivalent), then so is the class of Wojciechowski^{n} definable languages. In the proof, we use a construction for, given a Wojciechowski automaton, obtaining a Choueka automaton defining the same language. The proof of the equivalence is then immediate because the class of Choueka's regular expressions is include into the class of Wojciechowski's regular expressions. The other result is an algorithm of determinization for Wojciechowski automata. It is obtained using the previous construction, the equivalence between Choueka's deterministic and nondeterministic automata, and a second construction computing a deterministic Wojciechowski automaton given a deterministic Choueka automaton.

The paper is not self-contained: the reader is supposed to be familiar with the theory of ordinals and with the classical automata theory (on finite words). One can find the classical theory of ordinals in [11] ([10] can be seen as a translation in french of pieces of [11] and a more modern approach in [9]. All the traditional automata theory on finite words is in [5]. Finally, for the theory of automata on infinite words we refer to [3, 8, 12].

The paper is composed of three parts:

- In the first three sections we present an unified notation for definitions and results of Choueka and Wojciechowski.

- The second one is the proof of the equivalence of Choueka automata and Wojciechowski's ones.
- The third one is the algorithm of determinization of Wojciechowski automata.

2. Notation and basic definitions

We refer the reader to [9–11] for the theory of ordinals.

All along this paper the letters i, j, k, m, and n are integers ($\in \mathbb{N}$). Greek letters ($\alpha, \beta, \gamma \ldots$) are ordinals. The ξ letter is always a limit ordinal. Finally, the φ letter denotes a function.

We denote by *Succ* the class of successor ordinals, *Lim* the class of limit ordinals and $Ord = Succ \cup Lim \cup \{0\}$. Let α be an ordinal less than ω^ω, and $\omega^m \cdot a_m + \omega^{m-1} \cdot a_{m-1} + \cdots + \omega^0 \cdot a_0$, where $a_1, \ldots, a_m$ and m are ordinals less than ω, be the normal form of α. For short, we will write $\alpha = \sum_{i=m}^{0} \omega^i . a_i$. When we write $\alpha = \beta + \omega^\gamma$ we are talking about the unique decomposition of α, where $\beta \geqslant \omega^\gamma$ or $\beta = 0$. Thus, the *type* of α is the ordinal γ.

Let $\mathscr{S}$ be a finite set. An *α-sequence* $a_0 a_1 a_2 \ldots$ on $\mathscr{S}$ (i.e. $\forall \beta < \alpha\ a_\beta \in \mathscr{S}$) is a function $\varphi : \alpha \to \mathscr{S}$ such that $\forall \beta < \alpha\ \varphi(\beta) = a_\beta$. The *length* of an α-sequence φ, denoted by $|\varphi|$, is the ordinal α. For practical reasons, ${}_\beta\varphi$ stands for $\varphi(\beta)$. The sequence itself is denoted by $({}_\beta\varphi)_{\beta<\alpha}$, and $\varphi|_\gamma$ the restriction of φ to $\gamma < \alpha$. $\mathscr{S}^\#$ is the class of all sequences, transfinites, infinites or finites, on $\mathscr{S}$, $\mathscr{S}^\alpha$ the class of all α-sequences on $\mathscr{S}$ ($\alpha \in Ord$), and $\mathscr{S}^{<\alpha}$ the class of all sequences on $\mathscr{S}$ of length less than α ($\alpha \in Ord$). If φ is an ω-sequence on $\mathscr{S}$. $\mathrm{In}(\varphi)$ will denote the set of elements of $\mathscr{S}$ appearing infinitely often in $({}_\beta\varphi)_{\beta<\alpha}$.

A word of length α is an α-sequence on a finite set, usually denoted by Σ, called *alphabet*. Each element of an alphabet is a *letter*. The word of length 0, also called the *empty word*, is denoted by λ.

Let φ be an α-sequence and $\beta, \gamma \in Ord$ such that $\beta < \gamma \leqslant \alpha$. We denote by $\varphi[\beta, \gamma[$ the $(\gamma - \beta)$-sequence such that $\varphi[\beta, \gamma[(\delta) = \varphi(\beta + \delta)$ for all $\delta < \gamma - \beta$.

Let u be an α-sequence on a finite set Σ_u and v be a β-sequence on a finite set Σ_v. The *product* of u and v, defined $u \cdot v$, is the $(\alpha + \beta)$-sequence w on $\Sigma_u \cup \Sigma_v$ such that:

$$
{}_\gamma w = \begin{cases} {}_\gamma u & \text{si } 0 \leqslant \gamma < \alpha, \\ {}_{\gamma-\alpha} v & \text{si } \alpha \leqslant \gamma < \alpha + \beta. \end{cases}
$$

Given $(u_i)_{i<n}$ n words on a finite alphabet Σ we recursively define Π, the *generalized product*:

$$
\prod_{i=k}^{n} u_i = \begin{cases} u_k \cdot \prod_{i=k+1}^{n} u_i & \text{if } k \leqslant n, \\ \lambda & \text{otherwise.} \end{cases}
$$

A *k-regular expression* is a regular expression describing a set of words of finite length. An *ω-regular expression* is a regular expression describing a set of words of infinite length. An *α-regular expression* is a regular expression describing a set of words of finite, infinite and transfinite length.

Let $\mathcal{S}$ be a finite set. We will denote $[\mathcal{S}]^1$ or $[\mathcal{S}]$ the powerset of $\mathcal{S}$ without the empty set $\emptyset$. $[\mathcal{S}]^0$ stands for $\mathcal{S}$, $[\mathcal{S}]^{n+1}$ for $[[\mathcal{S}]^n]$, and $[\mathcal{S}]_0^n$ for $\bigcup_{i=0}^n [\mathcal{S}]^i$. Extending this notation, if $s \in \mathcal{S}$, $\{s\}^0$ stands for s and $\{s\}^{n+1} = \{\{s\}^n\}$. We remark that $\{s\}^n \in [\mathcal{S}]^n$.

If $X \in [\mathcal{S}]^m$, the *type* of X, $t(X)$, is the integer m.

Let $\mathcal{S}, \Sigma$ be two sets, $n, i \in \mathbb{N}$, $i \leqslant n+1$ and $\mathcal{M} : [\mathcal{S}]_0^n \times \Sigma \to \mathcal{S}$ be a function. We call the *restriction* of $\mathcal{M}$ to i, denoted by $\mathcal{M}|_i$, the function $\mathcal{M}|_i : [\mathcal{S}]_0^{i-1} \times \Sigma \to \mathcal{S}$ defined by $\forall q \in [\mathcal{S}]_0^{i-1} \forall \sigma \in \Sigma\, \mathcal{M}|_i(q, \sigma) = \mathcal{M}(q, \sigma)$. If $\mathcal{M}$ is a relation, $\mathcal{M}|_i$ is a relation too, defined by $(q \in [\mathcal{S}]_0^{i-1}, \sigma, p) \in \mathcal{M}|_i \Leftrightarrow (q, \sigma, p) \in \mathcal{M}$.

$\mathcal{L}(\mathcal{A})$ denote the set of words accepted by the automaton $\mathcal{A}$.

Let $\mathcal{S}$ and $\mathcal{S}'$ be two sets. A function from $\mathcal{S}$ to $\mathcal{S}'$ is also called a *projection.* A projection $p : \Sigma \to \Sigma'$ can naturally be extended to words: if $u, v \in \Sigma^\#$, $p(u \cdot v) = p(u) \cdot p(v)$, and to sets of words: if $\mathcal{E} \subseteq \Sigma^\#$, $p(\mathcal{E}) = \{p(x)/x \in \mathcal{E}\}$.

Let $m, n \in \mathbb{N}$, m sets $\mathcal{S}_1, \ldots, \mathcal{S}_m$, and $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_m$. We recursively define m projections $\mathrm{p}_i : [\mathcal{S}]^j \to [\mathcal{S}_i]^j$ $(1 \leqslant i \leqslant m, j \in \mathbb{N})$:

- If $(q_1, \ldots, q_m) \in \mathcal{S}$ then $\mathrm{p}_i((q_1, \ldots, q_m)) = q_i$.
- If $\{q_1, \ldots, q_k\} \in [\mathcal{S}]_1^n$ then $\mathrm{p}_i(\{q_1, \ldots, q_k\}) = \{\mathrm{p}_i(q_1), \ldots, \mathrm{p}_i(q_k)\}$.

3. Automata

We refer the reader to [5] for the classical automata theory on finite words, and to [3, 8, 12] for the automata theory on infinite words.

When we talk about *automata,* we mean *finite automata.* Infinite automata are not the subject of this paper. We will use the terms *$\mathcal{C}$-automaton* and *n-automaton* to point out a Choueka automaton, and *$\mathcal{W}$-automaton* for a Wojciechowski automaton.

3.1. $\mathcal{C}$-automata

Choueka automata are a generalization of Muller's ones. The behavior of both is the same for infinite (ω) words. Let us see how it works on $(\omega + 1)$-sequences: let $({}_\beta\varphi)_{\beta<\omega}$ be the run of a deterministic Choueka automaton on the ω first letters of the word (that is, $\varphi(0)$ is the initial state, $\varphi(1)$ the state reached from $\varphi(0)$ by the first letter of the word, ..., $\varphi(n)$ the state reached from $\varphi(n-1)$ by the nth letter of the word, ...). $\varphi(\omega)$ is the set of states appearing infinitely often in $({}_\beta\varphi)_{\beta<\omega}$. Considering $\varphi(\omega)$ as a state, $\varphi(\omega+1)$ if the state reached from $\varphi(\omega)$ by the ωth letter of the word. Using this idea, a run on an $\omega \cdot 2$-sequence looks like two glued runs on ω-sequences: the first one begins at the initial state of the automaton, and the beginning of the second one depends on the end of the first.

For ω^2-sequences, $\varphi(\omega^2)$ is the part of the powerset of the set of states such that $x \in \varphi(\omega^2)$ iff x appears infinitely often in $({}_{\omega\cdot k}\varphi)_{k<\omega}$, and so on. An α-sequence is said

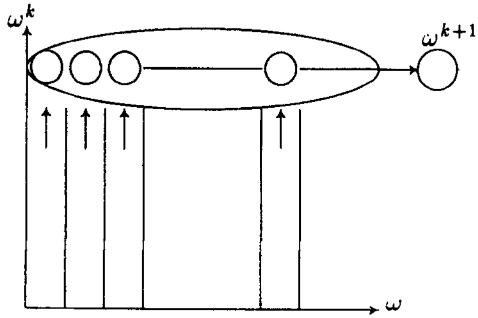


Fig. 1. The way $\mathscr{C}$-automata work.

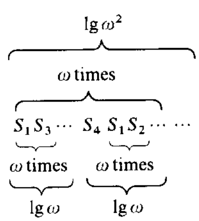
to be *accepted* by the automaton if $\varphi(\alpha)$ was defined to be a final state. We remark that the transition function is from $[\mathscr{S}]_0^n \times \Sigma \to \mathscr{S}$, where $\mathscr{S}$ is the set of states, Σ is a finite alphabet, and n is an integer, so the definition of a $\mathscr{C}$-automaton depends on the integer n, that's why we call such an automaton n-automaton. Of course it is impossible to define $\Sigma^{<\omega^\omega}$ using a $\mathscr{C}$-automaton, because the integer n forces the length of the longest word accepted by the automaton to be lower than ω^{n+1}. Fig. 1 sums up how a $\mathscr{C}$-automaton works on an ω^{k+1}-sequence:

The following is the formalization of the foregoing.

Definition 1. An α-sequence φ on $\mathscr{S}$ is said to be *continuous* if $\forall \beta \in Succ\ \varphi(\beta) \in \mathscr{S}$ and $\forall \beta \in Lim\ \varphi(\beta) = \mathrm{In}(\psi)$ where ψ is the unique ω-sequence such that for all $0 \leqslant i < \omega$ $\psi(i) = \varphi(\gamma + \omega^{n-1} \cdot i)$ if $\beta = \gamma + \omega^n$. A continuous α-sequence is then entirely defined by its nonlimit values.

Example 2.

- Let φ be a continuous $(\omega + 1)$-sequence on a set $\mathscr{S}$. $\varphi(\omega)$ is the set of elements of $\mathscr{S}$ appearing infinitely often in $({}_\beta\varphi)_{\beta<\omega}$.
- Defining the following $(\omega^2 + 1)$-sequence on $\mathscr{S} = \{s_1, s_2, s_3, s_4, s_5\}$:



we have $\varphi(\omega) = \{s_1, s_3\}$, $\varphi(\omega \cdot 2) = \{s_1, s_2\}$, $\varphi(\omega \cdot 3) = \{s_1, s_3\}$, $\varphi(\omega \cdot 4) = \{s_1, s_2\}$, $\ldots$, $\varphi(\omega^2) = \{\{s_1, s_2\}, \{s_1, s_3\}\}$.

Fig. 2. A Büchi's definable word.

Definition 3. A *n-automaton* $\mathcal{A}$ is a 5-uplet $\langle \mathcal{S}, \mathcal{M}, s^\star, \mathcal{F}, \Sigma \rangle$ with:

- $\mathcal{S}$ the finite set of states,
- $\mathcal{M} \subseteq |\mathcal{S}|_0^n \times \Sigma \times \mathcal{S}$ the relation[1] for transitions,
- $s^\star \in \mathcal{S}$ the initial state,
- $\mathcal{F} \subseteq [\mathcal{S}]_0^n$ the set of final "states" (from now on, we will call, for evident reasons, *state* any element of $[\mathcal{S}]_0^n$, but *set of states* still means $\mathcal{S}$),
- Σ a finite alphabet.

$\mathcal{M}:[\mathcal{S}]_0^n \times \Sigma \times \mathcal{S}$ can naturally be extended from letters to words defining $\mathcal{M}_e:[\mathcal{S}]_0^n \times \Sigma^{<\omega^{n+1}} \times [\mathcal{S}]_0^n$: if u is an α-sequence ($\alpha < \omega^{n+1}$) on Σ, $(s, u, s_f) \in \mathcal{M}_e$ iff there exists a continuous $(\alpha + 1)$-sequence φ on $\mathcal{S}$ such that $\varphi(0) = s$, $\varphi(\alpha) = s_f$ and $\forall \beta < \alpha, (\varphi(\beta), u(\beta), \varphi(\beta + 1)) \in \mathcal{M}$.

A word u is said to be accepted by $\mathcal{A}$ iff there exists $s_f \in \mathcal{F}$ such that $(s^*, u, s_f) \in \mathcal{M}_e$.

One can easily complete a $\mathcal{C}$-automaton. In the sequel, when we talk about $\mathcal{C}$-automata, we will mean *complete* $\mathcal{C}$-automata.

Definition 4. A subset A of $\mathcal{S}^\#$ is called *n-definable* (*$\mathcal{C}$-definable*) iff there exists an n-automaton ($\mathcal{C}$-automaton) $\mathcal{A}$ such that $\mathcal{L}(\mathcal{A}) = \mathrm{A}$.

Definition 5. A *run* of an n-automaton $\mathcal{A} = \langle \mathcal{S}, \mathcal{M}, s^\star, \mathcal{F}, \Sigma \rangle$ on a word $u \in \Sigma^\alpha$, $\alpha < \omega^{n+1}$, is a continuous $(\alpha + 1)$-sequence φ such that $\varphi(0) = s^\star$ and $\forall \beta < \alpha(\varphi(\beta), u(\beta), \varphi(\beta + 1)) \in \mathcal{M}$. If $\mathcal{A}$ is a deterministic automaton, then this sequence is unique. A run is said to be *accepting* if $\varphi(\alpha) \in \mathcal{F}$.

3.1.1. Characterization of $\mathcal{C}$-definable words

Unlike finite words, infinite words are not always definable by a finite automaton: an ω-sequence x on Σ is definable by a Büchi automaton iff it is ultimately periodic, that is, there exists $y, z \in \Sigma^*$ such that $x = y \cdot z^\omega$. Graphically, if half-a-line represents the infinite word (Fig. 2).

Choueka automata generalize this result to $1/2^n$ Euclidian n-dimensional spaces with integral coordinates. We assume without loss of generality that $\Sigma = \{a, b\}$.

An *arithmetic progression* is a set $\{a + nb/n = 0, 1, 2, \ldots \text{ and } a, b \in \mathbb{N}\}$. The progression is said to be *proper* if $b \neq 0$. A Cartesian product of n arithmetic progressions of which k exactly are proper ones is called a *n-periodic set of order k*. A set is *ultimately n-periodic of order k* iff it is a finite union of n-periodic sets of order k ($0 \leqslant k \leqslant n$).

[1] In Choueka's original definition, automata are deterministic (that is, transitions are not defined by a relation, but by a function), but he proves in [4] the equivalence between his deterministic and nondeterministic automata.

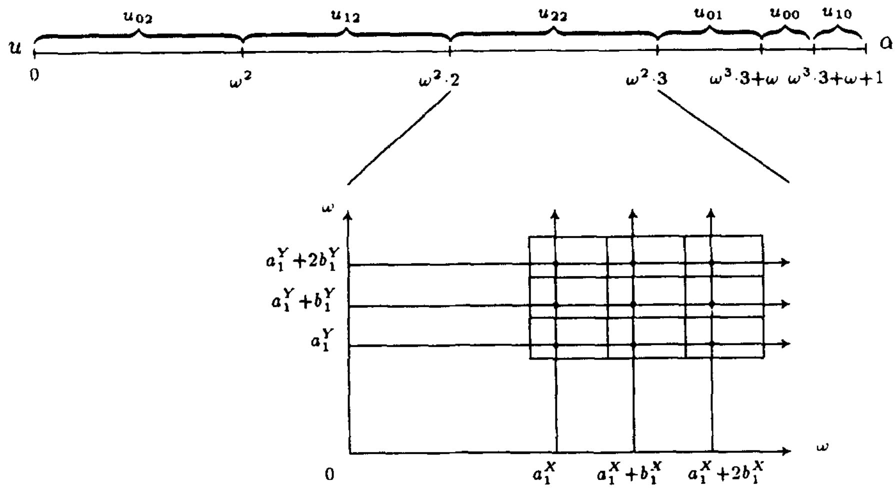


Fig. 3. A $\mathscr{C}$-definable word.

Given $u \in \Sigma^\alpha$, $\alpha \in Ord$, $a \in \Sigma$, we define $P_a(u)$ to be $\{\beta < \alpha /_\beta u = a\}$.

Let $\alpha = \sum_{i=m}^{0} \omega^i \cdot a_i$ be an ordinal less than ω^ω. By $Dec(\alpha)$ we mean the $(m+1)$-uplet $(a_0, \ldots, a_m)$. One can easily extend Dec to set of ordinals by $Dec_e\{\alpha_1, \ldots, \alpha_\beta\} = \{Dec(\alpha_1), \ldots, Dec(\alpha_\beta)\}$, $\beta \in Ord$.

Theorem 6 (Choueka [4]). *Let $\alpha = \sum_{i=m}^{0} \omega^i \cdot a_i$ and $u \in \{a, b\}^\alpha$. One can split u in factors using the following*:

$$u = \prod_{j=m}^{0} \prod_{i=0}^{a_j - 1} u\left[\left(\sum_{k=m}^{j+1} \omega^k \cdot a_k\right) + \omega^j \cdot i, \left(\sum_{k=m}^{j+1} \omega^k \cdot a_k\right) + \omega^j \cdot (i+1)\right[$$

$$\stackrel{\text{def}}{=} \prod_{j=m}^{0} \prod_{i=0}^{a_j - 1} u_{ij}.$$

We note that $|u_{ij}| = \omega^j$. $\{u\}$ is $\mathscr{C}$-definable iff $\forall u_{ij}\, Dec_e(P_a(u_{ij}))$ is an ultimately j-periodic set.

Example 7. $\alpha = \omega^2 \cdot 3 + \omega + 2$, $u \in \{a, b\}^\alpha$ (Fig. 3).

One can say that for each u_{ij} there exists a k-dimensional box ($k \leqslant j$) that can be translated infinitely often from its size in every k directions following axes of the $1/2^j$ Euclidian space with integral coordinates representing u_{ij}. In Fig. 3 one can see a unique 2-periodic set of order 2.

Example 8. $\alpha = \omega$, $u \in \{a, b\}^\omega$ (Fig. 4). This time, $Dec_e(P_a(u))$ is an union of two 1-periodic sets of order 1 and one 1-periodic set of order 0 (the biggest point does not appear anywhere else whereas the two others series of points are infinite).

Fig. 4. Dots represent the positions of a's, other letters are b's.

3.2. $\mathscr{W}$-automata

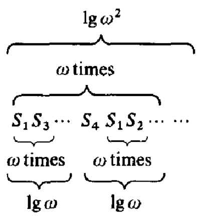
Wojciechowski's definition of automata is closer to ordinals than Choueka's. Unlike Choueka's, a $\mathscr{W}$-automaton can accept words of any length. The definition of the second transition function is based on the notion of cofinality of a set of ordinals with an ordinal.

Definition 9. Let $\xi \in Lim$ and $\mathscr{E}$ be a set of ordinals all lower than ξ. $\mathscr{E}$ is said to be *cofinal* with ξ iff $\forall \alpha < \xi\ \exists \beta \in \mathscr{E}\ \beta > \alpha$.

Now we can define the motion of continuous α-sequence for Wojciechowski's idea.

Definition 10. An α-sequence φ on $\mathscr{S}$ is said to be *continuous* if $\forall \beta \in Succ\ \varphi(\beta) \in \mathscr{S}$ and $\forall \beta \in Lim$, $\varphi(\beta) = \{s \in \mathscr{S} / \{\gamma < \beta / \varphi(\gamma) = s\}$ is cofinal with $\beta\}$. As for Choueka's definition of continuous α-sequences, a continuous α-sequence is entirely defined by its nonlimit values.

Example 11.
- Let φ be an $(\omega + 1)$-sequence on a set $\mathscr{S}$, $\varphi(\omega)$ is the set of elements of $\mathscr{S}$ appearing infinitely often in $({}_{\beta}\varphi)_{\beta<\omega}$.
- Defining the following $(\omega^2 + 1)$-sequence on $\mathscr{S} = \{s_1, s_2, s_3, s_4, s_5\}$:



we have $\varphi(\omega) = \{s_1, s_3\}$, $\varphi(\omega \cdot 2) = \{s_1, s_2\}$, $\varphi(\omega \cdot 3) = \{s_1, s_3\}$, $\varphi(\omega \cdot 4) = \{s_1, s_2\}$, $\ldots$, $\varphi(\omega^2) = \{s_1, s_2, s_3, s_4\}$.

Definition 12. A *$\mathscr{W}$-automaton* $\mathscr{A}$ is a 5-uplet $\langle \mathscr{S}, \mathscr{M}, s^\star, \mathscr{F}, \Sigma \rangle$ with:
- $\mathscr{S}$ the finite set of states,
- $\mathscr{M} \subseteq [\mathscr{S}] \cup \mathscr{S} \times \Sigma \times \mathscr{S}$ the transition relation.
- $s^\star \in \mathscr{S}$ the initial state,
- $\mathscr{F} \subseteq [\mathscr{S}] \cup \mathscr{S}$ the set of final "states" (from now on we will call, for evident reasons, *state* any element of $[\mathscr{S}] \cup \mathscr{S}$, but *set of states* still means $\mathscr{S}$),
- Σ is finite alphabet.

$\mathcal{M}: [\mathcal{S}] \cup \mathcal{S} \times \Sigma \times \mathcal{S}$ can naturally be extended from letters to words defining $\mathcal{M}_e: [\mathcal{S}] \cup \mathcal{S} \times \Sigma^{\#} \times [\mathcal{S}] \cup \mathcal{S}$: if u is an α-sequence on Σ, $(s, u, s_f) \in \mathcal{M}_e$ iff there exists a continuous $(\alpha + 1)$-sequence φ such that $\varphi(0) = s$, $\varphi(\alpha) = s_f$ and $\forall i < \alpha$, $(\varphi(i), u(i), \varphi(i + 1)) \in \mathcal{M}$.

The accepting condition of a word by a $\mathcal{W}$-automaton is the same as the one for Choueka automaton, the only difference being the definition of continuous sequences.

Definition 13. A subset A of $\mathcal{S}^{\#}$ is said to be *$\mathcal{W}$-definable* iff there exists a $\mathcal{W}$-automaton $\mathcal{A}$ such that $\mathcal{L}(\mathcal{A}) = $ A.

The definition of a run is the same as for Choueka automata.

The following few examples show how $\mathcal{W}$-automata work ($\Sigma = \{\sigma\}$). In all these Figs. 5–9 circles are shadowed when they represent states of the automaton not element of $\mathcal{S}$, but of $[\mathcal{S}]_1^n$.

Example 14.

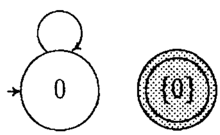


Fig. 5. A $\mathcal{W}$-automaton accepting σ^{ω}.

Example 15.

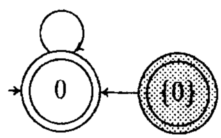


Fig. 6. A $\mathcal{W}$-automaton accepting $\sigma^{\#}$

One just have have to change the final states to accept only words of length $\alpha \in Succ \cup \{0\}$ or words of length $\alpha \in Lim$.

Example 16.

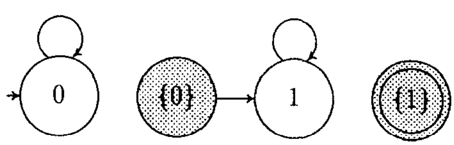


Fig. 7. A $\mathcal{W}$-automaton accepting $\sigma^{\omega \cdot 2}$.

Example 17.

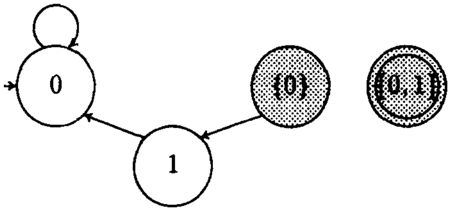


Fig. 8. A $\mathscr{W}$-automaton accepting σ^{ω^2}.

Example 18.

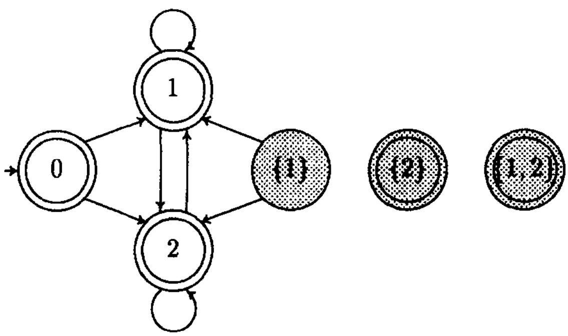


Fig. 9. A $\mathscr{W}$-automaton accepting all words of denumerable length.

Let u be an α-sequence on $\{\sigma\}$.

If $\alpha \in Succ$, the $(\alpha + 1)$-sequence φ defined by $\forall \beta \in Succ$, $\varphi(\beta) = 1$ is a run of the automaton on u, because for $\xi \in Lim$, $\xi < \alpha$ we have $\varphi(\xi) = \{1\}$, and $(1, \sigma, 1) \in \mathscr{M}$ and $(\{1\}, \sigma, 1) \in \mathscr{M}$.

If $\alpha \in Lim$, $\alpha < \omega_1$, α is the limit of an increasing progression of ordinals $(\beta_n)_{n<\omega}$. Let us assume, for each β_i of this progression, that $\varphi(\beta_i) = 2$ and $\varphi(\gamma) = 1$ for all others ordinals in $Succ$. At the end of u, the state reached is either $\{1, 2\}$ or $\{2\}$, which are both final states.

Now let us assume that $\alpha = \omega_1$ and u is accepted. Then, $2 \in \varphi(\omega_1)$. We define Γ to be $\{\gamma < \omega_1 / \varphi(\gamma) = 2\}$. Let γ_0 be the lowest element of Γ, γ_1 the element of Γ immediately greater than γ_0, and so on. As Γ is not a denumerable set, one can build an increasing progression of denumerable ordinals $(\gamma_i)_{i<\omega}$. Let β be the limit of this progression. As is well known, the limit of an increasing progression of denumerable ordinals is a denumerable ordinal, and ω_1 is not limit of such a progression, so $\beta < \omega_1$. As $\beta \in Lim$, $2 \in \varphi(\beta)$ and $(\varphi(\beta), \sigma, \varphi(\beta + 1))$ is not a transition, so u cannot be accepted.

One can verify that the shortest sequence accepted by a $\mathscr{W}$-automaton depends on the number of states of this automaton. Precisely, the length of this sequence is an ordinal belonging to $\{\alpha = \sum_{i=m}^{0} \omega^i \cdot a_i / \sum_{i=1}^{m} i \cdot a_i + a_0 \leqslant k\}$ where k is $|\mathscr{S}|$. We refer to [13] for the proof. It follows that the class of $\mathscr{W}$-definable sets is not closed under complementation and that deterministic and nondeterministic $\mathscr{W}$-automata are not equivalent.

4. α-regular expressions

In the sequel *$\mathscr{C}$-α-regular expression* stands for *Choueka's α-regular expression* and $\mathscr{W}$-α-regular expression for *Wojciechowski's α-regular expression.*

In this section, we sum up the proofs of the equivalence between regular expressions and automata for both ideas. Given $\mathscr{A}$ an automaton, $\mathscr{E}_A$ denotes the regular expression equivalent to $\mathscr{A}$.

4.1. $\mathscr{C}$-α-regular expressions

Definition 19. Let $\Sigma = \{\sigma_1, \ldots, \sigma_k\}$ be a finite alphabet. A *$\mathscr{C}$-α-regular expression* is a finite word on the alphabet $\Sigma \cup \{+, \cdot, {}^{\omega}, {}^{*}, (,), \lambda, \emptyset\}$ such that:
- $\emptyset$ is a $\mathscr{C}$-α-regular expression,
- λ is a $\mathscr{C}$-α-regular expression,
- $\sigma \in \Sigma$ is a $\mathscr{C}$-α-regular expression,
- if e_1 and e_2 are both $\mathscr{C}$-α-regular expressions, then so is $e_1 + e_2$,
- if e_1 and e_2 are both $\mathscr{C}$-α-regular expressions, then so is $e_1 \cdot e_2$,
- if e is a $\mathscr{C}$-α-regular expression, then so is (e),
- if e is a $\mathscr{C}$-α-regular expression, then so is e^*,
- if e is a $\mathscr{C}$-α-regular expression, then so is e^{ω}.

If e is a $\mathscr{C}$-α-regular expression, let $\bar{e}$ be the set of words denoted by e:
- $\bar{\emptyset} = \emptyset$,
- $\overline{e_1 + e_2} = \overline{e_1} \cup \overline{e_2}$,
- $\overline{e^*} = \bar{e}^*$,
- $\bar{\lambda} = \lambda$,
- $\overline{e_1 \cdot e_2} = \overline{e_1} \cdot \overline{e_2}$,
- $\bar{e} = \bar{e}^{\omega}$,
- $\Sigma \ni \bar{\sigma} = \sigma$,
- $\overline{(e)} = \bar{e}$.

All the operators have their usual meaning, but unlike in Büchi's ω-regular expressions, the ω operator is free (as the * operator) in $\mathscr{C}$-α-regular expressions.

4.1.1. *From $\mathscr{C}$-α-regular expressions to deterministic $\mathscr{C}$-automata*

In his original article, Choueka gave a method to pass from a $\mathscr{C}$-α-regular expression to a deterministic $\mathscr{C}$-automaton defining the same language.

Ideas for union and intersection are using the usual product technic (as for Kleene automata). The complement is obtained as usual by switching final states to nonfinal and nonfinal to final.

Formally, if $\mathscr{A} = \langle \mathscr{S}_A, \mathscr{M}_A, s_A^\star, \mathscr{F}_A, \Sigma \rangle$ is an n_A-automaton and $\mathscr{B} = \langle \mathscr{S}_B, \mathscr{M}_B, s_B^\star, \mathscr{F}_B, \Sigma \rangle$ an n_B-automaton, assuming for example that $n_A > n_B$ we define the n_A *product automaton* to be $\mathscr{C} = \langle \mathscr{S}_A \times \mathscr{S}_B, \mathscr{M}_C, (s_A^\star, s_B^\star), \mathscr{F}_C, \Sigma \rangle$, where $(q_1, \sigma, q_2) \in \mathscr{M}_C \Leftrightarrow (\mathrm{p}_1(q_1), \sigma, \mathrm{p}_1(q_2)) \in \mathscr{M}_A$ and $(\mathrm{p}_2(q_1), \sigma, \mathrm{p}_2(q_2)) \in \mathscr{M}_B$ if $q_1 \in [\mathscr{S}_A \times \mathscr{S}_B]_0^{n_A}$, $q_2 \in \mathscr{S}_A \times \mathscr{S}_B$ and $\sigma \in \Sigma$. One can easily show that $\mathscr{L}(\mathscr{C}) = \mathscr{L}(\mathscr{A}) \cup \mathscr{L}(\mathscr{B})$ if $q \in \mathscr{F}_C \Leftrightarrow \mathrm{p}_1(q) \in \mathscr{F}_A$ or $\mathrm{p}_2(q) \in \mathscr{F}_B$, $\mathscr{L}(\mathscr{C}) = \mathscr{L}(\mathscr{A}) \cap \mathscr{L}(\mathscr{B})$ if $q \in \mathscr{F}_C \Leftrightarrow \mathrm{p}_1(q) \in \mathscr{F}_A$ and $\mathrm{p}_2(q) \in \mathscr{F}_B$.

We note that if $\mathscr{A}$ and $\mathscr{B}$ are deterministic n-automaton, then so is $\mathscr{C}$.

Let $\mathscr{B} = \langle \mathscr{S}, \mathscr{M}, s^\star, \mathscr{F}, \Sigma \rangle$ be a deterministic n-automaton. The deterministic n-automaton $\mathscr{B}$ such that $\mathscr{L}(\mathscr{B})$ is the complement of $\mathscr{L}(\mathscr{A})$, denoted $\overline{\mathscr{L}(\mathscr{A})}$ is $\mathscr{B} = \langle \mathscr{S}, \mathscr{M}, s^\star, [\mathscr{S}]_0^n \backslash \mathscr{F}, \Sigma \rangle$.

In order to keep the determinism of an automaton, the obtention of product, Kleene closure (*) and ω operation uses a construction called *Weak Safra* because one can derivate it from Safra's proof of the equivalence between Büchi automata and Muller automata (but the weak Safra construction is not really due to Safra). Let $\mathscr{A} = \langle \mathscr{S}_A, \mathscr{M}_A, s_A^\star, \mathscr{F}_A, \Sigma \rangle$ and $\mathscr{B} = \langle \mathscr{S}_B, \mathscr{M}_B, s_B^\star, \mathscr{F}_B, \Sigma \rangle$ be two deterministic $\mathscr{C}$-automata, $m = |[\mathscr{S}_B]_0^n|$ ($\mathscr{B}$ is an n-automaton), and take $m + 2$ copies of $\mathscr{B}$. The deterministic automaton $\mathscr{C}$ such that $\mathscr{L}(\mathscr{C}) = \mathscr{L}(\mathscr{A}) \cdot \mathscr{L}(\mathscr{B})$ is made to have the $m + 2$ copies of $\mathscr{B}$ working simultaneously with $\mathscr{A}$. Each copy of $\mathscr{B}$ can be either *active* or *inactive* (ready to start (on the initial state) and insensitive to the input). At the beginning all copies of $\mathscr{B}$ are inactive, only $\mathscr{A}$ works ($\mathscr{A}$ already works). $\mathscr{C}$ actives an inactive copy of $\mathscr{B}$ when $\mathscr{A}$ is leaving a final state, after switching off all actives copies of $\mathscr{B}$ which are in the same state, except the first actived one. The word is accepted if it leads to a final state of a copy of $\mathscr{B}$.

Formally, let $\mathscr{A} = \langle \mathscr{S}_A, \mathscr{M}_A, s_A^\star, \mathscr{F}_A, \Sigma \rangle$ be a deterministic n_A-automaton and $\mathscr{B} = \langle \mathscr{S}_B, \mathscr{M}_B, s_B^\star, \mathscr{F}_B, \Sigma \rangle$ be a deterministic n_B-automaton. Assume now that $n_A > n_B$ and define $m = |[\mathscr{S}_B]^{n_A}|$ ($m = |[\mathscr{S}_B]^{n_B}|$ if $n_A \leqslant n_B$). In order to express the notion of inactivity of a copy of $\mathscr{B}$ we add a new state to $\mathscr{S}_B$: $\mathscr{S}'_B = \mathscr{S}_B \cup \{i\}$ (i stands for *i*nactive) and we extend the transition function $\mathscr{M}_B$: $\forall 0 \leqslant j \leqslant n_A \forall \sigma \in \Sigma\ \mathscr{M}'_B(\{i\}^j, \sigma) = i$ and $\mathscr{M}_B(q, \sigma) = q' \Rightarrow \mathscr{M}'_B(q, \sigma) = q'$. Now we are ready to begin the construction of $\mathscr{C} = \langle \mathscr{S}_C = \mathscr{S}_A \times (\mathscr{S}_B'^{m+2} \backslash \mathscr{S}_B^{m+2}), \mathscr{M}_C, (s_A^\star, i, \ldots, i), \mathscr{F}_C, \Sigma \rangle$, the weak Safra deterministic n_A-automaton. The transition function verifies $\mathscr{M}_C(q, \sigma) = (q'^A, q_1'^B, \ldots, q_{m+2}'^B)$ iff the three conditions below are true:

- $q'^A = \mathscr{M}_A(\mathrm{p}_1(q), \sigma)$.
- If $\nexists 1 < j < k / p_j(q) = \mathrm{p}_k(q)$ then $q_k'^B = \mathscr{M}_B(\mathrm{p}_k(q), \sigma)$ else $q_k'^B = i$.
- If $\mathrm{p}_1(q) \in \mathscr{F}_A$ then $q_p'^B = \mathscr{M}_B(s_B^\star, \sigma)$ with p the lowest integer such that $\mathrm{p}_p(q) = \{i\}^{t(q)}$.

To use the foregoing idea to build a deterministic $\mathscr{C}$-automaton $\mathscr{C}$ such that $\mathscr{L}(\mathscr{C}) = \mathscr{L}^*(\mathscr{A})$ one just have to take $|[\mathscr{S}_A]^n| + 2$ copies of $\mathscr{A}$ working simultaneously, in case $\mathscr{A} = \langle \mathscr{S}_A, \mathscr{M}_A, s_A^\star, \mathscr{F}_A, \Sigma \rangle$ is an n-automaton. An inactive copy of $\mathscr{A}$ is actived each time a copy of $\mathscr{A}$ is in a final state. For desactivation we keep the same rule as for the product construction. $\mathscr{C}$ is in a final state iff one of the copies of $\mathscr{A}$ is in a final state, or if $\mathscr{C}$ is in its initial state $s_C^\star = (s_A^\star, i, \ldots, i)$.

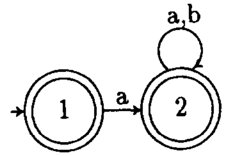


Fig. 10. The automaton accepting U^*.

The weak Safra construction cannot directly be used to obtain $\mathscr{L}^\omega(\mathscr{A})$ defining $\mathscr{F}_C = \{q \in [\mathscr{S}_C]_1^{n_A+1} / \exists 1 \leqslant i \leqslant |q| \, \exists 1 \leqslant j \leqslant m+2$ such that $q_i \in q$ and $p_j(q_j) \in \mathscr{F}_A\}$ because more than $\mathscr{L}^\omega(\mathscr{A})$ the set of words which have an infinity or prefixes in $\mathscr{L}^*(\mathscr{A})$, denoted by $\overrightarrow{\mathscr{L}^*(\mathscr{A})}$, is accepted, as is shown in the example below. The deterministic n-automaton accepting $\mathscr{L}^\omega(\mathscr{A})$, is build observing that[2] $U^\omega = U^* \cdot \vec{V}$, where the deterministic automaton defining V is obtained from U^*'s one applying determinism preserving operations (union, intersection, product and complementation: if $\mathscr{A}^* = \langle \mathscr{S} = \{s_1, \ldots, s_m\}, \mathscr{M}, s^\star, \mathscr{F}, \Sigma \rangle$ is a deterministic automaton such that $\mathscr{L}(\mathscr{A}^*) = U^*$, defining $\mathscr{A}_{ij}^* = \langle \mathscr{S}, \mathscr{M}, s_i, \{s_j\}, \Sigma \rangle$, $V_{ij} = \mathscr{L}(\mathscr{A}_{ij}^*)$, $W_{ij} = V_{ij} \cap V_{1j} \cap \Sigma^+$, $W_{ij}'^{\ddagger} = W_{ij} \cap \bigcap_{k=1}^m \overline{W_{ik} \cdot \Sigma^+}$, $V_i = (V_{1i} \cap U^*) \cdot (\bigcup_{j=1}^m W_{ij}')$, we have $V = \bigcup_i V_i$). If V is defined by a deterministic n-automaton $\langle \mathscr{S}, \mathscr{M}, s^\star, \mathscr{F}, \Sigma \rangle$ then the deterministic $n+1$-automaton accepting $\vec{V}$ is $\mathscr{A}' = \langle \mathscr{S}, \mathscr{M}, s^\star, \mathscr{F}', \Sigma \rangle$ with $\mathscr{F}' \subseteq [\mathscr{S}]_0^{n+1}$ and $q \in \mathscr{F}' \Leftrightarrow q \cap \mathscr{F} \neq \emptyset$.

The following is an example on finite words of the computation of the deterministic automaton accepting V from the deterministic automaton of U^* such that $U^\omega = U^* \cdot \vec{V}$.

Example 20. $U = a \cdot b^*$, $U^* = (a \cdot b^*)^* = \lambda + a \cdot (a+b)^*$, $U^\omega = (a \cdot b^*)^\omega$, $\vec{U} = a \cdot b^\omega$ (Fig. 10)

We remark that $U^\omega \not\ni a \cdot b^\omega \in \vec{U}^*$. Let us apply the algorithm.

$$V_1 = \emptyset, \qquad V_{21} = \emptyset, \qquad V_{12} = a \cdot (a+b)^*, \qquad V_{22} = (a+b)^*,$$

$$W_{11} = \emptyset, \qquad W_{21} = \emptyset, \qquad W_{12} = a \cdot (a+b)^*,$$

$$W_{22} = a \cdot (a+b)^* \cap (a+b)^* = a \cdot (a+b)^*,$$

$$W_{11}' = \emptyset, \quad W_{21}' = \emptyset,$$

$$W_{12}' = a \cdot (a+b)^* \cap \overline{a \cdot (a+b)^+} = a \cdot (a+b)^* \cap (\lambda + a + b \cdot (a+b)^*) = a.$$

$$W_{22}' = a \cdot (a+b)^* \cap (\lambda + a + b \cdot (a+b)^*) = a,$$

$$V = (a \cdot (a+b)^* \cap (a \cdot b)^*)^* \cdot a + ((a+b)^* \cap (a+b)^*)^* \cdot a$$

$$= (a \cdot b^*)^\dagger \cdot a + (a \cdot b^*)^* \cdot a = (a \cdot b^*)^* \cdot a.$$

So $(a \cdot b^*)^\omega = (a \cdot b^*)^* \cdot \overrightarrow{(a \cdot b^*)^* \cdot a}$

[2] Proof: [3].

‡ $W_{ij}' = \{x \in W_{ij} / \nexists k$ such that x has a proper prefix in $W_{ik}\}$.

The passage from a $\mathscr{C}$-α-regular expression to a deterministic $\mathscr{C}$-automaton defining the same language is then effective.

4.1.2. From deterministic $\mathscr{C}$-automata to $\mathscr{C}$-α-regular expressions

In this section *automaton* means *deterministic automaton.*

A n-automaton $\mathscr{A} = \langle \mathscr{S}\mathscr{M}, s^\star, \mathscr{F}, \Sigma \rangle$ is said to be of *type 0* if $\mathscr{F} \subseteq [\mathscr{S}]_0^n$ and $\mathscr{A}$ is neither of type 1 nor of type 2. We call $\mathscr{A}$ *automaton of type 1* if $\mathscr{F} \subseteq [\mathscr{S}]^n$ and $\mathscr{A}$ is not of type 2, *automaton of type 2* if $\mathscr{M} = \mathscr{M}|_n$ and $\mathscr{F} \subseteq [\mathscr{S}]^n$.

The computation of the $\mathscr{C}$-α-regular expression from a type 1 n-automaton $\mathscr{A} = \langle \mathscr{S}, \mathscr{F}, \mathscr{M}, s^\star, \Sigma \rangle$ uses Kleene's theorem: Let $Q = [\mathscr{S}]^n \cup \{s^\star\} = \{q_1 = s^\star, \dots, q_m\}$, for $1 \leqslant i,j \leqslant m$ $V_{ij}^0 = \{u \in \Sigma^{\omega^n} / M_e(q_i, u) = q_j\}$, and for $1 \leqslant k \leqslant m$ $V_{ij}^k = V_{ij}^{k-1} + V_{ik}^{k-1} \cdot (V_{kk}^{k-1})^* \cdot V_{kj}^{k-1}$. The $\mathscr{C}$-α-regular expression denoting $\mathscr{L}(\mathscr{A})$ is $\overset{q_i \in \mathscr{F}}{+} V_{1i}^m$.

Now we describe how to obtain a $\mathscr{C}$-α-regular expression from a type 2 $(n + 1)$-automaton $\mathscr{A} = \langle \mathscr{S}, \mathscr{F}, \mathscr{M}, s^\star, \Sigma \rangle$. The following algorithm is similar to the one which computes an ω-regular expression equivalent to a Muller automaton. Let us assume, without loss of generality, that $\mathscr{F}$ contains only one element $q = \{q_1, \dots, q_m\}$. Let $\mathscr{B} = \langle \mathscr{S}, \mathscr{M}, s^\star, \{q_1\}, \Sigma \rangle$ be a type 1 n-automaton and $(\mathscr{C}_i)$ be the family of type 1 n-automata: $\mathscr{C}_i = \langle \mathscr{S} \cup \{t^\star\}, \mathscr{M}_i, t^\star, \{q_{i+1}\}, \Sigma \rangle$ with $q_{m+1} = q_1, \forall \sigma \in \Sigma \mathscr{M}_i(t^\star, \sigma) = \mathscr{M}(q_i, \sigma)$, $\forall q \in [\mathscr{S}]_0^{n-2} \mathscr{M}_i(q, \sigma) = \mathscr{M}(q, \sigma)$ and $\forall 1 \leqslant j \leqslant m$ $\mathscr{M}_i(q_j, \sigma) = \mathscr{M}(q_j, \sigma)$. Let V be the $\mathscr{C}$-α-regular expression denoting $\mathscr{L}(\mathscr{A})$, V_B be the one denoting $\mathscr{L}(\mathscr{B})$ and V_{C_i}, the one for $\mathscr{L}(\mathscr{C}_i)$, then $V = V_B \cdot (V_{C_1} \cdots V_{C_m})^\omega$.

Now, the general case. Let us suppose that $\mathscr{A} = \langle \mathscr{S}, \mathscr{M}, s^\star, \mathscr{F}, \Sigma \rangle$ is a type 0 n-automaton and, for simplicity, $\mathscr{F} = \{q\}$. For each $X \in [\mathscr{S}]_0^n$, we define $\mathscr{A}_X = \langle \mathscr{S}, \mathscr{M}, s^\star, \{X\}, \Sigma \rangle$, $\mathscr{A}_X^m = \langle \mathscr{S}, \mathscr{M}|_{m+1}, s^\star, \{X\}, \Sigma \rangle$ if $X \in [\mathscr{S}]^m$, and $\mathscr{B}_{X,Y}^m = \langle \mathscr{S} \cup \{t^\star\}, \mathscr{M}|_{m+1} \cup \mathscr{M}', t^\star, \{Y\}, \Sigma \rangle$ where $t^\star$ is a new state, $Y \in [\mathscr{S}]^m$ and $\forall \sigma \in \Sigma \mathscr{M}'(t^\star, \sigma) = q \Leftrightarrow \mathscr{M}(X, \sigma) = q$. The relation below gives $\mathscr{E}_A$, a $\mathscr{C}$-α-regular expression denoting $\mathscr{L}(\mathscr{A})$ $(q \in [\mathscr{S}]^k)$:

$$\mathscr{E}_A = \mathscr{E}_{A_q^k} \overset{k < i \leqslant n}{+} \left(\overset{X \in [\mathscr{S}]^i}{+} (\mathscr{E}_{A_X} \cdot \mathscr{E}_{A_{X,q}^k}) \right).$$

Given an n-automaton $\mathscr{A}$, the construction of $\mathscr{E}_A$ is then effective.

4.2. Equivalence between deterministic n-automata and nondeterministic ones

Theorem 21 (Choueka [4]). *Let V denotes an n-definable language Σ and $p : \Sigma \to \Sigma'$ be a projection, then $p(V)$ is n-definable.*

Theorem 22 (Choueka [4]). *Nondeterministic automata are equivalent to deterministic ones.*

Proof. Given a nondeterministic n-automaton $\mathscr{A} = \langle \mathscr{S}, \mathscr{F}, \mathscr{M}, s^\star, \Sigma \rangle$ we build a deterministic one $\mathscr{B} = \langle \mathscr{S}, \mathscr{M}', s^\star, \mathscr{F}, \Sigma \times \mathscr{S} \rangle$ taking $\mathscr{M}'(q, (\sigma, q')) = q' \Leftrightarrow (q, \sigma, q') \in \mathscr{M}$. Let $\mathscr{E}_B$ be the $\mathscr{C}$-α-regular expression denoting $\mathscr{L}(\mathscr{B})$, and $p : \Sigma \times \mathscr{S} \to \Sigma$ the projection defined by $p(a, b) = a$. One can easily see that $p(\mathscr{E}_B)$ denotes $\mathscr{L}(\mathscr{A})$. As we know how to obtain a deterministic n-automaton equivalent to $p(\mathscr{E}_B)$ we have an algorithm determinizing $\mathscr{C}$-automata. □

Theorem 23 (Choueka [4]). *A subset of $\Sigma^{\#}$ is $\mathscr{C}$-definable iff it is definable by a $\mathscr{C}$-α-regular expression.*

Corollary 24. *Given $\mathscr{A}$ and $\mathscr{B}$ two $\mathscr{C}$-automata, the questions below are decidables:*
- $\mathscr{L}(\mathscr{A}) = \emptyset$?
- $\mathscr{L}(\mathscr{A}) \cap \mathscr{L}(\mathscr{B}) = \emptyset$?
- $\mathscr{L}(\mathscr{A}) = \mathscr{L}(\mathscr{B})$?
- $\mathscr{L}(\mathscr{A}) \subseteq \mathscr{L}(\mathscr{B})$?

Proof. The construction and simplification of $\mathscr{E}_A$ answers $\mathscr{L}(\mathscr{A}) = \emptyset$. The class of $\mathscr{C}$-definable languages is closed under complementation because deterministic and nondeterministic automata are equivalent. As is closed under union and intersection, $\mathscr{L}(\mathscr{A}) \cap \mathscr{L}(\mathscr{B}) = \emptyset$ is decidable and so is $(\mathscr{L}(\mathscr{A}) \cap \overline{\mathscr{L}(\mathscr{B})}) \cup (\mathscr{L}(\mathscr{B}) \cap \overline{\mathscr{L}(\mathscr{A})}) = \emptyset$, an other formulation for $\mathscr{L}(\mathscr{A}) = \mathscr{L}(\mathscr{B})$. The question $\mathscr{L}(\mathscr{A}) \subseteq \mathscr{L}(\mathscr{B})$ is equivalent to $(\mathscr{L}(\mathscr{B}) \cap \mathscr{L}(\mathscr{A})) = \mathscr{L}(\mathscr{A})$. □

4.3. $\mathscr{W}$-α-regular expressions

Definition 25. Let $\Sigma = \{\sigma_1, \ldots, \sigma_k\}$ be a finite alphabet. A *$\mathscr{W}$-α-regular expression* is a finite word on the alphabet $\Sigma \cup \{+, \cdot, {}^{\#}, {}^{\omega}, {}^{*}, (,), \lambda, \emptyset\}$ such that:
- $\emptyset$ is a $\mathscr{W}$-α-regular expression,
- λ is a $\mathscr{W}$-α-regular expression,
- $\sigma \in \Sigma$ is a $\mathscr{W}$-α-regular expression,
- if e_1 and e_2 are both $\mathscr{W}$-α-regular expressions, then so is $e_1 + e_2$,
- if e_1 and e_2 are both $\mathscr{W}$-α-regular expressions, then so is $e_1 \cdot e_2$,
- if e is a $\mathscr{W}$-α-regular expression, then so is (e),
- if e is a $\mathscr{W}$-α-regular expression, then so is e^*,
- if e is a $\mathscr{W}$-α-regular expression, then so is e^ω,
- if e is a $\mathscr{W}$-α-regular expression, then so is $e^{\#}$.

If e is a $\mathscr{W}$-α-regular expression, let $\bar{e}$ be the set of words denoted by e:
- $\bar{\emptyset} = \emptyset$,
- $\overline{e_1 + e_2} = \overline{e_1} \cup \overline{e_2}$,
- $\overline{e^*} = \bar{e}^*$,
- $\bar{\lambda} = \lambda$,
- $\overline{e_1 \cdot e_2} = \overline{e_1} \cdot \overline{e_2}$,

- $\overline{e^{\omega}} = \bar{e}^{\omega}$,
- $\Sigma \ni \bar{\sigma} = \sigma$,
- $\overline{(e)} = \bar{e}$,
- $\overline{e^{\#}} = \bar{e}^{\#}$

All the operators have their usual meaning, except the new one, $e^{\#}$, standing for *e is repeated finitely, infinitely or transfinitely often*. The $^{\#}$ operator is free and as for $\mathscr{C}$-α-regular expressions the $^{\omega}$ operator is free too.

4.3.1. From $\mathscr{W}$-α-regular expressions to $\mathscr{W}$-automata

Let $\mathscr{A} = \langle \mathscr{S}_A, \mathscr{M}_A, s_A^{\star}, \mathscr{F}_A, \Sigma \rangle$ and $\mathscr{B} = \langle \mathscr{S}_B, \mathscr{M}_B, s_B^{\star}, \mathscr{F}_B, \Sigma \rangle$ be two $\mathscr{W}$-automata.

The construction of the $\mathscr{W}$-automaton $\mathscr{C}$ such that $\mathscr{L}(\mathscr{C}) = \mathscr{L}(\mathscr{A}) \cup \mathscr{L}(\mathscr{B})$ (respectively $\mathscr{L}(\mathscr{C}) = \mathscr{L}(\mathscr{A}) \cap \mathscr{L}(\mathscr{B})$) uses the product technic: $\mathscr{C} = \langle \mathscr{S}_A \times \mathscr{S}_B, \mathscr{M}_C, (s_A^{\star}, s_B^{\star}), \mathscr{F}_C, \Sigma \rangle$ where $(q_1, \sigma, q_2) \in \mathscr{M}_C \Leftrightarrow (\mathrm{p}_1(q_1), \sigma, \mathrm{p}_1(q_2)) \in \mathscr{M}_A$ and $(\mathrm{p}_2(q_1), \sigma, \mathrm{p}_2(q_2)) \in \mathscr{M}_B$ if $q_1 \in [\mathscr{S}_A \times \mathscr{S}_B]_0^1$ and $q_2 \in \mathscr{S}_a \times \mathscr{S}_B$. We have $\mathscr{L}(\mathscr{C}) = \mathscr{L}(\mathscr{A}) \cup \mathscr{L}(\mathscr{B})$ (respectively, $\mathscr{L}(\mathscr{C}) = \mathscr{L}(\mathscr{A}) \cap \mathscr{L}(\mathscr{B})$) if $q \in \mathscr{F}_C \Leftrightarrow \mathrm{p}_1(q) \in \mathscr{F}_A$ or (respectively and) $\mathrm{p}_2(q) \in \mathscr{F}_B$. If $\mathscr{A}$ and $\mathscr{B}$ are deterministic automata, then so is $\mathscr{C}$.

To deal with product and Kleene closure one can use weak Safra construction, so the determinism of the resulting automaton depends on the determinism of $\mathscr{A}$ and $\mathscr{B}$.

A problem arises with determinism preserving when dealing with the ω operator. One can see $\mathscr{C}$-automata as automata made of stages: $q \in [\mathscr{S}]^i \Leftrightarrow q$ belongs to the ith stage of the automaton, and q can be reached by an other state of the same stage by a continuous $(\beta + \omega^i)$-sequence ($\beta \geqslant \omega^i$ or $\beta = 0$). According to the definition of continuous α-sequence, the moving up into the $(i+1)$th stage is obtained by repeating infinitely often states belonging to ith stage.

$\vdots$	$\vdots$
$\{\,\}^{i+1} \cdots \{\,\}^{i+1}$	$[\mathscr{S}]^{i+1}$
$\{\,\}^{i} \cdots \{\,\}^{i}$	$[\mathscr{S}]^{i}$
$\vdots$	$\vdots$
$\{\,\} \cdots \{\,\}$	$[\mathscr{S}]$
$s_1 \ldots s_m$	$\mathscr{S}$

Wojciechowski automata are made of only two stages. Thus, the idea of repeating infinitely often a final state in a stage (that is, the ω operator) is a little bit more difficult to express: in order to know if a stage $q \in [\mathscr{S}]$ have been repeated infinitely often new states have to be add to $\mathscr{S}$ such that all transitions from q are to these new states, and all transitions to these new states are from q. In other words, all these news states mark the passage by q. The infinite repetition of these new states means the infinite repetition of q, and is easy to express, because they all belong to the lowest stage of the automaton. Example 17 illustrates such a situation. We note that this construction is the only one available and implies the loss of determinism. More formally, the construction of the $\mathscr{W}$-automaton $\mathscr{C}$ such that $\mathscr{L}(\mathscr{C}) = \mathscr{L}^{\omega}(\mathscr{A})$ is given by the following: $\mathscr{C} = \langle \mathscr{S}_C = \mathscr{S}_A \times \{0, 1\}, \mathscr{M}_C, (s_A^\star, 0), \mathscr{F}_C, \Sigma \rangle$ where $(q_1, \sigma, q_2) \in \mathscr{M}_C$ iff one of the conditions below is true:

- $q_1 \in \mathscr{S}_C \cup [\mathscr{S}_A \times \{0\}]$, $q_2 \in \mathscr{S}_A \times \{0\}$ and $(\mathrm{p}_1(q_1), \sigma, \mathrm{p}_1(q_2)) \in \mathscr{M}_A$,
- $q_1 \in \mathscr{S}_C \cup [\mathscr{S}_A \times \{0\}]$, $\mathrm{p}_1(q_1) \in \mathscr{F}_A$, $q_2 \in \mathscr{S}_A \times \{1\}$, $(s_A^\star, \sigma, \mathrm{p}_1(q_2)) \in \mathscr{M}_A$.

Final states are $\mathscr{F}_C = \{q \in [\mathscr{S}_C] / 1 \in \mathrm{p}_2(q)\}$ union $\{q \in [\mathscr{S}_C]_0^1 / \mathrm{p}_1(q) \in \mathscr{F}_A\}$ if $s_A^\star \in \mathscr{F}_A$.

The construction of the $\mathscr{W}$-automaton $\mathscr{C}$ such that $\mathscr{L}(\mathscr{C}) = \mathscr{L}^{\#}(\mathscr{A})$ uses the forgoing memorization technic. We need a new initial state $s_C^\star$ in order to accept λ: $\mathscr{C} = \langle \mathscr{S}_C = \mathscr{S}_A \times \{0, 1\} \cup \{s_C^\star\}, \mathscr{M}_C, s_C^\star, \mathscr{F}_C, \Sigma \rangle$, where $(q_1, \sigma, q_2) \in \mathscr{M}_C$ iff one of the four following conditons is true:

- $q_1 = s_C^\star$, $q_2 \in \mathscr{S}_A \times \{0\}$, $(s_A^\star, \sigma, \mathrm{p}_1(q_2)) \in \mathscr{M}_A$,
- $q_1 \in \mathscr{S}_A \times \{0, 1\} \cup [\mathscr{S}_A \times \{0\}]$, $q_2 \in \mathscr{S}_A \times \{0\}$ and $(\mathrm{p}_1(q_1), \sigma, \mathrm{p}_1(q_2)) \in \mathscr{M}_A$,
- $q_1 \in \mathscr{S}_A \times \{0, 1\} \cup [\mathscr{S}_A \times \{0\}]$, $\mathrm{p}_1(q_1) \in \mathscr{F}_A$, $q_2 \in \mathscr{S}_A \times \{1\}$, $(s_A^\star, \sigma, \mathrm{p}_1(q_2)) \in \mathscr{M}_A$,
- $q_1 \in [\mathscr{S}_A \times \{0, 1\}]$, $1 \in \mathrm{p}_2(q_1)$, $q_2 \in \mathscr{S}_A \times \{1\}$, $(s_A^\star, \sigma, q_2) \in \mathscr{M}_A$.

Final states are $\mathscr{F}_C = \{s_C^\star\} \cup \{q \in \mathscr{S}_A \times \{0, 1\} \cup [\mathscr{S}_A \times \{0\}] / \mathrm{p}_1(q) \in \mathscr{F}_A\} \cup \{q \in [\mathscr{S}_A \times \{0, 1\}] / 1 \in \mathrm{p}_2(q)\}$.

4.3.2. From $\mathscr{W}$-automata to $\mathscr{W}$-α-regular expressions

Definition 26. Let $\mathscr{A} = \langle \mathscr{S}, \mathscr{M}, s^\star, \mathscr{F}, \Sigma \rangle$ be a $\mathscr{W}$-automaton, i and $f \in [\mathscr{S}]_0^1$, $P \in [\mathscr{S}]$. Let $\mathscr{E}_x(i, P, f)$ denote a $\mathscr{W}$-α-regular expression defining the set of words W such that a run φ of $\mathscr{A}$ on $u \in W$ begins with i, ends with f, and $\forall \beta \in Succ$, $0 < \beta < |u|\ \varphi(\beta) \in P$. By $\mathscr{E}\mathscr{E}_x(i, P, f)$ we denote a $\mathscr{W}$-α-regular expression representing the set of words W such that a run φ of $\mathscr{A}$ on $u \in W$ begins in i, ends in f, $\forall \beta \in Succ$ $0 < \beta < |u|\ \varphi(\beta) \in P$ and $\forall \xi \in Lim\ \xi < |u|\ \varphi(\xi) \neq P$.

Definition 27. Let $\mathscr{S}$ be a finite set and $P \in [\mathscr{S}]$ such that $P = \{s_1, s_2, \ldots, s_k\}$. Let $i \leqslant k$, $P^{(i)} = P \backslash \{s_i\}$, and $\mathscr{E}_i(P, f) = \mathscr{E}_x(s_1, P^{(2)}, s_2) \cdot \mathscr{E}_x(s_2, P^{(3)}, s_3) \cdots \mathscr{E}_x(s_{i-1}, P^{(i)}, s_i) \cdot \mathscr{E}_x(s_i, P^{(i \circ 1)}, f)$ where $i \circ 1 = i + 1$ if $i \neq k$, 1 otherwise. Let $\mathscr{E}_{\text{loop}}(P) = \mathscr{E}_x(s_1, P^{(2)}, s_2) \cdot \mathscr{E}_x(s_2, P^{(3)}, s_3) \cdots \mathscr{E}_k(s_{k-1}, P^{(k)}, s_k) \cdot \mathscr{E}_x(s_k, P^{(1)}, s_1)$.

Given a $\mathscr{W}$-automaton, the relations below, defined by induction on the number of elements of the set of states appearing in a run, provide an algorithm to compute a $\mathscr{W}$-α-regular expression equivalent to the $\mathscr{W}$-automaton:

(1) $\mathscr{E}_x(i, P, f) = \mathscr{E}\mathscr{E}_x(i, P, f) + \mathscr{E}\mathscr{E}_x(i, P, P) \cdot \mathscr{E}\mathscr{E}_x(P, P, P)^{\#} \cdot \mathscr{E}\mathscr{E}_x(P, P, f)$ if $f \neq P$,

(2) $\mathscr{E}_x(i, P, f) = \mathscr{E}\mathscr{E}_x(i, P, f) + \mathscr{E}\mathscr{E}_x(i, P, P) \cdot \mathscr{E}\mathscr{E}_x(P, P, P)^{\#}$ if $f = P$,

(3) $\mathscr{E}\mathscr{E}_x(i, P, f) = \mathscr{E}_x(i, P^{(1)}, f) + \mathscr{E}_x(i, P^{(1)}, s_1)\cdot\mathscr{E}_{\text{loop}}(P)^*\cdot\bigcup_{1\leqslant j\leqslant k}\mathscr{E}_j(P, f)$ if $f \neq P$,

(4) $\mathscr{E}\mathscr{E}_x(i, P, f) = \mathscr{E}_x(i, P^{(1)}, s_1)\cdot\mathscr{E}_{\text{loop}}(P)^{\omega}$ if $f = P$.

Theorem 28 (Wojciechowski [14]). A subset of $\Sigma^{\#}$ is $\mathscr{W}$-definable iff it is definable by a $\mathscr{W}$-α-regular expression.

5. Equivalence between $\mathscr{C}$-automata and $\mathscr{W}$-n-automata

In this section the equivalence between $\mathscr{C}$-automata and $\mathscr{W}$-automata is proved, assuming of course that the domain of $\mathscr{W}$-automata is restricted to the one of $\mathscr{C}$-automata.

Let $\mathscr{A}$ be a $\mathscr{W}$-automaton, n be an integer, $\mathscr{L}_1 = \mathscr{L}(\mathscr{A})$ and $\mathscr{L}_2 = \mathscr{L}(\mathscr{A}) \cap \Sigma^{<\omega^{n+1}}$. When we talk about $\mathscr{A}$ as a $\mathscr{W}$-automaton the language defined by $\mathscr{A}$ is $\mathscr{L}_1$. When we talk about $\mathscr{A}$ as a $\mathscr{W}$-n-automaton the language defined by $\mathscr{A}$ is $\mathscr{L}_2$.

First of all we give an algorithm taking a $\mathscr{W}$-n-automaton $\mathscr{A}$ as input and building a $\mathscr{C}$-automaton $\mathscr{A}$ defining the same language as $\mathscr{A}$ as output. Although the formalism is boring, the base idea, illustrated in the example below, is simple:

Example 29. Let $\Sigma = \{a, b\}$, $u = (ab^{\omega})^{\omega}$, $v = (b^{\omega})^{\omega}$ and $\mathscr{A}$ be the following $\mathscr{C}$-automaton:

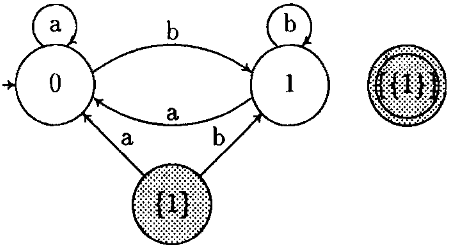


Fig. 11.

Let φ^u be the run of $\mathscr{A}$ on u and φ^v be the run of $\mathscr{A}$ on v. These figures represent φ^u and φ^v:

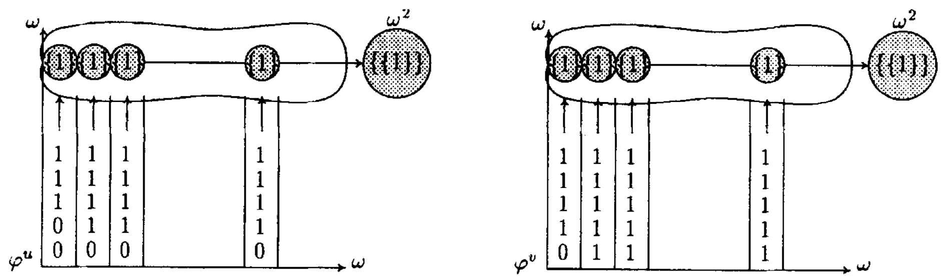


Fig. 12.

Assuming the states used during the run of $\mathscr{A}$ on $u\,[\omega\cdot i, \omega\cdot i + (k-1)[$ (respectively, $v[\omega\cdot i, \omega\cdot i + (k-1)[$) have been memorized in $\varphi^u(\omega\cdot i + k)$ $(0 \leqslant i, k < \omega)$ (respectively, $\varphi^v(\omega\cdot i + k)$) the two runs become:

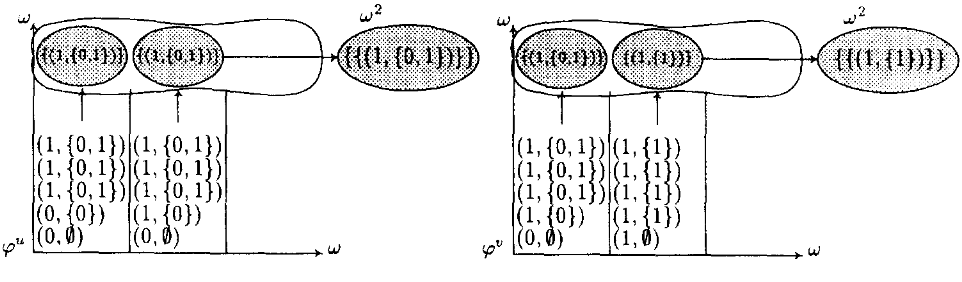


Fig. 13.

and the "memories" (or *histories*) associated (because runs are continuous sequences) with $\varphi^u(\omega^2)$ and $\varphi^v(\omega^2)$ allow us to distinguish between u and v after reading ω^2 letters. If $\xi \in Lim$, the union of histories in the projection of $\varphi(\xi)$ is the set of states s such that $\{\gamma/\varphi(\gamma) = s\}$ is cofinal with ξ, so we have expressed the notion of Wojciechowski's continuity, starting from a $\mathscr{C}$-automaton.

Definition 30. Let $\mathscr{S}$ be a finite set. We define the *flat* function $\mathscr{P}: [\mathscr{S}]^n \to [\mathscr{S}]$ by $\mathscr{P}(q = \{q_1, \ldots, q_k\}) = \bigcup_{i=1}^{k} \mathscr{P}(q_i)$ if $q \in [\mathscr{S}]^n$, $n > 0$ and $q_1, \ldots, q_k \in [\mathscr{S}]^{n-1}$, or $\mathscr{P}(q) = \{q\}$ if $q \in [\mathscr{S}]^0$.

Definition 31. Let φ be a run of an n-automaton $\mathscr{A} = \langle \mathscr{S}', \mathscr{M}, s^\star, \mathscr{F}, \Sigma \rangle$ such that $\forall \alpha \in Succ\ \varphi(\alpha) = (s^\alpha, h_k^\alpha, \ldots, h_0^\alpha)$, $s^\alpha \in \mathscr{S}$ a finite set, $h^\alpha_{i, 0 \leqslant i < k} \in [\mathscr{S}] \cup \{\emptyset\}$ and $k \in \mathbb{N}$. As a run is uniquely determined by its nonlimit values, we define φ' to be the unique continuous sequence (according to Choueka) such that $\forall \alpha \in Succ, \varphi(\alpha) = (s^\alpha, h_k^\alpha, \ldots, h_0^\alpha)$ $\Leftrightarrow \varphi'(\alpha) = s^\alpha$. The run φ' is called *projection of the run* φ *from* $\mathscr{S}'$ *to* $\mathscr{S}$.

Definition 32. $\mathscr{R}'(\{(s_1, h_n, \ldots, h_0), \ldots, (s_k, h_n, \ldots, h_0)\}) = \bigcup_{i=1}^{k} \{s_k\}$, $k, n \in \mathbb{N}$.

Definition 33. $\mathscr{R}_k(\{(s_1, h_n^1, \ldots, h_0^1), \ldots, (s_m, h_n^m, \ldots, h_0^m)\}) = \bigcup_{i=1}^{m} h_k^i$, $m, n \in \mathbb{N}$ and $k \leqslant n$.

Definition 34. $fst((a_1, \ldots, a_n)) = a_1$.

Definition 35. Let $\mathscr{A} = \langle \mathscr{S}, \mathscr{M}, s^\star, \mathscr{F}, \Sigma \rangle$ be a $\mathscr{W}$-automaton. Starting from $\mathscr{A}$ we build an n-automaton $\langle \mathscr{S}', \mathscr{M}', s^{\star\prime}, \mathscr{F}', \Sigma \rangle$ $(m \leqslant n)$:

- $\mathscr{S}' = \mathscr{S} \times \underbrace{([\mathscr{S}] \cup \{\emptyset\}) \times \cdots \times ([\mathscr{S}] \cup \{\emptyset\})}_{n-1 \text{ times}}$.
- $s^{\star\prime} = (s^\star, \underbrace{\emptyset, \ldots, \emptyset}_{n-1 \text{ times}})$.

- If $\sigma \in \Sigma$, then $((s, h_{n-2}, \ldots, h_0), \sigma, (s', h'_{n-2}, \ldots, h'_0)) \in \mathcal{M}'$ iff $s \in \mathcal{S}$, $s' \in \mathcal{S}$, $\forall i \leqslant n-2$ $h'_i = h_i \cup \{s\}$, and $(s, \sigma, s') \in \mathcal{M}$.
- If $\sigma \in \Sigma$, then $(X \in [\mathcal{S}']^m, \sigma, (s', h'_{n-2}, \ldots, h'_0)) \in \mathcal{M}'$ iff $s' \in \mathcal{S}$, $\forall i < m$ $h'_i = \emptyset$, $\forall i \geqslant m$ $h'_i = \mathcal{R}_i(\mathcal{P}(X))$ and

$$\begin{cases} (\mathcal{R}_{m-2}(\mathcal{P}(X)), \sigma, s') \in \mathcal{M} & \text{if } n \geqslant m > 1 \\ (\mathcal{R}'(X), \sigma, s') \in \mathcal{M} & \text{if } m = 1. \end{cases}$$

- If $s' = (s, h_{n-2}, \ldots, h_0) \in \mathcal{S}'$, $s' \in \mathcal{F}' \Leftrightarrow s \in \mathcal{F}$.
- If $X \in [\mathcal{S}']^m$,

$$X \in \mathcal{F}' \Leftrightarrow: \begin{cases} \mathcal{R}_{m-2}(\mathcal{P}(X)) \in \mathcal{F} & \text{if } n \geqslant m > 1 \\ \mathcal{R}'(X) \in \mathcal{F} & \text{if } m = 1. \end{cases}$$

We will call any $\mathcal{C}$-automaton obtained from a $\mathcal{W}$-automaton using this definition a *W2C-automaton.*

As is shown in the lemma below, adding histories does not change the infinite repetition of a state in runs of automata.

Lemma 36. *Let $\langle \mathcal{S}, \mathcal{M}, s^\star, \mathcal{F}, \Sigma \rangle$ be a $\mathcal{W}$-automaton and $\mathcal{A}$ the corresponding W2C-automaton. Assume $\mathcal{A}$ is an n-automaton. Let φ_1 be a run of $\mathcal{A}$ on a word $u \in \Sigma^{<\omega^{n+1}}$ and φ_2 the projection of φ_1 on $\mathcal{S}$. Then $s \in \mathcal{R}'(\mathcal{P}(\varphi_1(\beta + \omega^m))) \Leftrightarrow s \in \mathcal{P}(\varphi_2(\beta + \omega^m))$.*

Proof. According to Choueka's definition of continuous α-sequences, it is sufficient to show that $s \in \mathcal{R}'(\mathcal{P}(\varphi_1(\omega^m))) \Leftrightarrow s \in \mathcal{P}(\varphi_2(\omega^m))$. As φ_2 is obtained by projection of φ_1, $\Rightarrow$ is trivial. The other way, $\Leftarrow$, is proved by a two-step induction on m.

If $m = 1$, the definition of $\mathcal{P}$ implies $s \in \mathcal{P}(\varphi_2(\omega)) \Leftrightarrow s \in \varphi_2(\omega)$. So according to the definition of continuous α-sequences, $\Gamma = \{\gamma / \varphi_2(\gamma) = s\}$ is infinite. Let γ_0 be the smallest ordinal in Γ, γ_1 the smallest ordinal in $\Gamma \setminus \{\gamma_0\}$, and so on. We build the infinite serie $(\gamma_i)_{i<\omega}$. We have $\varphi_2(\gamma_j) = s \Leftrightarrow \exists h^j_{n-2}, \ldots, h^j_0 / \varphi_1(\gamma_j) = (s, h^j_{n-2}, \ldots, h^j_0)$. If $\varphi_1|_\omega = (s_0, h^0_{n-2}, \ldots, h^0_0), (s_1, h^1_{n-2}, \ldots, h^1_0), \ldots, (s_k, h^k_{n-2}, \ldots, h^k_0), \ldots$ the definition of histories implies $k > k' \Rightarrow \forall i, h^{k'}_i \subseteq h^k_i$, and, as $\mathcal{S}$ is a finite set, there exists k_l such that $\forall k > k_l \forall i \leqslant n-2$, $h^k_i = h^{k_l}_i$. Then, $\{\gamma / \varphi_1(\gamma) = (s, h^{k_l}_{n-2}, \ldots, h^{k_l}_0)\}$ is infinite and according to the definition of continuous α-sequences, $(s, h^{k_l}_{n-2}, \ldots, h^{k_l}_0) \in \varphi_1(\omega) = \mathcal{P}(\varphi_1(\omega)) \Rightarrow s \in \mathcal{R}'(\mathcal{P}(\varphi_1(\omega)))$.

Our induction hypothesis is (1): $s \in \mathcal{P}(\varphi_2(\omega^m)) \Rightarrow s \in \mathcal{R}'(\mathcal{P}(\varphi_1(\omega^m)))$. Now we show that $\forall k < \omega$ $s \in \mathcal{P}(\varphi_1(\omega^m \cdot k)) \Rightarrow s \in \mathcal{R}'(\mathcal{P}(\varphi_1(\omega^m \cdot k)))$. It is obvious for $k = 1$. Let us assume that $s \in \mathcal{P}(\varphi_2(\omega^m \cdot k)) \Rightarrow s \in \mathcal{R}'(\mathcal{P}(\varphi_1(\omega^m \cdot k)))$ in order to prove (2): $s \in \mathcal{P}(\varphi_1(\omega^m \cdot (k+1))) \Rightarrow s \in \mathcal{R}'(\mathcal{P}(\varphi_1(\omega^m \cdot (k+1))))$. $\omega^m \cdot (k+1) = \omega^m \cdot k + \omega^m$ and $\omega^m \cdot k > \omega^m$ so using the definition of continuous α-sequences again (2) is true if (1) is true, and (1) is true. So $\forall k < \omega$ $s \in \mathcal{P}(\varphi_2(\omega^m \cdot k)) \Rightarrow s \in \mathcal{R}'(\mathcal{P}(\varphi_1(\omega^m \cdot k)))$.

Now we show that $s \in \mathcal{P}(\varphi_2(\omega^{m+1})) \Rightarrow s \in \mathcal{R}'(\mathcal{P}(\varphi_1(\omega^{m+1})))$. If $s \in \mathcal{P}(\varphi_2(\omega^{m+1}))$ then $\exists X \in [\mathcal{S}]^m$ such that $X \in \varphi_2(\omega^{m+1})$ and $s \in \mathcal{P}(X)$, so by the definition of continuity $\{k / \varphi_2(\omega^m \cdot k) = X\}$ is infinite, i.e. $\{k / s \in \mathcal{P}(\varphi_2(\omega^m \cdot k))\}$ is infinite and by induction hypothesis $\{k/s \in \mathscr{R}'(\mathscr{P}(\varphi_1(\omega^m \cdot k)))\}$ is infinite so by the definition of continuous α-sequences $s \in \mathscr{R}'(\mathscr{P}(\varphi_1(\omega^{m+1})))$. □

Lemma 37. *Let* $u \in \Sigma^\alpha$, $\{\varphi_1^i\}$ *be the set of runs of a* $\mathscr{W}$*-automaton* $\mathscr{A} = \langle \mathscr{S}, \mathscr{M}, s^\star, \mathscr{F}, \Sigma \rangle$ *on* u *and* $\{\varphi_2^j\}$ *be the set of runs of the corresponding W2C n-automaton* $\mathscr{A}' = \langle \mathscr{S}', \mathscr{M}', s^{\star\prime}, \mathscr{F}', \Sigma \rangle$ *on* u. *Then* $\forall i \exists j$ *(and reciprocally,* $\forall j \exists i$*) such that:*

(A) *If* $\alpha \in Succ$, $\forall \beta \in Succ \leqslant \alpha$ $fst(\varphi_2^j(\beta)) = \varphi_1^i(\beta)$.

(B) *If* $\alpha \in Lim$ *such that the unique decomposition of* α *is* $\alpha = \beta + \omega^m$ *with* $m > 0$ *and* $\beta = 0$ *or* $\beta \geqslant \omega^m$, *then:*

$$\begin{cases} \mathscr{R}'(\varphi_2^j(\alpha)) = \varphi_1^i(\alpha) & \text{if } m = 1, \\ \mathscr{R}_{m-2}(\mathscr{P}(\varphi_2^j(\alpha))) = \varphi_1^i(\alpha) & \text{if } n \geqslant m > 1. \end{cases}$$

Proof. By transfinite induction.

If $\alpha = 0$, we have $fst(\varphi_2^j(0) = (s^\star, \emptyset, \ldots, \emptyset)) = s^\star = \varphi_1^j(0)$.

Let us assume now that the lemma is true for $\forall \beta \leqslant \alpha$, in order to show that this hypothesis implies that the lemma is also true for $\alpha + 1$. We know that, by induction hypothesis, given a $\mathscr{W}$-automaton, the corresponding W2C-automaton and a word u, if there exists a run of one of the two automata on u, there exists a run of the other on u such that they both verify (A) and (B). Let φ_1 and φ_2 be two such runs. We try to extend φ_1 and φ_2 by one more transition. If $\alpha \in Succ$, $fst(\varphi_2^j(\alpha)) = \varphi_1^i(\alpha)$ and according to the definition of the W2C-automaton $(\varphi_1^i(\alpha),_\alpha u, \varphi_1^i(\alpha+1)) \in \mathscr{M} \Leftrightarrow ((\varphi_1^i(\alpha), h_{n-2}, \ldots, h_0),_\alpha u, (\varphi_1^i(\alpha+1), h_{n-2} \cup \{\varphi_1^i(\alpha)\}, \ldots, h_0 \cup \{\varphi_1^i(\alpha)\})) \in \mathscr{M}'$, so we have $fst(\varphi_2^j(\alpha+1)) = \varphi_1^i(\alpha+1)$. If $\alpha \in Lim$ and $\alpha = \omega$, by induction hypothesis $\mathscr{R}'(\varphi_2(\alpha)) = \varphi_1(\alpha)$, and by the definition of $\mathscr{M}'(\mathscr{R}'(\varphi_2(\alpha)),_\alpha u, \varphi_1(\alpha+1)) \in \mathscr{M} \Leftrightarrow (\varphi_2(\alpha),_\alpha u, (\varphi_1(\alpha+1), \mathscr{R}_{n-2}(\mathscr{P}(\varphi_2(\alpha))), \ldots, \mathscr{R}_1(\mathscr{P}(\varphi_2(\alpha))), \emptyset) \in \mathscr{M}'$, and we have $fst(\varphi_2(\alpha+1)) = \varphi_1(\alpha+1)$. If $\alpha = \omega^m$, $m > 1$, by induction hypothesis $\varphi_1(\alpha) = \mathscr{R}_{m-2}(\mathscr{P}(\varphi_2(\alpha)))$ and by the definition of $\mathscr{M}'(\mathscr{R}_{m-2}(\mathscr{P}(\varphi_2(\alpha))),_\alpha u, \varphi_1(\alpha+1)) \in \mathscr{M} \Leftrightarrow (\varphi_2(\alpha),_\alpha u, (\varphi_1(\alpha+1), \mathscr{R}_{n-2}(\mathscr{P}(\varphi_2(\alpha))), \ldots, \mathscr{R}_m(\mathscr{P}(\varphi_2(\alpha))), \emptyset, \ldots, \emptyset)) \in \mathscr{M}'$ and $fst(\varphi_2(\alpha+1)) = \varphi_1(\alpha+1)$ once again.

We assume now that $\alpha \in Lim$ and the lemma is true $\forall \gamma < \alpha$. Let $\beta + \omega^m$ be the unique decomposition of α. Obviously, $\varphi_1^i(\alpha) = \{s/\{\gamma/\varphi_1^i(\gamma) = s\}$ is cofinal with $\alpha\} = \{s/\{\gamma > \beta/\varphi_1^i(\gamma) = s\}$ is cofinal with $\alpha\}$. This remark and Choueka's definition of continuous α-sequences allow us to restrict the proof to $\alpha = \omega^m$. If $\alpha = \omega$, we have $\mathscr{R}'(\varphi_2^j(\alpha)) = \varphi_1^i(\alpha)$ by immediate application of the previous lemma and because Choueka's definition of continuity and Wojciechowski's are identical for the case of ω-sequences. We now turn to the case $m > 1$, assuming $s \in \varphi_1^i(\omega^m)$, so $\{\gamma < \omega^m/\varphi_1^i(\gamma) = s\}$ is cofinal with ω^m. By induction hypothesis, $\varphi_1^i(\gamma) = s \Rightarrow fst(\varphi_2^j(\gamma)) = s$, so $\{\gamma < \omega^m/fst(\varphi_2^j(\gamma)) = s\}$ is cofinal with ω^m, and $\Gamma = \{k/\forall n - 1 > l \geqslant m - 2\ s \in \mathscr{R}_l(\mathscr{P}(\varphi_2^j(\omega^{m-1} \cdot k)))\}$ is infinite because histories h_i are cleaned only during the transitions from $\varphi_2^j(\omega^{i+1} \cdot k)$ to $\varphi_2^j(\omega^{i+1} \cdot k + 1)$, and because of the foregoing lemma. As Γ is infinite, using once again the foregoing lemma, it comes $s \in \mathscr{R}_l(\mathscr{P}(\varphi_2^j(\omega^m)))\ \forall l\ m - 2 \leqslant l < n - 1$ and in particular $s \in \mathscr{R}_{m-2}(\mathscr{P}(\varphi_2^j(\omega^m)))$.

Now let us show the converse. Let us assume that $s \in \mathscr{R}_{m-2}(\mathscr{P}(\varphi_2^j(\omega^m)))$. $\varphi_2^j(\omega^m) = \text{In}(\psi)$, where ψ is the unique continuous ω-sequence such that $\psi(i) = \varphi_2^j(\omega^{m-1} \cdot i)$, so $s \in \mathscr{R}_{m-2}(\mathscr{P}(\varphi_2^j(\omega^m))) \Rightarrow \exists X \in [\mathscr{S}]^{m-1} / s \in \mathscr{R}_{m-2}(\mathscr{P}(X))$ and $\{k < \omega / \varphi_2^j(\omega^{m-1} \cdot k) = X\}$ is infinite. As histories h_{m-2} are cleaned only during the transitions from $\varphi_2^j(\omega^{m-1} \cdot k)$ to $\varphi_2^j(\omega^{m-1} \cdot k + 1)$, and observing the way histories are filled, we have that $\{k < \omega / \exists \gamma\, \omega^{m-1} \cdot k < \gamma < \omega^{m-1} \cdot (k+1)\ fst(\varphi_2^j(\gamma)) = s\}$ is infinite, and so by induction hypothesis $\{k < \omega / \exists \gamma\, \omega^{m-1} \cdot k < \gamma < \omega^{m-1} \cdot (k+1)\ \varphi_1^i(\gamma) = s\}$ is infinite, so $\{\gamma / \varphi_1^i(\gamma) = s\}$ is cofinal with ω^m, this implies $s \in \varphi_1^i(\omega^m)$. □

Lemma 38. *The language accepted by a $\mathscr{W}$-n-automaton and the one accepted by the corresponding W2C-automaton are identical.*

Proof. Let $\mathscr{A} = \langle \mathscr{S}, \mathscr{M}, s^\star, \mathscr{F}, \Sigma \rangle$ be a $\mathscr{W}$-n-automaton and $\mathscr{A}' = \langle \mathscr{S}', \mathscr{M}', s^{\star\prime}, \mathscr{F}', \Sigma \rangle$ be the corresponding W2C-automaton. Let u be an α-sequence accepted by $\mathscr{A}$ and φ_1 an accepting run of $\mathscr{A}$ on u. Using the previous lemma there exists an accepting run φ_2 of $\mathscr{A}'$ on u verifying:

(A) If $\alpha \in Succ$, $\forall \beta \in Succ \leqslant \beta\ fst(\varphi_2(\beta)) = \varphi_1(\beta)$.

(B) If $\alpha \in Lim$ such that the unique decomposition of α is $\alpha = \beta + \omega^m$ with $m > 0$ and $\beta = 0$ or $\beta \geqslant \omega^m$, then:

$$\begin{cases} \mathscr{R}'(\varphi_2(\alpha)) = \varphi_1(\alpha) & \text{if } m = 1, \\ \mathscr{R}_{m-2}(\mathscr{P}(\varphi_2(\alpha))) = \varphi_1(\alpha) & \text{if } m > 1. \end{cases}$$

If $\alpha \in Succ$ we have $fst(\varphi_2(\alpha)) = \varphi_1(\alpha)$, but, according to the definition of $\mathscr{F}'$, $\varphi_1(\alpha) \in \mathscr{F} \Rightarrow (\varphi_1(\alpha), h_{n-2}, \ldots, h_0) \in \mathscr{F}'$ so u is accepted by $\mathscr{A}'$. If $\alpha = \beta + \omega^m$ and $m = 1$ we have $\varphi_1(\alpha) = \mathscr{R}'(\varphi_2(\alpha))$ and $\mathscr{R}'(\varphi_2(\alpha)) \in \mathscr{F} \Rightarrow \varphi_2(\alpha) \in \mathscr{F}'$ so u is accepted by $\mathscr{A}'$. If $m > 1$ we have $\mathscr{R}_{m-2}(\mathscr{P}(\varphi_2(\alpha))) = \varphi_1(\alpha)$ and $\mathscr{R}_{m-2}(\mathscr{P}(\varphi_2(\alpha))) \in \mathscr{F} \Rightarrow \varphi_2(\alpha) \in \mathscr{F}'$ so u is accepted by $\mathscr{A}'$. The proof of the converse is similar. □

Theorem 39. *A $\mathscr{W}$-n-automaton is equivalent to an n-automaton and reciprocally.*

Proof. $\mathscr{C}$-definable languages are representable by $\mathscr{C}$-α-regular expressions, the set of $\mathscr{C}$-α-regular expressions is include into $\mathscr{W}$-α-regular expressions' one, and $\mathscr{W}$-α-regular expressions are equivalent to $\mathscr{W}$-automata. Then, for each $\mathscr{C}$-automaton an equivalent $\mathscr{W}$-automaton can be build.

Conversely, given a $\mathscr{W}$-n-automaton, the corresponding W2C-automaton (a $\mathscr{C}$-automaton) always exists and defines the same language. □

Corollary 40. *The class of $\mathscr{W}$-n-definable languages is closed under complementation.*

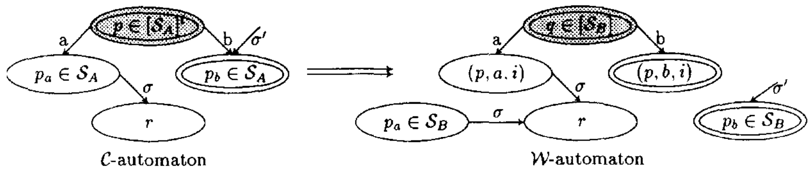


Fig. 14.

Proof. The class of $\mathscr{C}$-definable languages is closed under complementation and is the same as $\mathscr{W}$-n-definable languages' one. □

6. Equivalence between nondeterministic and deterministic $\mathscr{W}$-n-automata

First of all we give a construction computing a deterministic $\mathscr{W}$-automaton from a deterministic n-automaton such that the input and output automata are equivalent. The algorithm for determinizing $\mathscr{W}$-n-automata is then easy to find. Note that more than determinizing a $\mathscr{W}$-n-automaton the algorithm transforms a $\mathscr{W}$-n-automaton into a $\mathscr{W}$-automaton.

Let $\mathscr{A} = \langle \mathscr{S}_A, \mathscr{M}_A, s_A^\star, \mathscr{F}_A, \Sigma \rangle$ be a deterministic n-automaton. Fig. 14 explains the construction for each $p \in [\mathscr{S}]_1^n$:

The main idea is to memorize the passage by a state $p \in [\mathscr{S}_A]^i$, $0 < i \leqslant n$, by adding news states that can be reached only from p, in such a way that all transitions from p are to these new states. For evident reasons, these new states are called *witnesses states* (of the passage by p). In the previous figure (p, a, i) and (p, b, i) are witness of the passage by $p \in [\mathscr{S}_A]^i$. Then, we have to find an equivalent of p for Wojciechowski's definition of state in a limit point.

Definition 41. Let $\mathscr{A} = \langle \mathscr{S}_A, \mathscr{M}_A, s_A^\star, \mathscr{F}_A, \Sigma \rangle$ be a deterministic n-automaton. The algorithm below computes a deterministic $\mathscr{W}$-automaton $\mathscr{B} = \langle \mathscr{S}_B, \mathscr{M}_B, s_B^\star, \mathscr{F}_B, \Sigma \rangle$ such that $\mathscr{L}(\mathscr{A}) = \mathscr{L}(\mathscr{B})$:

$\mathscr{S}_B = \mathscr{S}_A$; $\mathscr{M}_B = \mathscr{M}_A|_1$; $\mathscr{F}_B = \mathscr{F}_A \cap [\mathscr{S}_A]_0^1$; $s_B^\star = s_A^\star$
$\forall p \in [\mathscr{S}_A]$/Definitions of continuity for Choueka et Wojciechowski are equivalent for ω-sequences/

$\forall \sigma \in \Sigma$
  $q = (p, \sigma, 1)$ new state $\notin \mathscr{S}_B$
  $\mathscr{S}_B \leftarrow \mathscr{S}_B \cup \{q\}$
  $\mathscr{M}_B(p, \sigma) = q$
  $\forall \sigma' \in \Sigma$
    $\mathscr{M}_B(q, \sigma') = \mathscr{M}_A(\mathscr{M}_A(p, \sigma), \sigma')$
  If $\mathscr{M}_A(p, \sigma) \in \mathscr{F}_A$
    $\mathscr{F}_B \leftarrow \mathscr{F}_B \cup \{q\}$

For $i = 2$ to n
 $\forall p \in [\mathscr{S}_A]^i = \{p_1, \ldots, p_k\}$
  $\forall \sigma \in \Sigma$
   (p, σ, i) new state $\notin \mathscr{S}_B$
   $\mathscr{S}_B \leftarrow \mathscr{S}_B \cup \{(p, \sigma, i)\}$
   If $\mathscr{M}_A(p, \sigma) \in \mathscr{F}_A$
    $\mathscr{F}_B \leftarrow \mathscr{F}_B \cup \{(p, \sigma, i)\}$
  $\forall q \in [\mathscr{S}_B]/(\forall j \in [1 \ldots k] \exists r \in q \exists \sigma \in \Sigma / r = (p_j, \sigma, i-1))$
   $\wedge (\nexists (r, \sigma, i-1) \in q / r \notin p) \wedge (\nexists (r, \sigma, k)/k > i-1))$
   $\forall \sigma \in \Sigma$
    $\mathscr{M}_B(q, \sigma) = (p, \sigma, i)$
    $\forall \sigma' \in \Sigma$
     $\mathscr{M}_B((q, \sigma, i), \sigma') = \mathscr{M}_A(\mathscr{M}_A(p, \sigma), \sigma')$
   If $p \in \mathscr{F}_A$
    $\mathscr{F}_B \leftarrow \mathscr{F}_B \cup \{q\}$

The $\mathscr{W}$-automaton obtained by the algorithm is called a *C2W-automaton*.

Before showing that the languages accepted by the input $\mathscr{C}$-automaton and the output $\mathscr{W}$-automaton are identical we need two lemmas on ordinals.

Lemma 42. *Let α be an ordinal of unique decomposition $\alpha = \beta + \omega^n$, $\omega > n > 0$, and $\omega > r \geqslant n$. There does not exist a set of ordinals of type r cofinal with α.*

Proof. The normal form of α is ($k \in \mathbb{N}$, $\forall 1 \leqslant i \leqslant k+1$ $m_i < \omega$ and $\alpha_{k+1} = n$)

$$\alpha = \omega^{\alpha_1} \cdot m_1 + \cdots + \omega^{\alpha_k} \cdot m_k + \omega^{\alpha_{k+1}} \cdot m_{k+1}.$$

Let γ be the ordinal of normal form

$$\gamma = \omega^{\alpha_1} \cdot m_1 + \cdots + \omega^{\alpha_k} \cdot m_k + \omega^{\alpha_{k+1}} \cdot (m_{k+1} - 1) + 1.$$

Obviously, $\gamma < \alpha$. The smallest ordinal β of type r bigger than γ is $\gamma + \omega^r$, and $\beta \geqslant \alpha$. So there does not exist an ordinal β such that $t(\beta) = r$ and $\gamma < \beta < \alpha$. $\square$

Lemma 43. *Let $\alpha = \beta + \omega^n$, $\omega > n > 0$ and Γ be a set of ordinals of type $n-1$ cofinal with α. Γ has an infinity of elements that can be written $\beta + \omega^{n-1} \cdot k$, $k < \omega$.*

Proof. Γ cofinal with $\alpha \Leftrightarrow \forall \sigma < \alpha \exists \gamma \in \Gamma\ \gamma > \sigma$, and $\alpha = \beta + \omega^n \Rightarrow \beta < \alpha$. Let $\gamma \in \Gamma$ such that $\gamma > \beta$. There exists an unique k, $0 < k < \omega$, such that $\gamma = \beta + \omega^{n-1} \cdot k$, $0 < k < \omega$, and as $\gamma < \alpha$ and Γ is cofinal with α there exists γ' in Γ such that $\gamma' > \gamma$ and γ' can be written in an unique way $\gamma = \beta + \omega^{n-1} \cdot k'$, $k < k' < \omega$. This operation can be infinitely repeated, because Γ is cofinal with α, so the lemma holds. $\square$

Lemma 44. *Let $\mathscr{A} = \langle \mathscr{S}_A, \mathscr{M}_A, s_A^\star, \mathscr{F}_A, \Sigma \rangle$ be a deterministic $\mathscr{C}$-automaton, $\mathscr{B} = \langle \mathscr{S}_B, \mathscr{M}_B, s_B^\star, \mathscr{F}_B, \Sigma \rangle$ be the C2W-automaton obtained from $\mathscr{A}$, u a word on Σ, φ_W^u the run of $\mathscr{B}$ on u and φ_C^u the run of $\mathscr{A}$ on u. Then, depending on α, one of the condition below is true:*

(1) $\varphi_W^u(\alpha) = \{(p_1, -, n-1), \ldots, (p_k, -, n-1)\} \cup P \Leftrightarrow \varphi_C^u(\alpha) = \{p_1, \ldots, p_k\}$ *if α can be written in a unique way $\alpha = \beta + \omega^n$, $n > 0$, $P \subseteq [\mathscr{S}_B]$ such that P contains only states with third part strictly less than $n-1$, $p_1, \ldots, p_k \in [\mathscr{S}_A]^{n-1}$ and – standing for any element of Σ,*

(2) $\varphi_W^u(\alpha) = (p, \sigma, n) \Leftrightarrow \varphi_C^u(\xi) = p$ *if* $\alpha = \xi + 1$, $t(\xi) = n$, $n > 0$.

(3) $\varphi_W^u(\alpha) = \varphi_C^u(\alpha)$ otherwise.

Proof. By transfinite induction on α.

If $\alpha = 0$, $\varphi_W^u(0) = s_B^\star = s_A^\star = \varphi_C^u(0)$.

Let $\alpha \in Ord$. Let us assume that the lemma is true for α, we show that it is true for $\alpha + 1$. If $\alpha \in Lim$ we have (1) and according to the construction (2) holds. If $\alpha \in Succ$, either (2) or (3) is true. If (3) then (3) holds again after reading one letter, because in this case $\mathscr{M}_B(q, \sigma) = \mathscr{M}_A(q, \sigma)$. If (2), $\alpha = \xi + 1$, $\varphi_W^u(\alpha + 1) = \mathscr{M}_B(\varphi_W^u(\alpha) = (p,{}_\xi u, t(\xi)),{}_\alpha u) = \mathscr{M}_A(\mathscr{M}_A(\varphi_C^u(\xi) = p,{}_\xi u),{}_\alpha u) = \varphi_C^u(\alpha + 1)$ so (3).

Now, let us assume that $\alpha \in Lim$ and that the lemma is true for $\beta < \alpha$. We show that this implies that the lemma is true for α.

Assume that $\varphi_C^u(\alpha) = \{p_1, \ldots, p_k\}$. We can suppose without loss of generality that $k = 1$. Let $\alpha = \beta + \omega^n$ be the unique decomposition of α. There exists an infinity of k such that $\varphi_C^u(\beta + \omega^{n-1} \cdot k) = p_1$ so $\{\beta + \omega^{n-1} \cdot k / k < \omega, \varphi_C^u(\beta + \omega^{n-1} \cdot k) = p_1\}$ is cofinal with α. By induction hypothesis, $p_1 = \varphi_C^u(\beta + \omega^{n-1} \cdot k) \Leftrightarrow \varphi_W^u(\beta + \omega^{n-1} \cdot k + 1) = (p_1,{}_{\beta + \omega^{n-1} \cdot k} u, n-1)$ so $\{\beta + \omega^{n-1} \cdot k + 1 / \varphi_W^u(\beta + \omega^{n-1} \cdot k + 1) = (p_1,{}_{\beta + \omega^{n-1} \cdot k} u, n-1)\}$ is cofinal with α, because Σ is a finite set. We can deduce $\exists \sigma \in \Sigma (p_1, \sigma, n-1) \in \varphi_W^u(\alpha)$.

Conversely, $(p_1, \sigma, n-1) \in \varphi_W^u(\alpha) \Rightarrow \{\gamma < \alpha / \varphi_W^u(\gamma) = (p_1, \gamma, n-1)\}$ is cofinal with α, that is to say, by induction hypothesis, that $\Gamma = \{\gamma < \alpha / \varphi_C^u(\gamma) = p_1\}$ is cofinal with α. As $p_1 \in [\mathscr{S}_A]^{n-1}$, $\forall \gamma \in \Gamma\, t(\gamma) = n-1$. According to Lemma 43 $\{\gamma \in \Gamma / \gamma = \beta + \omega^{n-1} \cdot k, k < \omega, \varphi_C^u(\gamma) = p_1\}$ is infinite so $p_1 \in \varphi_C^u(\alpha)$.

Let us assume now that $r = (q, \sigma, j) \in \varphi_C^u(\alpha)$ with $j > n-1$. Then $\{\gamma < \alpha / \varphi_W^u(\gamma) = r\}$ is cofinal with α, that is to say, by induction hypothesis, that $\Gamma = \{\gamma < \alpha / \varphi_C^u(\gamma) = q\}$ is cofinal with α. As $q \in [\mathscr{S}_A]^j$, $t(\gamma \in \Gamma) = j$, contradiction with Lemma 42.

Thus (1) is proved, so the lemma is proved. □

Theorem 45. *Let $\mathscr{A}$ be a deterministic $\mathscr{C}$-automaton and $\mathscr{B}$ be the corresponding C2W-automaton. Then $\mathscr{L}(\mathscr{A}) = \mathscr{L}(\mathscr{B})$.*

Proof. Let $\mathscr{A} = \langle \mathscr{S}_A, \mathscr{M}_A, s_A^\star, \mathscr{F}_A, \Sigma \rangle$, $\mathscr{B} = \langle \mathscr{S}_B, \mathscr{M}_B, s_B^\star, \mathscr{F}_B, \Sigma \rangle$, u an α-sequence such that $u \in \mathscr{L}(\mathscr{A})$, φ_C^u the run of $\mathscr{A}$ on u and φ_W^u the run of $\mathscr{B}$ on u. If $\alpha \in Lim$, $\alpha = \beta + \omega^n$, $\omega > n > 0$, let $\varphi_C^u(\alpha) = \{p_1, \ldots, p_k\}$. According to the construction, all the elements of $[\mathscr{S}_B]$ such that the longest line of the algorithm holds belong to $\mathscr{F}_B$ if $\{p_1, \ldots, p_k\} \in \mathscr{F}_A$, but $\varphi_W^u(\alpha)$ is one of these elements, so u is accepted by $\mathscr{B}$. If

$\alpha = \xi + 1$, $t(\xi) = n$, $\omega > n > 0$, $u \in \mathscr{L}(\mathscr{A}) \Rightarrow \varphi_C^u(\alpha) = \mathscr{M}_A(\varphi_C^u(\xi),_\xi u) \in \mathscr{F}_A$, following the construction $\mathscr{M}_A(\varphi_C^u(\xi),_\xi u) \in \mathscr{F}_A \Rightarrow (\varphi_C^u(\xi),_\xi u, n) \in \mathscr{F}_B$, and according to the previous Lemma $\varphi_W^u(\alpha) = (\varphi_C^u(\xi),_\xi u, n)$ so $u \in \mathscr{L}(\mathscr{B})$. Otherwise, $\varphi_C^u(\alpha) \in \mathscr{F}_A$ and as $\mathscr{F}_A \cap [\mathscr{S}_A]_0^1 \subseteq \mathscr{F}_B$ and $\varphi_C^u(\alpha) = \varphi_W^u(\alpha)$, u is accepted by $\mathscr{B}$.

The proof of correction can be made in a similar way. □

Corollary 46. *Nondeterministic $\mathscr{W}$-n-automaton are equivalent to deterministic ones.*

Proof. In order to determinize $\mathscr{W}$-n-automaton one have to build the corresponding W2C-automaton, to determinize it and to build the C2W-automaton from it, which is deterministic. □

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