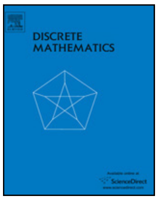
Discrete Mathematics 309 (2009) 4187–4191

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc



Note

# Pattern avoidance in “flattened” partitions

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ARTICLE INFO

*Article history:*
Received 13 August 2008
Received in revised form 27 October 2008
Accepted 6 November 2008
Available online 27 December 2008

*Keywords:*
Flatten partition
Pattern avoidance

ABSTRACT

To flatten a set partition (with apologies to *Mathematica* ®) means to form a permutation by erasing the dividers between its blocks. Of course, the result depends on how the blocks are listed. For the usual listing—increasing entries in each block and blocks arranged in increasing order of their first entries—we count the partitions of $[n]$ whose flattening avoids a single 3-letter pattern. Five counting sequences arise: a null sequence, the powers of 2, the Fibonacci numbers, the Catalan numbers, and the binomial transform of the Catalan numbers.

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## 1. Introduction

There is an extensive literature on pattern avoidance in permutations. Klazar [3–5] considered an analogous notion for set partitions and Sagan [7] introduced a second such notion based on restricted growth functions (see also [1,2]). Here we consider set partitions avoiding a permutation in the following sense. Suppose partitions $\Pi$ of $[n] = \{1, 2, \ldots, n\}$ are written in *standard increasing* form: increasing entries in each block and blocks arranged in increasing order of their first entries. Then we can define Flatten $(\Pi)$ to be the permutation of $[n]$ obtained by erasing the dividers between the blocks of $\Pi$. For example, $\Pi = 136\,/\,279\,/\,4\,/\,58$ is in standard increasing form and Flatten $(\Pi) = 136279458$. (The computer algebra system *Mathematica* ® implements this operation with the command `Flatten`). For a permutation $\pi$ on an initial segment of the positive integers (a pattern permutation) we say the partition $\Pi$ *avoids* $\pi$, or $\Pi$ is $\pi$-avoiding, if the permutation Flatten$(\Pi)$ avoids $\pi$ in the classical sense: a permutation $\sigma = (\sigma_i)_{i=1}^n$ of $[n]$ avoids a permutation $\pi = (\pi_i)_{i=1}^k$ of $[k]$ if there is no subsequence $(\sigma_{i_j})_{j=1}^k$ of $\sigma$ whose reduced form (replace smallest entry by 1, second smallest by 2, and so on) is $\pi$. We write $\Pi \vdash [n]$ if $\Pi$ is a partition of $[n]$. Set $\mathcal{A}(n; \pi) = \{\Pi \vdash [n] : \text{Flatten}\,(\Pi) \text{ avoids } \pi\}$. In Section 2, we count $\mathcal{A}(n; \pi)$ for all 3-letter pattern permutations $\pi$.

## 2. Set partitions avoiding a 3-letter pattern

### 2.1. 123-avoiding

This case is not very interesting; the counting sequence $(|\mathcal{A}(n; 123)|)_{n\geq 1}$ is $(1, 2, 1, 0, 0, 0, \ldots)$.

### 2.2. 132-avoiding

A partition $\Pi$ of $[n]$ is in $\mathcal{A}(n; 132)$ if and only if Flatten $(\Pi)$ is the identity permutation. This is because the first entry of Flatten $(\Pi)$ is always 1 and will be the ‘1’ of a 132 pattern unless Flatten $(\Pi)$ is an increasing sequence, that is, the identity permutation. So any subset of the $n - 1$ spaces between $1, 2, \ldots, n$ can serve as the dividers to form $\Pi$ and $|\mathcal{A}(n; 132)| = 2^{n-1}$.

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doi:10.1016/j.disc.2008.11.019

### 2.3. 213-avoiding

First, we claim a partition $\Pi$ of $[n]$ is in $\mathcal{A}(n;213)$ if and only if (i) the first block of $\Pi$ has the form $I \cup J$ with $I$ a nonempty initial segment of $[n]$ and $J$ a terminal segment of $[n]$ (possibly empty) disjoint from $I$, and (ii) the remaining blocks, when standardized, themselves form a 213-avoiding partition. (To *standardize* means to replace smallest entry by 1, second smallest by 2, and so on.)

Clearly, these two conditions are sufficient and condition (ii) is necessary. If condition (i) fails for $\Pi \in \mathcal{A}(n;213)$, let $a$ be the smallest element of $[n]$ not in the first block; $a$ is necessarily the first element of the second block. Because the condition fails there exist $b, c$ in $[n]$ with $c > b > a$, $b$ in the first block and $c$ in a later block. Hence $c$ occurs after $a$ and $bac$ is a 213-pattern in Flatten $(\Pi)$, a contradiction. So condition (i) is necessary also.

Now let $u(n) = |\mathcal{A}(n;213)|$ and set $u(n,k) = |\{\Pi \in \mathcal{A}(n;213) : \text{ first block of } \Pi \text{ has size } k\}|$. Clearly, $u(n,n) = 1$ and for $1 \le k \le n$, the first block is determined by $I$, and there are $k$ choices for $I$, namely, $([i])_{i=1}^{k}$. Hence we have the system of equations,

$$\begin{aligned}
&u(n,n) = 1 \quad \text{for } n \ge 1\\
&u(n,k) = ku(n-k) \quad \text{for } 1 \le k < n\\
&u(n) = \sum_{k=1}^{n} u(n,k) \quad \text{for } n \ge 1,
\end{aligned}$$

with solution involving the Fibonacci numbers ($F_{-1} := 1, F_0 = 0,\ F_1 = 1$)

$$\begin{aligned}
&u(n,j) = jF_{2n-2j-1} \quad \text{for } 1 \le j < n, \text{ and}\\
&u(n) = F_{2n-1}.
\end{aligned}$$

### 2.4. 231-avoiding

This case gives rise to the Catalan numbers via Touchard's identity [8],

$$C_n = \sum_{k\ge 0} \binom{n-1}{2k} 2^{n-1-2k} C_k. \tag{1}$$

For a permutation $p$ of $[n]$, a *descent terminator* is an entry smaller than its immediate predecessor and, by convention, the first entry is also considered a descent terminator. A *right-to-left (R–L) minimum* of $p$ is an entry smaller than all the entries after it. Clearly, for a partition in standard increasing form and its associated permutation, {descent terminators} $\subseteq$ {block initiators} $\subseteq$ {R–L minima}. For $\Pi \vdash [n]$, let $M(\Pi)$ denote the set of R–L minima of Flatten $(\Pi)$ that are not descent terminators, and set $\mathcal{A}(n,k\,;231) = \{\Pi \in \mathcal{A}(n;231) : |M(\Pi)| = k\}$. We claim $|\mathcal{A}(n,k\,;231)| = \binom{n-1}{k} 2^k C_{\frac{n-1-k}{2}}$ where $C_n$ is the Catalan number and $C_n := 0$ when $n$ is not an integer. Touchard's identity (1) then implies $|\mathcal{A}(n;231)| = C_n$.

To establish the claim, it suffices to show

$$|\mathcal{A}(n,0\,;231)| = C_{\frac{n-1}{2}} \quad \text{for } n \ge 1, \text{ and} \tag{2}$$

$$|\mathcal{A}(n,k\,;231)| = \binom{n-1}{k} 2^k |\mathcal{A}(n-k,0\,;231)| \quad \text{for } 1 \le k < n. \tag{3}$$

To show (2), let $\Pi \in \mathcal{A}(n,0\,;231)$. Then the R–L minima and descent terminators of $p :=$ Flatten $(\Pi)$ coincide. The last entry of $p$ is certainly an R–L minimum, hence a descent terminator, and so it must form a singleton block in $\Pi$. Each non-last block has length $\le 2$ because if $(a,b,c,\ldots)$ is a block of length $\ge 3$, then $bcd$ is a 231-pattern where $d$ is the first entry of the next block: certainly $b < c$ and we also have $d < b$ because if $b < d$, then $b <$ *all* entries that follow it. This would make $b$ an R–L minimum that was not a descent terminator, a contradiction. On the other hand, each non-last block has length $\ge 2$ because a non-last singleton block would imply that the first entry of the next block was an R–L minimum that was not a descent terminator. Hence all but the last block have length 2 and so $n$ is odd, say $n = 2r+1$, and $\Pi$ is of the form $a_1b_1/a_2b_2/\ldots/a_rb_r/a_{r+1}$.

Clearly, $a_1 = 1$. Also, $a_2 = 2$ because otherwise, since $a_2$ is an R–L minimum, 2 would occur to the left of $a_2$ and this would force $b_1 = 2$. But then $a_2$ would be an R–L minimum that was not a descent terminator. Next, we claim $a_{i+2} \le 2i+2$ for $1 \le i \le r-1$. Suppose contrariwise that $a_{i+2} > 2i+2$ for some $i$. Then none of $3, 4, \ldots, 2i+2$ can occur after $a_{i+2}$ because $a_{i+2}$ is an R–L minimum. This forces the first $i+1$ blocks to consist of the first $2i+2$ positive integers leaving $b_{i+1}$ an R–L minimum, which is not possible. Hence the sequence $(c_i)_{i=1}^{r-1}$ with $c_i := a_{i+2} - 2$ satisfies

$$1 \le c_1 < c_2 < \cdots < c_{r-1}, \quad \text{and} \quad c_i \le 2i \quad \text{for } 1 \le i \le r-1. \tag{4}$$

**Fig. 1.** Algorithmic construction of inverse map.

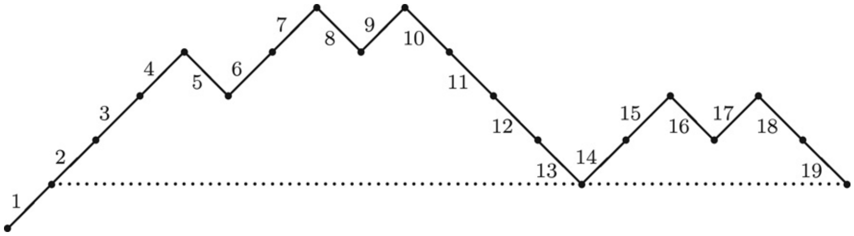

**Fig. 2.** Graphical construction of inverse map.

We have exhibited a map from $\mathcal{A}(2r+1, 0\,; 231)$ to sequences $(c_i)_{i=1}^{r-1}$ satisfying (4). This map is in fact a bijection and here is its inverse. Given such a sequence, for example with $r = 9$, $(c_i) = (1, 2, 4, 5, 7, 12, 13, 15)$, we can immediately recover the $a_i$'s and must determine the $b_i$'s (blank squares in Fig. 1).

Fill in the blank squares using $B = [2r+1] \setminus (a_i)_{i=1}^{r+1}$ from right to left as follows. Place the smallest element of $B$ that exceeds $a_{r+1}$ in the $b_r$ square and, in general, place the smallest not-yet-placed element of $B$ that exceeds $a_{i+1}$ in the $b_i$ square. The example has $B = \{5, 8, 10, 11, 12, 13, 16, 18, 19\}$, yielding $(b_i)_{i=1}^{r} = (13, 12, 5, 11, 8, 10, 19, 16, 18)$.

We remark that there is a nice graphical way to visualize the result of this algorithmic procedure using Dyck paths. Recall that the Catalan number $C_r$ counts sequences $(c_i)_{i=1}^{r-1}$ satisfying (4) [9, Ex. 6.19, item t]. Indeed, given a Dyck path of semilength $r$ let $c_i$ denote the number of steps preceding the $(i+1)$th upstep for $1 \le i \le r-1$. This is a bijection from Dyck $r$-paths to the sequences $(c_i)_{i=1}^{r-1}$ satisfying (4). So, sketch the Dyck path corresponding to the sequence $(c_i)_{i=1}^{r-1}$, prepend an upstep, and number all $2r+1$ steps in order from left to right, as in Fig. 2 for our running example.

Every upstep in a Dyck path has a matching downstep: the first one encountered directly east from the upstep or, more precisely, the terminal downstep of the shortest Dyck subpath starting at the upstep. The $a_i$'s are evident in the augmented Dyck path as the labels on the upsteps, and the $b_i$'s are also discernible: $b_i$ is the label on the matching downstep for the *next* upstep after $a_i$, $1 \le i \le r$. It is now clear that the $a_i$'s are increasing and that $a_i < b_i > a_{i+1}$ for $1 \le i \le r$; hence $(a_i)_{i=1}^{r+1}$ is both the set of R–L minima and the set of descent terminators in Flatten $(\Pi)$ and so $|M(\Pi)| = 0$. It is also easy to verify that $\Pi$ is 231-avoiding. Indeed, since all entries following $a_i$ are $> a_i$, the first two entries of a putative 231 pattern would have to be $b$'s, say $b_i < b_j$ with $i < j$, and $a_j$ would be the last upstep preceding $b_i$ (or else $b_j$ would be $< b_i$). Hence, for all $k > j$, upstep $a_k$ occurs after $b_i$ and so $b_k > a_k > b_i$ for $k > j$. Since $b_i$ is the '2' of the 231 pattern and we have just seen that all later entries are larger than $b_i$, no entry after $b_j$ can serve as the '1' of the pattern. We conclude that the partition $a_1 b_1 / a_2 b_2 / \cdots / a_r b_r / a_{r+1}$ is in $\mathcal{A}(n, 0\,; 231)$ as required.

To prove (3), consider $\Pi \in \mathcal{A}(n, k\,; 231)$. Let $K$ denote the set of R–L minima that are not descent terminators in Flatten $(\Pi)$. Thus $|K| = k$ and $K \subseteq [2, n]$. Let $L$ denote the set of elements in $K$ that initiate a block in $\Pi$. Thus $L \subseteq K$. Let $\Pi_0$ denote the partition obtained from $\Pi$ by deleting each element $i$ of $K$ from its block and, if $i$ is also in $L$, concatenating this block with the currently preceding block. Then $\Pi_0 \in \mathcal{A}(n-k, 0\,; 231)$. For example, $\Pi = 1 / 24 / 37 / 568$ yields $K = \{2, 6, 8\}$, $L = \{2\}$, and $\Pi_0 = \text{standardize}\,(14 / 37 / 5) = 13 / 25 / 4$. An example where three consecutive blocks are concatenated to form $\Pi_0$ is $\Pi = 1 / 2 / 35 / 4$ with $K = \{2, 3\}$, $L = \{3\}$, and $\Pi_0 = \text{standardize}\,(15 / 4) = 13 / 2$. We claim the map $\mathcal{A}(n, k\,; 231) \longrightarrow (K, L, \Pi_0)$ is a bijection to all triples $(K, L, \Pi_0)$ with $K$ a $k$-element subset of $[2, n]$, $L$ an arbitrary subset of $K$, and $\Pi_0$ a partition in $\mathcal{A}(n-k, 0\,; 231)$, and (3) then follows. To establish this claim, suppose given such a triple $(K, L, \Pi_0)$, and build up $\Pi$ as follows from $\Pi_0$. For each $a \in K$ in turn from smallest to largest, locate the last block in the current partition whose first entry is $< a$; then, to get the next partition, after adding 1 to each entry $\ge a$ insert $a$ into the located block at the appropriate position to ensure an increasing block. The end result will be a partition of $[n]$ in which the descent terminators are the block initiators and no element of $K$ is a block initiator. Finally, for each element of $K$ that is in the subset $L$, place a divider just before that element so that it initiates a block. This procedure yields $\Pi$ and shows that the map is invertible.

### 2.5. 312-avoiding

We claim a partition $\Pi$ of $[n]$ is in $\mathcal{A}(n; 312)$ if and only if (i) the first block of $\Pi$ is all of $[n]$ or has the form $I \setminus \{a\}$ where $I$ is an initial segment of $[n]$ of length $\ge 2$ and $a \ge 2$ is in $I$, and (ii) the remaining blocks, when standardized, themselves form a 312-avoiding partition.

The conditions are sufficient because if they hold and a 312 pattern involved the first block, then only the '3' could occur in the first block leaving the '1' and '2' to occur in later blocks. This however is impossible because at most one letter smaller than the '3' is missing from the first block. So we merely need to show that condition (i) is necessary. Suppose then that condition (i) is not met. Let $c$ denote the largest entry in the first block and $a$ the smallest letter missing from the first block.

Then by supposition there is a letter $b$ missing from the first block with $a < b < c$. Since $a$ must be the first entry of the second block, $b$ occurs after $a$ and $cab$ is a 312 pattern in Flatten ($\Pi$), a contradiction.

Now, if the first block has length $k < n$, there are exactly $k$ choices for $a$, namely, $2, 3, \ldots, k+1$. This observation leads to the very same recurrence relation as in the 213-avoiding case, and another Fibonacci counting sequence: $|\mathcal{A}(n; 312)| = F_{2n-1}$.

### 2.6. 321-avoiding

This case is counted by the binomial transform of the Catalan numbers: $|\mathcal{A}(n+1; 321)| = \sum_{k=0}^{n} \binom{n}{k} C_k$. Our proof is quite similar to that of the 231-avoiding case but with Touchard's identity replaced by the following one involving the Riordan numbers $R_n$,

$$\sum_{k=0}^{n} \binom{n}{k} 2^k R_{n-k} = \sum_{k=0}^{n} \binom{n}{k} C_k, \tag{5}$$

where $R_n := \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} C_j$. The identity (5) is easily proved by reversing the order of summation after substituting for $R_{n-k}$.

The Riordan number $R_n$ (A005043 in OEIS) is well known to count, among other things, Dyck $n$-paths with no short descents. (A 'descent' is a maximal sequence of contiguous downsteps and 'short' means of length 1.) Mimicking Section 2.4, define $\mathcal{A}(n, k; 321) = \{\Pi \in \mathcal{A}(n; 321) : |M(\Pi)| = k\}$. We claim $|\mathcal{A}(n, k; 321)| = \binom{n-1}{k} 2^k R_{n-1-k}$ for $0 \le k \le n-1$, and the identity (5) then implies $|\mathcal{A}(n; 321)| = \sum_{k=0}^{n-1} \binom{n-1}{k} C_k$.

To establish the claim, it suffices to show

$$|\mathcal{A}(n, 0; 321)| = R_{n-1} \tag{6}$$

$$|\mathcal{A}(n, k; 321)| = \binom{n-1}{k} 2^k |\mathcal{A}(n-k, 0; 321)| \quad \text{for } 1 \le k \le n-1 \tag{7}$$

To prove assertion (6) there is a bijection (essentially due to Krattenthaler [6]) from $\mathcal{A}(n, 0; 321)$ to Dyck $(n-1)$-paths with no short descents, illustrated with $n = 9$:

partition in $\mathcal{U}(n, 0; 321)$
136 / 278 / 49 / 5 →

erase dashes to
form 321 − avoiding
permutation $p$
136278495 →

form complement
$n + 1 - p$ of $p$
974832615 →

reverse $(n + 1 - p)$
516238479 →

delete last entry
(necessarily $n$)
51623847 →

list left-to-right
maxima $(m_i)_{i=1}^{k}$ and
their locations$(\ell_i)_{i=1}^{k}$
$\boldsymbol{m} = (5, 6, 8)$, $\boldsymbol{\ell} = (1, 3, 6)$ →

form differences
$a_i = m_i - m_{i-1}$ and $d_i = \ell_{i+1} - \ell_i$
($a_0 := 0$ and $\ell_{k+1} = n$)
$\boldsymbol{a} = (5, 1, 2)$, $\boldsymbol{d} = (2, 3, 3)$ →

form Dyck path with ascent lengths
$(a_i)_{i=1}^{k}$ and descent lengths $(d_i)_{i=1}^{k}$
*uuuuuddudddduuddd*

Bijection $\mathcal{A}(n, 0; 321) \longrightarrow$ to Dyck $(n-1)$ − paths with no short descents

The proof of assertion (7) uses the same bijection $\Pi \to (K, L, \Pi_0)$ as in the proof of (3) and is omitted.

It would be interesting to investigate permutation-avoidance for other canonical representations of a set partition where less familiar counting sequences seem to arise.

## Acknowledgements

I thank two anonymous referees for a careful reading of the paper and several helpful suggestions.

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