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Coherent configurations and triply regular association schemes obtained from spherical designs

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A B S T R A C T

Delsarte, Goethals and Seidel showed that if X is a spherical t-design with degree s satisfying $t \geqslant 2s-2$, X carries the structure of an association scheme. Also Bannai and Bannai showed that the same conclusion holds if X is an antipodal spherical t-design with degree s satisfying $t = 2s-3$. As a generalization of these results, we prove that a union of spherical designs with a certain property carries the structure of a coherent configuration. We derive triple regularity of tight spherical 4-, 5-, 7-designs, mutually unbiased bases, linked systems of symmetric designs with certain parameters.

1. Introduction

Spherical codes and designs were studied by Delsarte, Goethals and Seidel [11]. There are two important parameters of finite set X in the unit sphere S^{d-1}, that is, strength t and degree s. In the paper [11], it is shown that $t \geqslant 2s-2$ implies X carries an s-class association scheme. Recently Bannai and Bannai [2] have shown that if X is antipodal and $t = 2s-3$, then X carries an s-class association scheme.

Coherent configurations, that were introduced by D.G. Higman [12], are known as a generalization of association schemes. In Section 2, as an analogue of these results, we give a certain sufficient condition for a union of spherical designs to carry the structure of a coherent configuration. Our proof is based on the method of Delsarte, Goethals and Seidel [11, Theorem 7.4].

In Section 3, we consider triply regular association schemes which were introduced in connection with spin models by F. Jaeger [14] and have higher regularity than ordinary association schemes. Triple regularity is equivalent to the condition that the partition consisting of subconstituents rela-

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tive to any point of the association scheme carries a coherent configuration whose parameters are independent of the point. In order to show that a symmetric association scheme is triply regular, we embed the scheme to the unit sphere S^{d-1} by a primitive idempotent. This embedding has a partition of derived designs in S^{d-2} for arbitrary point in the association scheme. Applying the main theorem of this paper to the union of derived designs, we obtain a sufficient condition for triple regularity of a symmetric association scheme.

In Sections 3–6, we consider tight spherical 4, 5, 7-designs, mutually unbiased bases (MUB), and linked symmetric designs with certain parameters. We note that tight spherical t-designs are classified except for $t = 4, 5, 7$. It is known that a tight spherical design, MUB, and a linked system of symmetric designs carry a symmetric association scheme [11, Theorem 7.4], [2, Theorem 1.1], [18]. We will show that these symmetric association schemes are triply regular using our main theorem.

2. Coherent configurations obtained from spherical designs

Let X be a finite set, we define $\mathrm{diag}(X \times X) = \{(x, x) \mid x \in X\}$. Let $\{f_i\}_{i\in I}$ be a set of relations on X, we define $f_i^t = \{(y, x) \mid (x, y) \in f_i\}$. $(X, \{f_i\}_{i\in I})$ is a coherent configuration if the following properties are satisfied:

(1) $\{f_i\}_{i\in I}$ is a partition of $X \times X$,
(2) $f_i^t = f_{i^*}$ for some $i^* \in I$,
(3) $f_i \cap \mathrm{diag}(X \times X) \neq \emptyset$ implies $f_i \subset \mathrm{diag}(X \times X)$,
(4) for $i, j, k \in I$, the number $|\{z \in X \mid (x, z) \in f_i,\ (z, y) \in f_j\}|$ is independent of the choice of $(x, y) \in f_k$.

If moreover $f_0 = \mathrm{diag}(X \times X)$ and $i^* = i$ for all $i \in I$, then we call $(X, \{f_i\}_{i\in I})$ a symmetric association scheme.

Let $X_1, \dots, X_n$ be finite subsets of S^{d-1}. We denote by $\coprod_{i=1}^n X_i$ the disjoint union of $X_1, \dots, X_n$. We denote by $\langle x, y\rangle$ the inner product of $x, y \in \mathbb{R}^d$. We define the nontrivial angle set $A(X_i, X_j)$ between X_i and X_j by

$$A(X_i, X_j) = \bigl\{\langle x, y\rangle \bigm| x \in X_i,\ y \in X_j,\ x \neq \pm y\bigr\},$$

and the angle set $A'(X_i, X_j)$ between X_i and X_j by

$$A'(X_i, X_j) = \bigl\{\langle x, y\rangle \bigm| x \in X_i,\ y \in X_j,\ x \neq y\bigr\}.$$

If $i = j$, then $A(X_i, X_i)$ (resp. $A'(X_i, X_i)$) is abbreviated $A(X_i)$ (resp. $A'(X_i)$).

We define the intersection numbers on X_j for $x, y \in S^{d-1}$ by

$$p^j_{\alpha,\beta}(x, y) = \bigl|\bigl\{z \in X_j \bigm| \langle x, z\rangle = \alpha,\ \langle y, z\rangle = \beta\bigr\}\bigr|.$$

For a positive integer t, a finite nonempty set X in the unit sphere S^{d-1} is called a spherical t-design in S^{d-1} if the following condition is satisfied:

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \frac{1}{|S^{d-1}|}\int\limits_{S^{d-1}} f(x)\,d\sigma(x)$$

for all polynomials $f(x) = f(x_1, \dots, x_d)$ of degree not exceeding t. Here $|S^{d-1}|$ denotes the volume of the sphere S^{d-1}. When X is a t-design and not a $(t+1)$-design, we call t its strength.

We define the Gegenbauer polynomials $\{Q_k(x)\}_{k=0}^{\infty}$ on S^{d-1} by

$$Q_0(x) = 1, \qquad Q_1(x) = dx,$$
$$\frac{k+1}{d+2k}Q_{k+1}(x) = xQ_k(x) - \frac{d+k-3}{d+2k-4}Q_{k-1}(x).$$

Let $\mathrm{Harm}(\mathbb{R}^d)$ be the vector space of the harmonic polynomials over $\mathbb{R}$ and $\mathrm{Harm}_l(\mathbb{R}^d)$ be the subspace of $\mathrm{Harm}(\mathbb{R}^d)$ consisting of homogeneous polynomials of total degree l. Let $\{\phi_{l,1},\dots,\phi_{l,h_l}\}$ be an orthonormal basis of $\mathrm{Harm}_l(\mathbb{R}^d)$ with respect to the inner product

$$\langle\phi,\psi\rangle=\frac{1}{|S^{d-1}|}\int_{S^{d-1}}\phi(x)\psi(x)\,d\sigma(x).$$

Then the addition formula for the Gegenbauer polynomial holds [11, Theorem 3.3]:

Lemma 2.1. $\sum_{i=1}^{h_l}\phi_{l,i}(x)\phi_{l,i}(y)=Q_l(\langle x,y\rangle)$ *for any* $l\in\mathbb{N}$, $x,y\in S^{d-1}$.

We define the l-th characteristic matrix of a finite set $X\subset S^{d-1}$ as the $|X|\times h_l$ matrix

$$H_l=\bigl(\phi_{l,i}(x)\bigr)_{\substack{x\in X\\ 1\leqslant i\leqslant h_l}}.$$

A criterion for t-designs using Gegenbauer polynomials and the characteristic matrices is known [11, Theorems 5.3, 5.5].

Lemma 2.2. *Let X be a finite set in S^{d-1}. The following conditions are equivalent*:

(1) *X is a t-design,*
(2) $\sum_{x,y\in X}Q_k(\langle x,y\rangle)=0$ *for any* $k\in\{1,\dots,t\}$,
(3) $H_k^tH_l=\delta_{k,l}|X|I$ *for* $0\leqslant k+l\leqslant t$.

We define $\{f_{\lambda,l}\}_{l=0}^{\lambda}$ as the coefficients of Gegenbauer expansion of x^λ for any nonnegative integers λ, i.e., $x^\lambda=\sum_{l=0}^{\lambda}f_{\lambda,l}Q_l(x)$, and let $F_{\lambda,\mu}(x)=\sum_{l=0}^{\min\{\lambda,\mu\}}f_{\lambda,l}f_{\mu,l}Q_l(x)$, where λ,μ are nonnegative integers.

The following three lemmas are used to prove Theorem 2.6 by using uniqueness of the solution of linear equations. Let A be a square matrix of size n. For index sets $I,J\subset\{1,\dots,n\}$, we denote the submatrix that lies in the rows of A indexed by I and the columns indexed by J as $A(I,J)$ and the complement of I as I'. If $I=\{i\}$ and $J=\{j\}$, then $A(I,J)$ is abbreviated $A(i,j)$. A lemma which relates a minor of A^{-1} to that of A is the following:

Lemma 2.3. *(See [13, p. 21].) Let A be a nonsingular matrix, and let I, J be index sets of rows and columns of A with $|I|=|J|$. Then*

$$\det A^{-1}\bigl(I',J'\bigr)=(-1)^{\sum_{i\in I}i+\sum_{j\in J}j}\frac{\det A(J,I)}{\det A}.$$

We define the k-th elementary symmetric polynomial $e_k(x_1,\dots,x_n)$ in n valuables $x_1,\dots,x_n$ by

$$e_k(x_1,\dots,x_n)=\begin{cases}1 & \text{if } k=0,\\ \sum_{1\leqslant i_1<\dots<i_k\leqslant n}x_{i_1}x_{i_2}\cdots x_{i_k} & \text{if } k\geqslant 1.\end{cases}$$

We define the polynomial $a_\lambda(x_1,\dots,x_n)$ for a partition $\lambda=(\lambda_1,\dots,\lambda_n)$ by

$$a_\lambda(x_1,\dots,x_n)=\sum_{\sigma\in S_n}\epsilon(\sigma)x_{\sigma(1)}^{\lambda_1}\cdots x_{\sigma(n)}^{\lambda_n},$$

and the Schur function $S_\lambda(x_1,\dots,x_n)$ by

$$S_\lambda(x_1,\dots,x_n)=\frac{a_{\lambda+\delta}(x_1,\dots,x_n)}{a_\lambda(x_1,\dots,x_n)},$$

where $\delta=(n-1,n-2,\dots,1,0)$.

Lemma 2.4. *Let A be a square matrix of order n with (i,j) entry α_j^{i-1}, where $\alpha_1,\dots,\alpha_n$ are distinct. Then*

$$A^{-1}(i,j)=(-1)^{i+j}\frac{e_{n-j}(\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n)}{\prod_{1\leqslant k<i}(\alpha_i-\alpha_k)\prod_{i<l\leqslant n}(\alpha_l-\alpha_i)}.$$

Proof. Putting $\lambda=(1^{n-j},0^{j-1})$, we have by [16, p. 42],

$$\begin{aligned}A^{-1}(i,j)&=(-1)^{i+j}\frac{\det A(\{j\}',\{i\}')}{\det A}\\&=(-1)^{i+j}\frac{a_{\lambda+\delta}(\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n)}{\det A}\\&=\frac{(-1)^{i+j}}{\prod_{1\leqslant k<i}(\alpha_i-\alpha_k)\prod_{i<l\leqslant n}(\alpha_l-\alpha_i)}\frac{a_{\lambda+\delta}(\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n)}{a_{\delta}(\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n)}\\&=\frac{(-1)^{i+j}}{\prod_{1\leqslant k<i}(\alpha_i-\alpha_k)\prod_{i<l\leqslant n}(\alpha_l-\alpha_i)}S_\lambda(\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n)\\&=\frac{(-1)^{i+j}}{\prod_{1\leqslant k<i}(\alpha_i-\alpha_k)\prod_{i<l\leqslant n}(\alpha_l-\alpha_i)}e_{n-j}(\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n).\end{aligned}$$ □

Lemma 2.5. *Let A be a square matrix of order n with (i,j) entry α_j^{i-1} and let B be a square matrix of order m with (i,j) entry β_j^{i-1}, where $\alpha_1,\dots,\alpha_n$ and $\beta_1,\dots,\beta_m$ are distinct. Let J,I be index sets of rows and columns, respectively, of $A\otimes B$ such that $J'=\{(n-1,m),(n,m-1),(n,m)\}$, $I'=\{(i_1,j_1),(i_2,j_2),(i_3,j_3)\}$. Then*

$$\begin{aligned}&\frac{\det(A\otimes B)(J,I)}{\det A\otimes B}\\&\quad=\pm\frac{\alpha_{i_1}\beta_{j_2}+\alpha_{i_2}\beta_{j_3}+\alpha_{i_3}\beta_{j_1}-\alpha_{i_1}\beta_{j_3}-\alpha_{i_2}\beta_{j_1}-\alpha_{i_3}\beta_{j_2}}{\prod_{1\leqslant r\leqslant 3}(\prod_{1\leqslant k<i_r}(\alpha_{i_r}-\alpha_k)\prod_{i_r<l\leqslant n}(\alpha_l-\alpha_{i_r})\prod_{1\leqslant k<j_r}(\beta_{j_r}-\beta_k)\prod_{j_r<l\leqslant m}(\beta_l-\beta_{j_r}))}.\end{aligned}$$

Proof. We define $f(i,j)=\prod_{1\leqslant k<i}(\alpha_i-\alpha_k)\prod_{i<l\leqslant n}(\alpha_l-\alpha_i)\prod_{1\leqslant k<j}(\beta_j-\beta_k)\prod_{j<l\leqslant m}(\beta_l-\beta_j)$. Using Lemmas 2.3 and 2.4,

$$\begin{aligned}&\frac{\det(A\otimes B)(J,I)}{\det A\otimes B}\\&\quad=\pm\det(A\otimes B)^{-1}\big(I',J'\big)\\&\quad=\pm\det\big(A^{-1}\otimes B^{-1}\big)\big(I',J'\big)\\&\quad=\pm\det\begin{pmatrix}\frac{(-1)^{i_1+n-1+j_1+m}\sum_{i\neq i_1}\alpha_i}{f(i_1,j_1)} & \frac{(-1)^{i_1+n+j_1+m-1}\sum_{j\neq j_1}\beta_j}{f(i_1,j_1)} & \frac{(-1)^{i_1+n+j_1+m}}{f(i_1,j_1)}\\ \frac{(-1)^{i_2+n-1+j_2+m}\sum_{i\neq i_2}\alpha_i}{f(i_2,j_2)} & \frac{(-1)^{i_2+n+j_2+m-1}\sum_{j\neq j_2}\beta_j}{f(i_2,j_2)} & \frac{(-1)^{i_2+n+j_2+m}}{f(i_2,j_2)}\\ \frac{(-1)^{i_3+n-1+j_3+m}\sum_{i\neq i_3}\alpha_i}{f(i_3,j_3)} & \frac{(-1)^{i_3+n+j_3+m-1}\sum_{j\neq j_3}\beta_j}{f(i_3,j_3)} & \frac{(-1)^{i_3+n+j_3+m}}{f(i_3,j_3)}\end{pmatrix}\\&\quad=\pm\frac{1}{\prod_{1\leqslant r\leqslant 3}f(i_r,j_r)}\det\begin{pmatrix}\sum_{i\neq i_1}\alpha_i & \sum_{j\neq j_1}\beta_j & 1\\ \sum_{i\neq i_2}\alpha_i & \sum_{j\neq j_2}\beta_j & 1\\ \sum_{i\neq i_3}\alpha_i & \sum_{j\neq j_3}\beta_j & 1\end{pmatrix}\\&\quad=\pm\frac{1}{\prod_{1\leqslant r\leqslant 3}f(i_r,j_r)}\det\begin{pmatrix}\alpha_{i_1} & \beta_{j_1} & 1\\ \alpha_{i_2} & \beta_{j_2} & 1\\ \alpha_{i_3} & \beta_{j_3} & 1\end{pmatrix}\\&\quad=\pm\frac{\alpha_{i_1}\beta_{j_2}+\alpha_{i_2}\beta_{j_3}+\alpha_{i_3}\beta_{j_1}-\alpha_{i_1}\beta_{j_3}-\alpha_{i_2}\beta_{j_1}-\alpha_{i_3}\beta_{j_2}}{\prod_{1\leqslant r\leqslant 3}(\prod_{1\leqslant k<i_r}(\alpha_{i_r}-\alpha_k)\prod_{i_r<l\leqslant n}(\alpha_l-\alpha_{i_r})\prod_{1\leqslant k<j_r}(\beta_{j_r}-\beta_k)\prod_{j_r<l\leqslant m}(\beta_l-\beta_{j_r}))}.\end{aligned}$$

□

The following is the main theorem of this paper.

Theorem 2.6. *Let $X_i \subset S^{d-1}$ be a spherical t_i-design for $i \in \{1, \dots, n\}$. Assume that $X_i \cap X_j = \emptyset$ or $X_i = X_j$, and $X_i \cap (-X_j) = \emptyset$ or $X_i = -X_j$ for $i, j \in \{1, \dots, n\}$. Let $s_{i,j} = |A(X_i, X_j)|$, $s^*_{i,j} = |A'(X_i, X_j)|$ and $A(X_i, X_j) = \{\alpha^1_{i,j}, \dots, \alpha^{s_{i,j}}_{i,j}\}$, $\alpha^0_{i,j} = 1$, when $-1 \in A'(X_i, X_j)$, we define $\alpha^{s^*_{i,j}}_{i,j} = -1$. We define $R^k_{i,j} = \{(x, y) \in X_i \times X_j \mid \langle x, y\rangle = \alpha^k_{i,j}\}$. If one of the following holds depending on the choice of $i, j, k \in \{1, \dots, n\}$:*

(1) $s_{i,j} + s_{j,k} - 2 \leqslant t_j$,
(2) $s_{i,j} + s_{j,k} - 3 = t_j$ *and for any $\gamma \in A(X_i, X_k)$ there exist $\alpha \in A(X_i, X_j)$, $\beta \in A(X_j, X_k)$ such that the number $p^j_{\alpha,\beta}(x, y)$ is independent of the choice of $x \in X_i$, $y \in X_k$ with $\gamma = \langle x, y\rangle$,*
(3) $s_{i,j} + s_{j,k} - 4 = t_j$ *and for any $\gamma \in A(X_i, X_k)$ there exist $\alpha, \alpha' \in A(X_i, X_j)$, $\beta, \beta' \in A(X_j, X_k)$ such that $\alpha \neq \alpha'$, $\beta \neq \beta'$ and the numbers $p^j_{\alpha,\beta}(x, y)$, $p^j_{\alpha,\beta'}(x, y)$ and $p^j_{\alpha',\beta}(x, y)$ are independent of the choice of $x \in X_i$, $y \in X_k$ with $\gamma = \langle x, y\rangle$,*

then $(\coprod_{i=1}^n X_i, \{R^k_{i,j} \mid 1 \leqslant i, j \leqslant n,\ 1 - \delta_{X_i,X_j} \leqslant k \leqslant s^*_{i,j}\})$ is a coherent configuration. The parameters of this coherent configuration are determined by $A(X_i, X_j)$, $|X_i|$, t_i, δ_{X_i,X_j}, $\delta_{X_i,-X_j}$, and when $s_{i,j} + s_{j,k} - 3 = t_j$ (resp. $s_{i,j} + s_{j,k} - 4 = t_j$), the numbers $p^j_{\alpha,\beta}(x, y)$ (resp. $p^j_{\alpha,\beta}(x, y)$, $p^j_{\alpha',\beta}(x, y)$, $p^j_{\alpha,\beta'}(x, y)$) which are assumed be independent of (x, y) with $\langle x, y\rangle = \gamma$.

Proof. Let $x \in X_i$, $y \in X_k$ be such that $\gamma = \langle x, y\rangle$. It is sufficient to show that the number $p^j_{\alpha,\beta}(x, y)$ depends only on γ and does not depend on the choice of $x \in X_i$, $y \in X_k$ satisfying $\gamma = \langle x, y\rangle$.

For the ease of notation, let $\alpha_l = \alpha^l_{i,j}$ and $\beta_m = \alpha^m_{j,k}$.

We define a mapping $\phi_l : S^{d-1} \to \mathbb{R}^{h_l}$ by $\phi_l(x) = (\varphi_{l,1}(x), \dots, \varphi_{l,h_l}(x))$. Let H_l be the l-th characteristic matrix of X_j. For any nonnegative integers λ and μ satisfying $\lambda + \mu \leqslant t_j$, we calculate

$$\left(\sum_{l=1}^{\lambda} f_{\lambda,l}\phi_l(x)H_l^t\right)\left(\sum_{m=1}^{\mu} f_{\mu,m}H_m\phi_m(y)^t\right)$$

in two different ways.

First we use Lemma 2.2 and Lemma 2.1 in turn, to obtain the following equality:

$$\begin{aligned}\left(\sum_{l=1}^{\lambda} f_{\lambda,l}\phi_l(x)H_l^t\right)\left(\sum_{m=1}^{\mu} f_{\mu,m}H_m\phi_m(y)^t\right) &= |X_j|\sum_{l=1}^{\min\{\lambda,\mu\}} f_{\lambda,l}f_{\mu,l}\phi_l(x)\phi_l(y)^t \\ &= |X_j|\sum_{l=1}^{\min\{\lambda,\mu\}} f_{\lambda,l}f_{\mu,l}Q_l\big(\langle x, y\rangle\big) \\ &= |X_j|F_{\lambda,\mu}\big(\langle x, y\rangle\big). \qquad (2.1)\end{aligned}$$

Next using Lemma 2.1, we obtain the following equality:

$$\begin{aligned}&\left(\sum_{l=1}^{\lambda} f_{\lambda,l}\phi_l(x)H_l^t\right)\left(\sum_{m=1}^{\mu} f_{\mu,m}H_m\phi_m(y)^t\right) \\ &= \sum_{z\in X_j}\left(\sum_{l=1}^{\lambda} f_{\lambda,l}\phi_l(x)\phi_l(z)^t\right)\left(\sum_{m=1}^{\mu} f_{\mu,m}\phi_m(z)\phi_m(y)^t\right) \\ &= \sum_{z\in X_j}\left(\sum_{l=1}^{\lambda} f_{\lambda,l}Q_l\big(\langle x, z\rangle\big)\right)\left(\sum_{m=1}^{\mu} f_{\mu,m}Q_m\big(\langle z, y\rangle\big)\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{z\in X_j} \langle x,z\rangle^\lambda \langle z,y\rangle^\mu \\
&= \sum_{\substack{\alpha\in A'(X_i,X_j)\\ \beta\in A'(X_j,X_k)}} \alpha^\lambda\beta^\mu p^j_{\alpha,\beta}(x,y) + p^j_{1,1}(x,y) + \sum_{m=1}^{s^*_{j,k}} \beta_m^\mu p^j_{1,\beta_m}(x,y) + \sum_{l=1}^{s^*_{i,j}} \alpha_l^\lambda p^j_{\alpha_l,1}(x,y) \\
&= \sum_{l=1}^{s_{i,j}}\sum_{m=1}^{s_{j,k}} \alpha_l^\lambda\beta_m^\mu p^j_{\alpha_l,\beta_m}(x,y) \\
&\quad + p^j_{1,1}(x,y) + (-1)^\mu p^j_{1,-1}(x,y) + (-1)^\lambda p^j_{-1,1}(x,y) + (-1)^\lambda(-1)^\mu p^j_{-1,-1}(x,y) \\
&\quad + \sum_{m=1}^{s_{j,k}} \beta_m^\mu p^j_{1,\beta_m}(x,y) + \sum_{l=1}^{s_{i,j}} \alpha_l^\lambda p^j_{\alpha_l,1}(x,y) + \sum_{m=1}^{s_{j,k}} (-1)^\lambda \beta_m^\mu p^j_{-1,\beta_m}(x,y) \\
&\quad + \sum_{l=1}^{s_{i,j}} \alpha_l^\lambda (-1)^\mu p^j_{\alpha_l,-1}(x,y) \\
&= \sum_{l=1}^{s_{i,j}}\sum_{m=1}^{s_{j,k}} \alpha_l^\lambda\beta_m^\mu p^j_{\alpha_l,\beta_m}(x,y) + G^{i,j,k}_{\lambda,\mu}(\gamma),
\end{aligned}
\tag{2.2}
$$

where

$$
\begin{aligned}
G^{i,j,k}_{\lambda,\mu}(t) = {}& \delta_{1,t}\delta_{X_i,X_j}\delta_{X_j,X_k} + (-1)^\mu \delta_{-1,t}\delta_{X_i,X_j}\delta_{X_j,-X_k} \\
& + (-1)^\lambda \delta_{-1,t}\delta_{X_i,-X_j}\delta_{X_j,X_k} + (-1)^{\lambda+\mu}\delta_{1,t}\delta_{X_i,-X_j}\delta_{X_j,-X_k} \\
& + (1-\delta_{1,t})(1-\delta_{-1,t})\big(\delta_{X_i,X_j} t^\mu + \delta_{X_j,X_k} t^\lambda + \delta_{X_i,-X_j}(-1)^\lambda(-t)^\mu \\
& + \delta_{X_j,-X_k}(-t)^\lambda(-1)^\mu\big).
\end{aligned}
$$

We obtain from (2.1) and (2.2):

$$
\sum_{l=1}^{s_{i,j}}\sum_{m=1}^{s_{j,k}} \alpha_l^\lambda\beta_m^\mu p^j_{\alpha_l,\beta_m}(x,y) = |X_j| F_{\lambda,\mu}\big(\langle x,y\rangle\big) - G^{i,j,k}_{\lambda,\mu}\big(\langle x,y\rangle\big).
\tag{2.3}
$$

In the case where i, j, k satisfy the assumption (1), for $0 \leqslant \lambda \leqslant s_{i,j}-1$ and $0 \leqslant \mu \leqslant s_{j,k}-1$, (2.3) yields a system of $s_{i,j}s_{j,k}$ linear equations whose unknowns are

$$
\big\{ p^j_{\alpha_l,\beta_m}(x,y) \;\big|\; 1 \leqslant l \leqslant s_{i,j},\ 1 \leqslant m \leqslant s_{j,k} \big\}.
$$

Its coefficient matrix $A \otimes B$ is nonsingular, where

$$
A = \begin{pmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_{s_{i,j}} \\ \vdots & \ddots & \vdots \\ \alpha_1^{s_{i,j}-1} & \cdots & \alpha_{s_{i,j}}^{s_{i,j}-1} \end{pmatrix}, \qquad
B = \begin{pmatrix} 1 & \cdots & 1 \\ \beta_1 & \cdots & \beta_{s_{j,k}} \\ \vdots & \ddots & \vdots \\ \beta_1^{s_{j,k}-1} & \cdots & \beta_{s_{j,k}}^{s_{j,k}-1} \end{pmatrix}.
$$

Therefore $p^j_{\alpha_l,\beta_m}(x,y)$ for $1 \leqslant l \leqslant s_{i,j}$, $1 \leqslant m \leqslant s_{j,k}$ depends only on γ and does not depend on the choice of x, y satisfying $\gamma = \langle x,y\rangle$, and is determined by $A(X_i,X_j)$, $A(X_j,X_k)$, γ, $|X_j|$, t_j, δ_{X_i,X_j}, δ_{X_j,X_k}, $\delta_{X_i,-X_j}$, $\delta_{X_j,-X_k}$.

In the case where i, j, k satisfy (2) i.e., for $\langle x,y\rangle = \gamma \in A(X_i,X_k)$, there exist $\alpha_{l^*} \in A(X_i,X_j)$, $\beta_{m^*} \in A(X_j,X_k)$ such that the number $p^j_{\alpha_{l^*},\beta_{m^*}}(x,y)$ is uniquely determined. The linear equation (2.3) is the following:

$$\sum_{\substack{1\leqslant l\leqslant s_{i,j}\\ 1\leqslant m\leqslant s_{j,k}\\ (l,m)\neq(l^*,m^*)}} \alpha_l^\lambda \beta_m^\mu p^j_{\alpha_l,\beta_m}(x,y) = |X_j| F_{\lambda,\mu}\big(\langle x,y\rangle\big) - G^{i,j,k}_{\lambda,\mu}\big(\langle x,y\rangle\big) - \alpha_{l^*}^\lambda \beta_{m^*}^\mu p^j_{\alpha_{l^*},\beta_{m^*}}(x,y). \tag{2.4}$$

For $0\leqslant\lambda\leqslant s_{i,j}-1$, $0\leqslant\mu\leqslant s_{j,k}-1$ and $(\lambda,\mu)\neq(s_{i,j}-1,s_{j,k}-1)$, (2.4) yields a system of $s_{i,j}s_{j,k}-1$ linear equations whose unknowns are

$$\big\{p^j_{\alpha_l,\beta_m}(x,y)\ \big|\ 1\leqslant l\leqslant s_{i,j},\ 1\leqslant m\leqslant s_{j,k},\ (l,m)\neq\big(l^*,m^*\big)\big\}.$$

The coefficient matrix C_1 of these linear equations is the submatrix obtained by deleting the $(s_{i,j},s_{j,k})$-row and (l^*,m^*)-column of $A\otimes B$. Using Lemma 2.4 the determinant of C_1 is, up to sign,

$$\begin{aligned}\det C_1 &= \pm\big((s_{i,j},s_{j,k}),\big(l^*,m^*\big)\big)\text{-cofactor of } A\otimes B\\ &= \pm\big(\big(l^*,m^*\big),\big((s_{i,j},s_{j,k})\big)\text{-entry of } (A\otimes B)^{-1}\big)\det A\otimes B\\ &= \pm\big(\big(l^*,s_{i,j}\big)\text{-entry of } A^{-1}\big)\times\big(\big(m^*,s_{j,k}\big)\text{-entry of } B^{-1}\big)\det A\otimes B\\ &= \pm\frac{\det A\otimes B}{\prod_{1\leqslant k<l^*}(\alpha_{l^*}-\alpha_k)\prod_{l^*<l\leqslant s_{i,j}}(\alpha_l-\alpha_{l^*})\prod_{1\leqslant k<m^*}(\beta_{m^*}-\beta_k)\prod_{m^*<l\leqslant s_{j,k}}(\beta_l-\beta_{m^*})}.\end{aligned}$$

Hence C_1 is nonsingular.

Therefore $p^j_{\alpha_l,\beta_m}(x,y)$ for $1\leqslant l\leqslant s_{i,j}$, $1\leqslant m\leqslant s_{j,k}$, $(l,m)\neq(l^*,m^*)$ depends only on γ and does not depend on the choice of x, y satisfying $\gamma=\langle x,y\rangle$, and is determined by $A(X_i,X_j)$, $A(X_j,X_k)$, γ, $|X_j|$, t_j, δ_{X_i,X_j}, δ_{X_j,X_k}, $\delta_{X_i,-X_j}$, $\delta_{X_j,-X_k}$, the number $p^j_{\alpha_{l^*},\beta_{m^*}}(x,y)$ which is assumed be independent of (x,y) with $\langle x,y\rangle=\gamma$.

In the case where i,j,k satisfy (3) i.e., for $\langle x,y\rangle=\gamma\in A(X_i,X_k)$ there exist $\alpha_{l_1},\alpha_{l_2}\in A(X_i,X_j)$, $\beta_{m_1},\beta_{m_2}\in A(X_j,X_k)$ such that the numbers $p^j_{\alpha_{l_1},\beta_{m_1}}(x,y)$, $p^j_{\alpha_{l_1},\beta_{m_2}}(x,y)$, $p^j_{\alpha_{l_2},\beta_{m_1}}(x,y)$ are uniquely determined. The linear equation (2.3) is the following:

$$\begin{aligned}&\sum_{\substack{1\leqslant l\leqslant s_{i,j}\\ 1\leqslant m\leqslant s_{j,k}\\ (l,m)\neq(l_1,m_1),(l_1,m_2),(l_2,m_1)}} \alpha_l^\lambda\beta_m^\mu p^j_{\alpha_l,\beta_m}(x,y)\\ &\quad= |X_j|F_{\lambda,\mu}\big(\langle x,y\rangle\big) - G^{i,j,k}_{\lambda,\mu}\big(\langle x,y\rangle\big) - \alpha_{l_1}^\lambda\beta_{m_1}^\mu p^j_{\alpha_{l_1},\beta_{m_1}}(x,y)\\ &\qquad - \alpha_{l_1}^\lambda\beta_{m_2}^\mu p^j_{\alpha_{l_1},\beta_{m_2}}(x,y) - \alpha_{l_2}^\lambda\beta_{m_1}^\mu p^j_{\alpha_{l_2},\beta_{m_1}}(x,y).\end{aligned}\tag{2.5}$$

For $0\leqslant\lambda\leqslant s_{i,j}-1$, $0\leqslant\mu\leqslant s_{j,k}-1$ and $(\lambda,\mu)\neq(s_{i,j}-2,s_{j,k}-1),(s_{i,j}-1,s_{j,k}-2),(s_{i,j}-1,s_{j,k}-1)$, (2.5) yields a system of $s_{i,j}s_{j,k}-3$ linear equations whose unknowns are

$$\big\{p^j_{\alpha_l,\beta_m}(x,y)\ \big|\ 1\leqslant l\leqslant s_{i,j},\ 1\leqslant m\leqslant s_{j,k},\ (l,m)\neq(l_1,m_1),(l_1,m_2),(l_2,m_1)\big\}.$$

The coefficient matrix C_2 of these linear equations is the submatrix obtained by deleting the $(s_{i,j}-1,s_{j,k})$, $(s_{i,j},s_{j,k}-1)$, $(s_{i,j},s_{j,k})$-rows and (l_1,m_1), (l_1,m_2), (l_2,m_1)-columns of $A\otimes B$. Let J, I be index sets of rows and columns, respectively, of $A\otimes B$ such that

$$J'=\big\{(s_{i,j}-1,s_{j,k}),(s_{i,j},s_{j,k}-1),(s_{i,j},s_{j,k})\big\}$$

and

$$I'=\big\{(l_1,m_1),(l_1,m_2),(l_2,m_1)\big\}.$$

Setting $(i_1,j_1),(i_2,j_2),(i_3,j_3)$ to be $(l_1,m_1),(l_1,m_2),(l_2,m_1)$ respectively, we have

$$\alpha_{i_1}\beta_{j_2}+\alpha_{i_2}\beta_{j_3}+\alpha_{i_3}\beta_{j_1}-\alpha_{i_1}\beta_{j_3}-\alpha_{i_2}\beta_{j_1}-\alpha_{i_3}\beta_{j_2}=(\alpha_{l_1}-\alpha_{l_2})(\beta_{m_1}-\beta_{m_2}).$$

Hence C_2 is nonsingular by Lemma 2.5. Therefore $p^j_{\alpha_l,\beta_m}(x,y)$ for $1 \leqslant l \leqslant s_{i,j}$, $1 \leqslant m \leqslant s_{j,k}$, $(l,m) \neq (l_1,m_1),(l_1,m_2),(l_2,m_1)$ depends only on γ and does not depend on the choice of x, y satisfying $\gamma = \langle x,y\rangle$, and is determined by $A(X_i,X_j)$, $A(X_j,X_k)$, γ, $|X_j|$, t_j, δ_{X_i,X_j}, δ_{X_j,X_k}, $\delta_{X_i,-X_j}$, $\delta_{X_j,-X_k}$, the numbers $p^j_{\alpha,\beta}(x,y)$, $p^j_{\alpha',\beta}(x,y)$, $p^j_{\alpha,\beta'}(x,y)$ which are assumed be independent of (x,y) with $\langle x,y\rangle = \gamma$. □

Several results are known for the case $n=1$ are derived from Theorem 2.6. We consider the case where $n=1$ and $X=X_1$ is a t-design of degree s. Then $t_1=t$ and

$$s_{1,1} = \begin{cases} s-1 & \text{if } X \text{ is antipodal,} \\ s & \text{if } X \text{ otherwise.} \end{cases}$$

Suppose $t \geqslant 2s-2$. If X is antipodal, then $t_1 \geqslant 2s_{1,1}$, and if X is not antipodal, then $t_1 \geqslant 2s_{1,1}-2$. Thus X satisfies the assumption (1) of Theorem 2.6, and hence X carries a symmetric association scheme. So Theorem 2.6 contains the first half of [11, Theorem 7.4] as a special case.

Suppose $t=2s-3$ and $p_{\gamma,\gamma}(x,y)$ is uniquely determined for any fixed $\gamma = \langle x,y\rangle \in A'(X)$. If X is antipodal, then $t_1 = 2s_{1,1}-1$, and if X is not antipodal, then $t_1 = 2s_{1,1}-3$. Thus X also satisfies the assumption (1) or (2) of Theorem 2.6, and hence X carries a symmetric association scheme. So Theorem 2.6 contains the second half of [11, Theorem 7.4] as a special case.

Suppose that $t=2s-3$. If X is antipodal, then $t_1 = 2s_{1,1}-1$. Thus X satisfies the assumption (1) of Theorem 2.6, and hence X carries a symmetric association scheme. So Theorem 2.6 contains [2, Theorem 1.1] as a special case.

Next, we consider triple regularity of a symmetric association scheme. This concept was introduced in connection with spin models [14].

Definition 2.7. Let $(X,\{R_i\}_{i=0}^d)$ be a symmetric association scheme. Then the association scheme X is said to be triply regular if, for all $i,j,k,l,m,n \in \{0,1,\dots,d\}$, and for all $x,y,z \in X$ such that $(x,y) \in R_i$, $(y,z) \in R_j$, $(z,x) \in R_k$, the number $p^{i,j,k}_{l,m,n} := |\{w \in X \mid (w,x) \in R_m,\ (w,y) \in R_n,\ (w,z) \in R_l\}|$ depends only on i,j,k,l,m,n and not on x,y,z.

Let $(X,\{R_i\}_{i=0}^d)$ be an association scheme. We define the i-th subconstituent with respect to $z \in X$ by $R_i(z) := \{y \in X \mid (z,y) \in R_i\}$. We denote by $R^k_{i,j}(z)$ the restriction of R_k to $R_i(z) \times R_j(z)$. The following lemma gives an equivalent definition of a triply regular association scheme. We omit its easy proof.

Lemma 2.8. *A symmetric association scheme $(X,\{R_i\}_{i=0}^d)$ is triply regular if and only if for all $z \in X$, $(\bigcup_{i=1}^d R_i(z), \{R^k_{i,j}(z) \mid 1 \leqslant i,j \leqslant d,\ 0 \leqslant k \leqslant d,\ p^k_{i,j} \neq 0\})$ is a coherent configuration whose parameters are independent of z.*

Let X be a spherical t-design in S^{d-1} with degree s, and $A'(X) = \{\alpha_1,\dots,\alpha_s\}$. For $z \in X$ and $i \in \{1,\dots,s\}$, $X_i(z)$ will denote the orthogonal projection of $\{y \in X \mid \langle y,z\rangle = \alpha_i\}$ to $z^\perp = \{y \in \mathbb{R}^d \mid \langle y,z\rangle = 0\}$, rescaled to lie in S^{d-2} in $z^\perp$. $X_i(z)$ is called the derived design. In fact $X_i(z)$ is a $(t+1-s^*)$-design by [11, Theorem 8.2], where $s^* = |A'(X) \setminus \{-1\}|$. We define $\alpha^k_{i,j} = \frac{\alpha_k - \alpha_i\alpha_j}{\sqrt{(1-\alpha_i^2)(1-\alpha_j^2)}}$. If $\langle x,z\rangle = \alpha_i$, $\langle y,z\rangle = \alpha_j$ and $\langle x,y\rangle = \alpha_k$, then the inner product of the orthogonal projection of x, y to $z^\perp$ rescaled to lie in S^{d-2}, is $\alpha^k_{i,j}$.

Corollary 2.9. *Let $X \subset S^{d-1}$ be a finite set and $A'(X) = \{\alpha_1,\dots,\alpha_s\}$. Assume that $(X,\{R_k\}_{k=0}^s)$ is a symmetric association scheme, where $R_k = \{(x,y) \in X \times X \mid \langle x,y\rangle = \alpha_k\}$ $(0 \leqslant k \leqslant s)$ and $\alpha_0 = 1$. Then*

(1) *$A(X_i(z),X_j(z)) = \{\alpha^k_{i,j} \mid 0 \leqslant k \leqslant s,\ p^k_{i,j} \neq 0,\ \alpha^k_{i,j} \neq \pm 1\}$.*
(2) *$X_i(z) = X_j(z)$ or $X_i(z) \cap X_j(z) = \emptyset$, and $X_i(z) = -X_j(z)$ or $X_i(z) \cap -X_j(z) = \emptyset$ for any $z \in X$ and any $i,j \in \{1,\dots,s\}$. And $\delta_{X_i(z),X_j(z)}$, $\delta_{X_i(z),-X_j(z)}$ are independent of $z \in X$.*
(3) *$X_i(z)$ has the same strength for all $z \in X$.*

Moreover if the assumptions (1), (2) *or* (3) *of Theorem* 2.6 *are satisfied for* $\{X_i(z)\}_{i=1}^s$, *and when* (i, j, k) *satisfies* (2) (*resp.* (3)) *the numbers* $p^j_{\alpha,\beta}(x, y)$ (*resp.* $p^j_{\alpha,\beta}(x, y)$, $p^j_{\alpha,\beta'}(x, y)$, $p^j_{\alpha',\beta}(x, y)$) *which are assumed to be independent of* (x, y) *with* $\gamma = \langle x, y\rangle$ *being independent of the choice of* z, *then* $(X, \{R_k\}_{k=0}^s)$ *is a triply regular association scheme.*

Proof. Let $z \in X$. (1) is immediate from the definition of $\alpha^k_{i,j}$.

We define $R^k_{i,j}(z) = \{(x, y) \in X_i(z) \times X_j(z) \mid \langle x, y\rangle = \alpha^k_{i,j}\}$. Then

$$\begin{aligned}
&\bigl\{\langle x, y\rangle \bigm| x \in X_i(z),\ y \in X_j(z)\bigr\} \ni \pm 1\\
&\quad\Leftrightarrow\quad \exists k\quad \alpha^k_{i,j} = \pm 1 \quad\text{and}\quad p^k_{i,j} \neq 0\\
&\quad\Leftrightarrow\quad \exists k\quad \alpha^k_{i,j} = \pm 1, \quad\text{and}\\
&\qquad \forall x \in X_i(z)\ \exists y \in X_j(z) \quad\text{s.t.}\quad (x, y) \in R^k_{i,j}(z) \quad\text{and}\\
&\qquad \forall y \in X_j(z)\ \exists x \in X_i(z) \quad\text{s.t.}\quad (x, y) \in R^k_{i,j}(z)\\
&\quad\Leftrightarrow\quad X_i(z) = \pm X_j(z).
\end{aligned}$$

Since

$$\bigl\{\langle x, y\rangle \bigm| x \in X_i(z),\ y \in X_j(z)\bigr\} = \bigl\{\alpha^k_{i,j} \bigm| 0 \leqslant k \leqslant s,\ p^k_{i,j} \neq 0\bigr\}$$

is independent of $z \in X$, (2) holds.

By Lemma 2.2, $X_i(z)$ is a spherical t-design if and only if $\sum_{x,y\in X_i(z)} Q_k(\langle x, y\rangle) = 0$ for $k = 1, \dots, t$. Since the number of $y \in X_i(z)$ satisfying $\langle x, y\rangle = \frac{\alpha_j-\alpha_i^2}{1-\alpha_i^2}$ is $p^i_{i,j}$ for any $x \in X_i(z)$, the latter condition is equivalent to $\sum_{0\leqslant j\leqslant s} Q_k(\frac{\alpha_j-\alpha_i^2}{1-\alpha_i^2})p^i_{i,j} = 0$ for $k = 1, \dots, t$, which is independent of z. Hence $X_i(z)$ has the same strength for all $z \in X$. Therefore (3) holds.

Moreover if the assumptions (1), (2) or (3) of Theorem 2.6 are satisfied for $\{X_i(z)\}_{i=1}^s$, then $(\coprod_{i=1}^s X_i(z), \{R^k_{i,j}(z) \mid 0 \leqslant i, j, k \leqslant s,\ \ p^k_{i,j} \neq 0\})$ is a coherent configuration. Clearly, $|X_i(z)|$ is independent of $z \in X$. Also, $A(X_i(z), X_j(z))$ is independent of $z \in X$ by (1), t_i is independent of $z \in X$ by (3), and $\delta_{X_i(z),X_j(z)}$, $\delta_{X_i(z),-X_j(z)}$ are independent of $z \in X$ by (2). It follows from Theorem 2.6 that the parameters of the coherent configuration are independent of $z \in X$. Therefore, $(X, \{R_k\}_{k=0}^s)$ is a triply regular association scheme by Lemma 2.8. □

3. Tight designs

Let X be a t-design in S^{d-1}. It is known [11, Theorems 5.11, 5.12] that there is a lower bound for the size of a spherical t-design in S^{d-1}. Namely, if X is a spherical t-design, then

$$|X| \geqslant \binom{d + t/2 - 1}{t/2} + \binom{n + t/2 - 2}{t/2 - 1}$$

if t is even, and

$$|X| \geqslant 2\binom{d + (t - 3)/2}{(t - 1)/2}$$

if t is odd. If X is a t-design for which one of the lower bounds is attained, then X is called a tight t-design. It was proved in [3,4,11] that if X is a tight t-design with degree s in S^{d-1}, then the following statements hold:

(1) if t is even, then $t = 2s$,
(2) if t is odd, then $t = 2s - 1$ and X is antipodal,
(3) if $d = 2$, then X is the regular $(t + 1)$-gon,
(4) if $d \geqslant 3$, then $t \leqslant 5$ or $t = 7$, 11.

If X is a tight 11-design in S^{d-1} where $d \geqslant 3$, then $d = 24$ and X is the set of minimum vectors of the Leech lattice [6]. We consider tight 4-, 5-, 7-designs in S^{d-1} where $d \geqslant 3$.

Let $X \subset S^{d-1}$ be a tight $2s$-design, and let $A'(X) = \{\alpha_i \mid 1 \leqslant i \leqslant s\}$. For any $z \in X$, $X_i(z)$ is a $t_i := t + 1 - s^* = (s+1)$-design in S^{d-2}. Then the degrees $s_{i,j} = |A(X_i(z), X_j(z))|$ satisfy $s_{i,j} \leqslant s$, and the following holds:

$$\begin{aligned} 2s - 2 \leqslant s + 1 \quad &\Leftrightarrow \quad s \leqslant 3 \\ &\Leftrightarrow \quad t = 2, 4, 6. \end{aligned}$$

In particular, if $t = 4$, then $s_{i,j} + s_{j,k} - 2 \leqslant t_j$ holds, i.e., the assumption (1) of Theorem 2.6 holds for all i, j, k. By Corollary 2.9, we obtain the following result.

Corollary 3.1. *Every tight* 4*-design carries a triply regular association scheme.*

The same argument shows that a spherical 3-design with degree 2 i.e., a strongly regular graph with $a_1^* = 0$ carries a triply regular association scheme. This is already known (see [10]).

Let $X \subset S^{d-1}$ be a tight $(2s-1)$-design, and let $A'(X) = \{\alpha_i \mid 1 \leqslant i \leqslant s\}$ where $\alpha_s = -1$. For any $z \in X$ and $i \neq s$, $X_i(z)$ is a $t_i := t + 1 - s^* = (s+1)$-design in S^{d-2}.

Then the degrees $s_{i,j} = |A(X_i(z), X_j(z))|$ satisfy $s_{i,j} \leqslant s - 1$, and the following holds:

$$\begin{aligned} 2s - 4 \leqslant s + 1 \quad &\Leftrightarrow \quad s \leqslant 5 \\ &\Leftrightarrow \quad t = 1, 3, 5, 7, 9. \end{aligned}$$

In particular, if $t = 5, 7$, then $s_{i,j} + s_{j,k} - 2 \leqslant t_j$ holds, i.e., the assumption (1) of Theorem 2.6 holds for all i, j, k. By Corollary 2.9, we obtain the following result.

Corollary 3.2. *Every tight* 5*- or* 7*-design carries a triply regular association scheme.*

The same argument shows that an antipodal spherical 3-design with degree 3 carries a triply regular association scheme i.e., subconstituents of a Taylor graph are strongly regular graphs. This is already known (see [7, Theorem 1.5.3]).

Although one might wonder if the tight 11-design is triply regular or not, it is shown that the tight 11-design is not triply regular using Magma as follows. Let X be the tight 11-design in S^{23}. The angle set of X is

$$A'(X) = \left\{\frac{1}{2}, \frac{1}{4}, 0, -\frac{1}{4}, -\frac{1}{2}, -1\right\}.$$

We set

$$\alpha_0 = 1, \qquad \alpha_1 = \frac{1}{2}, \qquad \alpha_2 = \frac{1}{4}, \qquad \alpha_3 = 0, \qquad \alpha_4 = -\frac{1}{4}, \qquad \alpha_5 = -\frac{1}{2}, \qquad \alpha_6 = -1,$$

and we define $R_k = \{(x, y) \in X \times X \mid \langle x, y\rangle = \alpha_k\}$. Let G be the automorphism group of X, and let G_z be the one-point stabilizer of $z \in X$. Then G_z transitively acts on $R_3(z)$ and the number of the orbits of G_z on $R_3(z) \times R_3(z)$ is 8. Let $\Omega_0, \dots, \Omega_7$ be those orbits. Renumbering the index, we have

$$\Omega_i = R_{3,3}^i(z) \quad \text{for } i \in \{0, 1, 2, 4, 5, 6\}, \qquad \Omega_3 \cup \Omega_7 = R_{3,3}^3(z).$$

Since the permutation character is multiplicity free, we obtain a commutative association scheme $\mathfrak{X}'$ with the first and second eigenmatrices given by

$$P=\begin{pmatrix}1 & 2464 & 22\,528 & 422\,240 & 22\,528 & 2464 & 1 & 924\\ 1 & 182 & -368 & 0 & 368 & -182 & -1 & 0\\ 1 & 1232 & 5632 & 0 & -5632 & -1232 & -1 & 0\\ 1 & 532 & 448 & -1920 & 448 & 532 & 1 & -42\\ 1 & 28 & -128 & 240 & -128 & 28 & 1 & -42\\ 1 & 64 & -272 & 240 & -272 & 64 & 1 & 174\\ 1 & -20 & 64 & -96 & 64 & -20 & 1 & 6\\ 1 & -10 & 16 & 0 & -16 & 10 & -1 & 0\end{pmatrix},$$

$$Q=\begin{pmatrix}1 & 2277 & 23 & 275 & 12\,650 & 2024 & 31\,625 & 44\,275\\ 1 & \frac{2691}{16} & \frac{23}{2} & \frac{475}{8} & \frac{575}{4} & \frac{368}{7} & -\frac{14\,375}{56} & -\frac{2875}{16}\\ 1 & -\frac{4761}{128} & \frac{23}{4} & \frac{175}{32} & -\frac{575}{8} & -\frac{391}{16} & \frac{2875}{32} & \frac{4025}{128}\\ 1 & 0 & 0 & -\frac{25}{2} & \frac{575}{8} & \frac{23}{2} & -\frac{575}{8} & 0\\ 1 & \frac{4761}{128} & -\frac{23}{4} & \frac{175}{32} & -\frac{575}{32} & -\frac{391}{16} & \frac{2875}{32} & -\frac{4025}{128}\\ 1 & -\frac{2691}{16} & -\frac{23}{2} & \frac{475}{8} & \frac{575}{4} & \frac{368}{7} & -\frac{14\,375}{56} & \frac{2875}{16}\\ 1 & -2277 & -23 & 275 & 12\,650 & 2024 & 31\,625 & -44\,275\\ 1 & 0 & 0 & -\frac{25}{2} & -575 & \frac{2668}{7} & \frac{2875}{14} & 0\end{pmatrix}.$$

Note $4050E_2$ is the gram matrix of $X_3(z)$. If $(R_3(z), \{R^i_{3,3}(z) \mid 0 \leqslant i \leqslant 6\})$ is an association scheme, then it is a fusion scheme of $\mathfrak{X}'$. But, by Bannai–Muzychuk criterion in [1] and [19], $(R_3(z), \{R^i_{3,3}(z) \mid 0 \leqslant i \leqslant 6\})$ is not a fusion scheme. Therefore X is not triply regular.

4. Derived designs of Q-polynomial association schemes

The reader is referred to [5] for the basic information on Q-polynomial association schemes. The following lemma is used to prove Lemma 4.2.

Lemma 4.1. *Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme of class d. Let $B_i = (p^k_{i,j})$ be its i-th intersection matrix, and $Q = (q_j(i))$ be the second eigenmatrix of $\mathfrak{X}$. Then*

$$\left(Q^t B_i\right)(h,i) = \frac{k_i q_h(i)^2}{m_h} \quad (0 \leqslant h, i \leqslant d).$$

Proof. See [5, p. 73, (4.2) and Theorem 3.5(i)]. □

The following lemma gives a property of derived designs of the embedding of a Q-polynomial association scheme into the first eigenspace.

Lemma 4.2. *Let $(X, \{R_i\}_{i=0}^s)$ be a Q-polynomial association scheme, and we identify X as the image of the embedding into the first eigenspace by $E_1 = \frac{1}{|X|}\sum_{j=0}^s \theta^*_j A_j$. Then, for $i \in \{1, \dots, s\}$ with $\theta^*_i \neq -\theta^*_0$, the derived design $X_i(z)$ is a 2-design in $S^{\theta^*_0 - 2}$ for any $z \in X$ if and only if $a^*_1(\theta^*_i + 1) = 0$.*

Proof. The angle set of $X_i(z)$ consists of

$$\frac{\frac{\theta^*_k}{\theta^*_0} - \frac{\theta^{*2}_i}{\theta^{*2}_0}}{1 - (\frac{\theta^*_i}{\theta^*_0})^2} = \frac{\theta^*_0\theta^*_k - \theta^{*2}_i}{\theta^{*2}_0 - \theta^{*2}_i} \quad \left(0 \leqslant k \leqslant s,\ p^k_{i,i} \neq 0\right).$$

Thus, Lemma 2.2 implies that $X_i(z)$ is a 2-design in $S^{\theta^*_0 - 2}$ if and only if

$$\sum_{j=0}^{s} Q_k\left(\frac{\theta_0^*\theta_j^* - \theta_i^{*2}}{\theta_0^{*2} - \theta_i^{*2}}\right)p_{i,j}^i = 0 \quad (k = 1, 2),$$

where $Q_k(x)$ is the Gegenbauer polynomial of degree k in $S^{\theta_0^*-2}$.

Since $Q_1(x) = (\theta_0^* - 1)x$, $\sum_{j=0}^{s} p_{i,j}^i = k_i$ and

$$\begin{aligned}\sum_{j=0}^{s} \theta_j^* p_{i,j}^i &= \left(Q^t B_i\right)(1, i) \\ &= \frac{k_i q_1(i)^2}{m_1} \quad \text{(by Lemma 4.1)} \\ &= \frac{k_i \theta_i^{*2}}{\theta_0^*},\end{aligned} \tag{4.1}$$

we have

$$\begin{aligned}\sum_{j=0}^{s} Q_1\left(\frac{\theta_0^*\theta_j^* - \theta_i^{*2}}{\theta_0^{*2} - \theta_i^{*2}}\right)p_{i,j}^i &= \frac{\theta_0^* - 1}{\theta_0^{*2} - \theta_i^{*2}}\left(\theta_0^* \sum_{j=0}^{s} \theta_j^* p_{i,j}^i - \theta_i^{*2} \sum_{j=0}^{s} p_{i,j}^i\right) \\ &= 0.\end{aligned}$$

Since $Q_2(x) = \frac{\theta_0^*+1}{2}((\theta_0^* - 1)x^2 - 1)$, $\sum_{j=0}^{s} p_{i,j}^i = k_i$, (4.1) and

$$\begin{aligned}\sum_{j=0}^{s} \theta_j^{*2} p_{i,j}^i &= \sum_{j=0}^{s}\left(c_2^* q_2(j) + a_1^* q_1(j) + b_0^* q_0(j)\right)p_{i,j}^i \\ &= c_2^*\left(Q^t B_i\right)(2, i) + a_1^* \frac{k_i \theta_i^{*2}}{\theta_0^*} + \theta_0^* k_i \quad \left(\text{by (4.1)}\right) \\ &= c_2^* \frac{k_i q_2(i)^2}{m_2} + k_i\left(\frac{a_1^*\theta_i^{*2}}{\theta_0^*} + \theta_0^*\right) \quad \text{(by Lemma 4.1)} \\ &= k_i\left(\frac{((\theta_i^* - a_1^*)\theta_i^* - \theta_0^*)^2}{(\theta_0^* - a_1^*)\theta_0^* - \theta_0^*} + \frac{a_1^*\theta_i^{*2}}{\theta_0^*} + \theta_0^*\right),\end{aligned}$$

we have

$$\begin{aligned}&\sum_{j=0}^{s} Q_2\left(\frac{\theta_0^*\theta_j^* - \theta_i^{*2}}{\theta_0^{*2} - \theta_i^{*2}}\right)p_{i,j}^i \\ &\quad = \frac{\theta_0^* - 1}{(\theta_0^{*2} - \theta_i^{*2})^2}\left(\theta_0^{*2} \sum_{j=0}^{s} \theta_j^{*2} p_{i,j}^i - 2\theta_0^*\theta_i^{*2} \sum_{j=0}^{s} \theta_j^* p_{i,j}^i + \theta_i^{*4} \sum_{j=0}^{s} p_{i,j}^i\right) - k_i \\ &\quad = \frac{k_i a_1^*(\theta_i^* + 1)^2 \theta_0^*}{(\theta_0^* + \theta_i^*)^2(\theta_0^* - a_1^* - 1)}.\end{aligned}$$

Therefore $X_i(z)$ is a 2-design in $S^{\theta_0^*-2}$ if and only if $a_1^*(\theta_i^* + 1) = 0$. □

5. Real mutually unbiased bases

Definition 5.1. Let $M = \{M_i\}_{i=1}^{f}$ be a collection of orthonormal bases of $\mathbb{R}^d$. M is called real mutually unbiased bases (MUB) if any two vectors x and y from different bases satisfy $\langle x, y\rangle = \pm 1/\sqrt{d}$.

It is known that the number f of real mutually unbiased bases in $\mathbb{R}^d$ can be at most $d/2+1$. We call M a maximal MUB if this upper bound is attained. Constructions of maximal MUB are known only for $d=2^{m+1}$, m odd [8]. Throughout this section, we assume $M=\{M_i\}_{i=1}^f$ is an MUB, put $X^{(i)}=M_i\cup(-M_i)$ and $X=M\cup(-M)$. The angle set of X is

$$A'(X)=\left\{\frac{1}{\sqrt{d}},0,-\frac{1}{\sqrt{d}},-1\right\}.$$

We set

$$\alpha_0=1,\qquad \alpha_1=\frac{1}{\sqrt{d}},\qquad \alpha_2=0,\qquad \alpha_3=-\frac{1}{\sqrt{d}},\qquad \alpha_4=-1,$$

and we define $R_k=\{(x,y)\in X\times X\mid \langle x,y\rangle=\alpha_k\}$.

Since $X^{(i)}$ is a spherical 3-design in S^{d-1} for any $i\in\{1,\dots,f\}$, X is also a spherical 3-design in S^{d-1}. It is shown in [15] that $(X,\{R_k\}_{k=0}^4)$ is a Q-polynomial association scheme with $a_1^*=0$. X is imprimitive and the set $\{X^{(1)},\dots,X^{(f)}\}$ is a system of imprimitivity with respect to the equivalence relation $R_0\cup R_2\cup R_4$.

By Lemma 4.2, for any $z\in X$ the derived design $X_i=X_i(z)$ is a 2-design in S^{d-2}. We define $s_{i,j}=|A(X_i,X_j)|$. Then the matrix $(s_{i,j})_{\substack{1\leqslant i\leqslant 3\\1\leqslant j\leqslant 3}}$ is

$$\begin{pmatrix}3&2&3\\2&1&2\\3&2&3\end{pmatrix}.$$

If $s_{i,j}+s_{j,k}-2\leqslant 2$, that is, when

$$(i,j,k)\in\big\{(1,2,1),(1,2,2),(1,2,3),(2,1,2),(2,2,1),(2,2,2),\\(2,2,3),(2,3,2),(3,2,1),(3,2,2),(3,2,3)\big\},$$

then the assumption (1) of Theorem 2.6 holds. We remark that X_2 is in fact a 3-design because X_2 is a cross polytope in $\mathbb{R}^{d-1}$, but this fact does not improve the proof.

The following lemma is used to determine intersection numbers of derived designs obtained from MUB.

Lemma 5.2. *We define $X_i(x,\alpha)=\{w\in X_i\mid\langle x,w\rangle=\alpha\}$, and $X_i(x,\alpha;y,\beta)=X_i(x,\alpha)\cap X_i(y,\beta)$. Then the following equalities hold*:

(1) $X_i(x,-\alpha)=X_i(-x,\alpha)$,
(2) $-X_i(x,\alpha)=X_{4-i}(x,-\alpha)$,
(3) $|X_i(x,\alpha;y,\beta)|=|X_i(-x,-\alpha;y,\beta)|=|X_i(x,\alpha;-y,-\beta)|=|X_{4-i}(x,-\alpha;y,-\beta)|$.

Proof. (1) and (2) are immediate from the definition.

By (1), $X_i(x,\alpha;y,\beta)=X_i(-x,-\alpha;y,\beta)=X_i(x,\alpha;-y,-\beta)$ holds. By (2), $-X_i(x,\alpha;y,\beta)=X_{4-i}(x,-\alpha;y,-\beta)$ holds. This proves (3). □

If $s_{i,j}+s_{j,k}-3=2$, that is, when

$$(i,j,k)\in\big\{(1,1,2),(1,3,2),(2,1,1),(2,1,3),(2,3,1),(2,3,3),(3,1,2),(3,3,2)\big\},\qquad(5.1)$$

Lemma 5.2 implies that the intersection numbers on $X_j(z)$ for $x\in X_i(z)$, $y\in X_k(z)$ are determined by the intersection numbers on $X_1(z)$ for $x'\in X_1(z)$, $y'\in X_2(z)$. And the intersection numbers $p^1_{\alpha^2_{1,1},\alpha^1_{1,2}}(x,y)$, $p^1_{\alpha^2_{1,1},\alpha^3_{1,2}}(x,y)$ for $x,y\in X_1(z)$ are uniquely determined by $\gamma=\langle x,y\rangle$ as follows:

$$p^1_{\alpha^2_{1,1},\alpha^1_{1,2}}(x,y)=\begin{cases}\frac{d}{2}-1 & \text{if } \langle x,y\rangle=\alpha^1_{1,2},\\ \frac{d}{2} & \text{if } \langle x,y\rangle=\alpha^3_{1,2},\end{cases}\qquad p^1_{\alpha^2_{1,1},\alpha^3_{1,2}}(x,y)=\begin{cases}\frac{d}{2} & \text{if } \langle x,y\rangle=\alpha^1_{1,2},\\ \frac{d}{2}-1 & \text{if } \langle x,y\rangle=\alpha^3_{1,2}.\end{cases}$$

Table 1
The values of $p^1_{\alpha,\beta}(x,y)$, where $x \in X_1,\ y \in X_1$.

(α,β)	$p^1_{\alpha,\beta}(x,y)$
$(\alpha^2_{1,1},\alpha^2_{1,1})$	$\begin{cases} 0 & \text{if } \langle x,y\rangle = \alpha^1_{1,1} \\ d-2 & \text{if } \langle x,y\rangle = \alpha^2_{1,1} \\ 0 & \text{if } \langle x,y\rangle = \alpha^3_{1,1} \end{cases}$
$(\alpha^2_{1,1},\alpha^1_{1,1})$, $(\alpha^1_{1,1},\alpha^2_{1,1})$	$\begin{cases} \frac{d+\sqrt{d}}{2}-1 & \text{if } \langle x,y\rangle = \alpha^1_{1,1} \\ 0 & \text{if } \langle x,y\rangle = \alpha^2_{1,1} \\ \frac{d+\sqrt{d}}{2} & \text{if } \langle x,y\rangle = \alpha^3_{1,1} \end{cases}$
$(\alpha^2_{1,1},\alpha^3_{1,1})$, $(\alpha^3_{1,1},\alpha^2_{1,1})$	$\begin{cases} \frac{d-\sqrt{d}}{2} & \text{if } \langle x,y\rangle = \alpha^1_{1,1} \\ 0 & \text{if } \langle x,y\rangle = \alpha^2_{1,1} \\ \frac{d-\sqrt{d}}{2}-1 & \text{if } \langle x,y\rangle = \alpha^3_{1,1} \end{cases}$

These numbers are independent of $z \in X$. Hence the assumption (2) of Theorem 2.6 holds for (i,j,k) in (5.1).

If $s_{i,j}+s_{j,k}-4=2$, that is, when

$$(i,j,k) \in \{(1,1,1),(1,1,3),(1,3,1),(1,3,3),(3,1,1),(3,1,3),(3,3,1),(3,3,3)\}, \tag{5.2}$$

Lemma 5.2 implies that the intersection numbers on $X_j(z)$ for $x \in X_i(z),\ y \in X_k(z)$ are determined by the intersection numbers on $X_1(z)$ for $x' \in X_1(z),\ y' \in X_1(z)$. And the intersection numbers $\{p^1_{\alpha,\beta}(x,y) \mid \alpha = \alpha^2_{1,1} \text{ or } \beta = \alpha^2_{1,1}\}$ are given in Table 1. These numbers are independent of $z \in X$. Hence the assumption (3) of Theorem 2.6 holds for (i,j,k) in (5.2). By Corollary 2.9, we obtain the following result.

Corollary 5.3. *Every MUB carries a triply regular association scheme.*

6. Linked systems of symmetric designs

Definition 6.1. Let $(\Omega_i,\Omega_j,I_{i,j})$ be an incidence structure satisfying $\Omega_i \cap \Omega_j = \emptyset$, $I^t_{j,i} = I_{i,j}$ for any distinct integers $i,j \in \{1,\dots,f\}$. We put $\Omega = \bigcup_{i=1}^f \Omega_i$, $I = \bigcup_{i\neq j} I_{i,j}$. (Ω, I) is called a linked system of symmetric (v,k,λ) designs if the following conditions hold:

(1) for any distinct integers $i,j \in \{1,\dots,f\}$, $(\Omega_i,\Omega_j,I_{i,j})$ is a symmetric (v,k,λ) design,
(2) for any distinct integers $i,j,l \in \{1,\dots,f\}$, and for any $x \in \Omega_i, y \in \Omega_j$, the number of $z \in \Omega_l$ incident with both x and y depends only on whether x and y are incident or not, and does not depend on i,j,l.

We define the integers σ,τ by

$$\left|\{z \in \Omega_l \mid (x,z) \in I_{i,l},\ (y,z) \in I_{j,l}\}\right| = \begin{cases} \sigma & \text{if } (x,y) \in I_{i,j}, \\ \tau & \text{if } (x,y) \notin I_{i,j}, \end{cases}$$

where $i,j,l \in \{1,\dots,f\}$ are distinct and $x \in \Omega_i,\ y \in \Omega_j$.

By [9, Theorem 1], we may assume that

$$\sigma = \frac{1}{v}\left(k^2 - \sqrt{n}(v-k)\right), \qquad \tau = \frac{k}{v}(k+\sqrt{n}),$$

where $n = k-\lambda$. It is easy to see that $(\Omega, \{R_i\}_{i=0}^3)$ is a 3-class association scheme, where

$$R_0 = \{(x,x) \mid x \in \Omega\},$$
$$R_1 = \{(x,y) \mid x \in \Omega_i,\ y \in \Omega_j,\ (x,y) \in I_{i,j} \text{ for some } i \neq j\},$$
$$R_2 = \{(x,y) \mid x, y \in \Omega_i,\ x \neq y \text{ for some } i\},$$
$$R_3 = \{(x,y) \mid x \in \Omega_i,\ y \in \Omega_j,\ (x,y) \notin I_{i,j} \text{ for some } i \neq j\}.$$

We note that the second eigenmatrix Q is given in [18] as follows:

$$Q = \begin{pmatrix} 1 & v-1 & (f-1)(v-1) & f-1 \\ 1 & -\sqrt{\frac{(v-1)(v-k)}{k}} & \sqrt{\frac{(v-1)(v-k)}{k}} & -1 \\ 1 & -1 & -f+1 & f-1 \\ 1 & \sqrt{\frac{(v-1)k}{v-k}} & -\sqrt{\frac{(v-1)k}{v-k}} & -1 \end{pmatrix},$$

and hence the Krein matrix $B_1^* = (q_{1,j}^k)_{\substack{0\leqslant j\leqslant 3\\ 0\leqslant k\leqslant 3}}$ is given as follows:

$$B_1^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ v-1 & \frac{k(v-k)(v-2)+(f-1)(2k-v)\sqrt{k(v-k)(v-1)}}{fk(v-k)} & \frac{k(v-k)(v-2)+(v-2k)\sqrt{k(v-k)(v-1)}}{fk(v-k)} & 0 \\ 0 & \frac{(f-1)(k(v-k)(v-2)+(v-2k)\sqrt{k(v-k)(v-1)})}{fk(v-k)} & \frac{(f-1)k(v-k)(v-2)+(2k-v)\sqrt{k(v-k)(v-1)}}{fk(v-k)} & v-1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore $(\Omega, \{R_i\}_{i=0}^3)$ is a Q-polynomial association scheme. $(\Omega, \{R_i\}_{i=0}^3)$ is imprimitive and the set $\{\Omega_1, \dots, \Omega_f\}$ is a system of imprimitivity with respect to the equivalence relation $R_0 \cup R_2$.

In the rest of this section, we assume that $a_1^* = 0$ i.e., $f = 1 + \frac{(v-2)\sqrt{k(v-k)}}{(v-2k)\sqrt{v-1}}$ (see [18] and [20]). Examples of linked systems of symmetric designs satisfying this assumption are constructed for $(v,k,\lambda) = (2^{2m}, 2^{2m-1} - 2^{m-1}, 2^{2m-2} - 2^{m-1})$ with $f = 2^{2m-1}$ for any $m > 1$ [9]. The linked system of symmetric $(2^{2m}, 2^{2m-1} - 2^{m-1}, 2^{2m-2} - 2^{m-1})$ designs constructs real MUB, see [17].

Let X be the embedding of Ω into the first eigenspace. The angle set of X is

$$A'(X) = \left\{ \frac{\theta_k^*}{\theta_0^*} \;\middle|\; 1 \leqslant k \leqslant 3 \right\},$$

and we set $\alpha_k = \theta_k^*/\theta_0^*$. We consider the derived design $X_i(z)$ for $z \in X$. By $a_1^* = 0$, Lemma 4.2 implies that $X_i(z)$ is a 2-design in S^{v-3}. We define $s_{i,j} = |A'(X_i(z), X_j(z))|$. Then the matrix $(s_{i,j})_{\substack{1\leqslant i\leqslant 3\\ 1\leqslant j\leqslant 3}}$ is

$$\begin{pmatrix} 3 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \end{pmatrix}.$$

Since $\{\Omega_1, \dots, \Omega_f\}$ is a system of imprimitivity, we obtain Table 2 and Table 3.

Table 2
The values of $p^j_{\alpha,\beta}(x,y)$, where $x \in X_i(z)$, $y \in X_l(z)$.

(i,j,l)	(α,β)	$p^j_{\alpha,\beta}(x,y)$	(i,j,l)	(α,β)	$p^j_{\alpha,\beta}(x,y)$
$(1,1,2)$	$(\alpha^2_{1,1}, \alpha^1_{1,2})$	$\begin{cases} \lambda - 1 & \langle x,y\rangle = \alpha^1_{1,2} \\ \lambda & \langle x,y\rangle = \alpha^3_{1,2} \end{cases}$	$(2,1,1)$	$(\alpha^1_{2,1}, \alpha^2_{1,1})$	$\begin{cases} \lambda - 1 & \langle x,y\rangle = \alpha^1_{2,1} \\ \lambda & \langle x,y\rangle = \alpha^3_{2,1} \end{cases}$
$(1,3,2)$	$(\alpha^2_{1,3}, \alpha^1_{3,2})$	$\begin{cases} k - \lambda & \langle x,y\rangle = \alpha^1_{1,2} \\ k - \lambda & \langle x,y\rangle = \alpha^3_{1,2} \end{cases}$	$(2,3,1)$	$(\alpha^1_{2,3}, \alpha^2_{3,1})$	$\begin{cases} k - \lambda & \langle x,y\rangle = \alpha^1_{2,1} \\ k - \lambda & \langle x,y\rangle = \alpha^3_{2,1} \end{cases}$
$(3,1,2)$	$(\alpha^2_{3,1}, \alpha^1_{1,2})$	$\begin{cases} \lambda & \langle x,y\rangle = \alpha^1_{3,2} \\ \lambda & \langle x,y\rangle = \alpha^3_{3,2} \end{cases}$	$(2,1,3)$	$(\alpha^1_{2,1}, \alpha^2_{1,3})$	$\begin{cases} \lambda & \langle x,y\rangle = \alpha^1_{2,3} \\ \lambda & \langle x,y\rangle = \alpha^3_{2,3} \end{cases}$
$(3,3,2)$	$(\alpha^2_{3,3}, \alpha^1_{3,2})$	$\begin{cases} k - \lambda - 1 & \langle x,y\rangle = \alpha^1_{3,2} \\ k - \lambda & \langle x,y\rangle = \alpha^3_{3,2} \end{cases}$	$(2,3,3)$	$(\alpha^1_{2,3}, \alpha^2_{3,3})$	$\begin{cases} k - \lambda - 1 & \langle x,y\rangle = \alpha^1_{2,3} \\ k - \lambda & \langle x,y\rangle = \alpha^3_{2,3} \end{cases}$

Table 3
The values of $p^j_{\alpha,\beta}(x,y)$, where $x \in X_i(z),\ y \in X_l(z)$.

(i,j,l)	(α,β)	$p^j_{\alpha,\beta}(x,y)$	(i,j,l)	(α,β)	$p^j_{\alpha,\beta}(x,y)$
	$(\alpha^2_{1,1},\alpha^2_{1,1})$	$\begin{cases} 0 & \langle x,y\rangle=\alpha^1_{1,1} \\ k-2 & \langle x,y\rangle=\alpha^2_{1,1} \\ 0 & \langle x,y\rangle=\alpha^3_{1,1} \end{cases}$		$(\alpha^2_{1,3},\alpha^2_{3,3})$	$\begin{cases} 0 & \langle x,y\rangle=\alpha^1_{1,3} \\ v-k-1 & \langle x,y\rangle=\alpha^2_{1,3} \\ 0 & \langle x,y\rangle=\alpha^3_{1,3} \end{cases}$
$(1,1,1)$	$(\alpha^2_{1,1},\alpha^1_{1,1})$	$\begin{cases} \sigma-1 & \langle x,y\rangle=\alpha^1_{1,1} \\ 0 & \langle x,y\rangle=\alpha^2_{1,1} \\ \sigma & \langle x,y\rangle=\alpha^3_{1,1} \end{cases}$	$(1,3,3)$	$(\alpha^2_{1,3},\alpha^1_{3,3})$	$\begin{cases} k-\tau & \langle x,y\rangle=\alpha^1_{1,3} \\ 0 & \langle x,y\rangle=\alpha^2_{1,3} \\ k-\tau & \langle x,y\rangle=\alpha^3_{1,3} \end{cases}$
	$(\alpha^1_{1,1},\alpha^2_{1,1})$	$\begin{cases} \sigma-1 & \langle x,y\rangle=\alpha^1_{1,1} \\ 0 & \langle x,y\rangle=\alpha^2_{1,1} \\ \sigma & \langle x,y\rangle=\alpha^3_{1,1} \end{cases}$		$(\alpha^1_{1,3},\alpha^2_{3,3})$	$\begin{cases} k-\sigma-1 & \langle x,y\rangle=\alpha^1_{1,3} \\ 0 & \langle x,y\rangle=\alpha^2_{1,3} \\ k-\sigma & \langle x,y\rangle=\alpha^3_{1,3} \end{cases}$
	$(\alpha^2_{1,1},\alpha^2_{1,3})$	$\begin{cases} 0 & \langle x,y\rangle=\alpha^1_{1,3} \\ k-1 & \langle x,y\rangle=\alpha^2_{1,3} \\ 0 & \langle x,y\rangle=\alpha^3_{1,3} \end{cases}$		$(\alpha^2_{3,1},\alpha^2_{1,3})$	$\begin{cases} 0 & \langle x,y\rangle=\alpha^1_{3,3} \\ k & \langle x,y\rangle=\alpha^2_{3,3} \\ 0 & \langle x,y\rangle=\alpha^3_{3,3} \end{cases}$
$(1,1,3)$	$(\alpha^2_{1,1},\alpha^1_{1,3})$	$\begin{cases} \tau-1 & \langle x,y\rangle=\alpha^1_{1,3} \\ 0 & \langle x,y\rangle=\alpha^2_{1,3} \\ \tau & \langle x,y\rangle=\alpha^3_{1,3} \end{cases}$	$(3,1,3)$	$(\alpha^2_{3,1},\alpha^1_{1,3})$	$\begin{cases} \tau & \langle x,y\rangle=\alpha^1_{3,3} \\ 0 & \langle x,y\rangle=\alpha^2_{3,3} \\ \tau & \langle x,y\rangle=\alpha^3_{3,3} \end{cases}$
	$(\alpha^1_{1,1},\alpha^2_{1,3})$	$\begin{cases} \sigma & \langle x,y\rangle=\alpha^1_{1,3} \\ 0 & \langle x,y\rangle=\alpha^2_{1,3} \\ \sigma & \langle x,y\rangle=\alpha^3_{1,3} \end{cases}$		$(\alpha^1_{3,1},\alpha^2_{1,3})$	$\begin{cases} \tau & \langle x,y\rangle=\alpha^1_{3,3} \\ 0 & \langle x,y\rangle=\alpha^2_{3,3} \\ \tau & \langle x,y\rangle=\alpha^3_{3,3} \end{cases}$
	$(\alpha^2_{1,3},\alpha^2_{3,1})$	$\begin{cases} 0 & \langle x,y\rangle=\alpha^1_{1,1} \\ v-k & \langle x,y\rangle=\alpha^2_{1,1} \\ 0 & \langle x,y\rangle=\alpha^3_{1,1} \end{cases}$		$(\alpha^2_{3,3},\alpha^2_{3,1})$	$\begin{cases} 0 & \langle x,y\rangle=\alpha^1_{3,1} \\ v-k-1 & \langle x,y\rangle=\alpha^2_{3,1} \\ 0 & \langle x,y\rangle=\alpha^3_{3,1} \end{cases}$
$(1,3,1)$	$(\alpha^2_{1,3},\alpha^1_{3,1})$	$\begin{cases} k-\sigma & \langle x,y\rangle=\alpha^1_{1,1} \\ 0 & \langle x,y\rangle=\alpha^2_{1,1} \\ k-\sigma & \langle x,y\rangle=\alpha^3_{1,1} \end{cases}$	$(3,3,1)$	$(\alpha^2_{3,3},\alpha^1_{3,1})$	$\begin{cases} k-\tau-1 & \langle x,y\rangle=\alpha^1_{3,1} \\ 0 & \langle x,y\rangle=\alpha^2_{3,1} \\ k-\tau & \langle x,y\rangle=\alpha^3_{3,1} \end{cases}$
	$(\alpha^1_{1,3},\alpha^2_{3,1})$	$\begin{cases} k-\sigma & \langle x,y\rangle=\alpha^1_{1,1} \\ 0 & \langle x,y\rangle=\alpha^2_{1,1} \\ k-\sigma & \langle x,y\rangle=\alpha^3_{1,1} \end{cases}$		$(\alpha^1_{3,3},\alpha^2_{3,1})$	$\begin{cases} k-\tau & \langle x,y\rangle=\alpha^1_{3,1} \\ 0 & \langle x,y\rangle=\alpha^2_{3,1} \\ k-\tau & \langle x,y\rangle=\alpha^3_{3,1} \end{cases}$
	$(\alpha^2_{3,1},\alpha^2_{1,1})$	$\begin{cases} 0 & \langle x,y\rangle=\alpha^1_{3,1} \\ k-1 & \langle x,y\rangle=\alpha^2_{3,1} \\ 0 & \langle x,y\rangle=\alpha^3_{3,1} \end{cases}$		$(\alpha^2_{3,3},\alpha^2_{3,3})$	$\begin{cases} 0 & \langle x,y\rangle=\alpha^1_{3,3} \\ v-k-2 & \langle x,y\rangle=\alpha^2_{3,3} \\ 0 & \langle x,y\rangle=\alpha^3_{3,3} \end{cases}$
$(3,1,1)$	$(\alpha^2_{3,1},\alpha^1_{1,1})$	$\begin{cases} \sigma & \langle x,y\rangle=\alpha^1_{3,1} \\ 0 & \langle x,y\rangle=\alpha^2_{3,1} \\ \sigma & \langle x,y\rangle=\alpha^3_{3,1} \end{cases}$	$(3,3,3)$	$(\alpha^2_{3,3},\alpha^1_{3,3})$	$\begin{cases} k-\tau-1 & \langle x,y\rangle=\alpha^1_{3,3} \\ 0 & \langle x,y\rangle=\alpha^2_{3,3} \\ k-\tau & \langle x,y\rangle=\alpha^3_{3,3} \end{cases}$
	$(\alpha^1_{3,1},\alpha^2_{1,1})$	$\begin{cases} \tau-1 & \langle x,y\rangle=\alpha^1_{3,1} \\ 0 & \langle x,y\rangle=\alpha^2_{3,1} \\ \tau & \langle x,y\rangle=\alpha^3_{3,1} \end{cases}$		$(\alpha^1_{3,3},\alpha^2_{3,3})$	$\begin{cases} k-\tau-1 & \langle x,y\rangle=\alpha^1_{3,3} \\ 0 & \langle x,y\rangle=\alpha^2_{3,3} \\ k-\tau & \langle x,y\rangle=\alpha^3_{3,3} \end{cases}$

If $s_{i,j} + s_{j,l} - 2 \leqslant 2$, that is, when

$$(i,j,l) \in \{(1,2,1),(1,2,2),(1,2,3),(2,1,2),(2,2,1),(2,2,2),\\ (2,2,3),(2,3,2),(3,2,1),(3,2,2),(3,2,3)\},$$

then the assumption (1) of Theorem 2.6 holds.

If $s_{i,j} + s_{j,l} - 3 = 2$, that is, when

$$(i,j,l) \in \{(1,1,2),(1,3,2),(2,1,1),(2,1,3),(2,3,1),(2,3,3),(3,1,2),(3,3,2)\}, \tag{6.1}$$

Table 2 implies that the numbers $p^j_{\alpha^2_{i,j},\alpha^1_{j,l}}(x,y)$ or $p^j_{\alpha^1_{i,j},\alpha^2_{j,l}}(x,y)$ are independent of $z \in X$ and $(x,y) \in X_i(z) \times X_l(z)$ with $\gamma = \langle x,y\rangle$. Hence the assumption (2) of Theorem 2.6 holds for (i,j,l) in (6.1).

If $s_{i,j} + s_{j,l} - 4 = 2$, that is, when

$$(i,j,l) \in \{(1,1,1),(1,1,3),(1,3,1),(1,3,3),(3,1,1),(3,1,3),(3,3,1),(3,3,3)\}, \tag{6.2}$$

Table 3 implies the numbers $p^j_{\alpha^2_{i,j},\alpha^2_{j,l}}(x,y)$, $p^j_{\alpha^2_{i,j},\alpha^1_{j,l}}(x,y)$ and $p^j_{\alpha^1_{i,j},\alpha^2_{j,l}}(x,y)$ are independent of $z \in X$ and $(x,y) \in X_i(z) \times X_l(z)$ with $\gamma = \langle x,y\rangle$. Hence the assumption (3) of Theorem 2.6 holds for (i,j,l) in (6.2). By Corollary 2.9, we obtain the following result.

Corollary 6.2. *Every linked system of symmetric design satisfying $f = 1 + \frac{(v-2)\sqrt{k(v-k)}}{(v-2k)\sqrt{v-1}}$ carries a triply regular association scheme.*

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