Bounds on the Exponent of Primitivity Which Depend on the Spectrum and the Minimal Polynomial

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ABSTRACT

Suppose A is an $n \times n$ nonnegative *primitive* matrix whose minimal polynomial has degree m. We conjecture that the well-known bound on the exponent of primitivity $(n-1)^2+1$, due to Wielandt, can be replaced by $(m-1)^2+1$. The only case for which we cannot prove the conjecture is when $m \geqslant 5$, the number of distinct eigenvalues of A is $m-1$ or m, and the directed graph of A has no circuits of length shorter than $m-1$, but at least one of its vertices lies on a circuit of length not shorter than m. We show that $m(m-1)$ is always a bound on the exponent, this being an improvement on Wielandt's bound when $m < n$. For the case in which A is also symmetric, the bound which we obtain is $2(m-1)$. To obtain our results we prove a lemma which shows that for a (general) *nonnegative* matrix, the number of its distinct eigenvalues is an upper bound on the length of the shortest circuit in its directed graph.

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1. INTRODUCTION

Recall that an $n \times n$ nonnegative matrix A is called *primitive* if for some positive integer N the matrix A^N is positive, or, in notation, $A^N \gg 0$. The *index of primitivity* of A is defined to be

$$\gamma(A) = \min\{k \in \mathbf{Z}_+ : Z^k \gg 0\},$$

where $\mathbf{Z}_+$ denotes the set of positive integers. The celebrated upper bound on $\gamma(A)$ due to Wielandt [27] is

$$\gamma(A) \leqslant (n-1)^2 + 1 =: w_n. \tag{1.1}$$

There is much history and research about finding good bounds for $\gamma(A)$. First, the rough bound of $2n^2 - 2n$ appears already in the works of Frobenius [8, p. 463], and apparently Wielandt stated (1.1) in [27] without proof, but he furnished an example that for some matrices the bound is sharp. The first published proofs of (1.1) appeared in Rosenblatt [21], Holladay and Varga [13], and Pták [20]. Second, there are many improvements of Wielandt's bound for special classes of matrices. For example, it has long been known that if A is (also) symmetric, then

$$\gamma(A) \leqslant 2(n-1). \tag{1.2}$$

It is impossible to list all those that have made contributions to the study of the exponent of primitivity, but let us name some major contributors in alphabetical order: Brualdi and Ross [2], Dulmage and Mendelsohn [4–6], Heap and Lynn [12], Lewin [14–16], Lewin and Vitek [17], Liu, McKay, Wormwald, and Zhang [18], Sedláček [22], Shao [23–25], and Zhang [28]. An additional paper worth mentioning is that of Moon and Moser [19], in which they show that as n increases, almost all (0, 1) matrices are not only primitive, but have an index of primitivity at most 2. For a recent survey of known results concerning the bounds on the exponent of primitivity see Brualdi and Ryser [3].

If one looks at many of the above papers, one sees that they mostly employ graph theory and/or number theory to obtain specific bounds on the special classes of matrices under consideration. A natural question is the following: Let the degree of the minimal polynomial of A be $m = m_A$. Since, as is well known (cf. Gantmacher [10]),

$$\begin{aligned} I + A + \cdots + A^{m-1} &\gg 0, \\ A + A^2 + \cdots + A^m &\gg 0, \end{aligned} \tag{1.3}$$

so that in the directed graph of A, $\Gamma(A)$, any vertex has access to any vertex in a path of length not exceeding m, is there a function $f(\cdot)$ such that

$$\gamma(A) \leqslant f(m)? \tag{1.4}$$

Actually, an immediate consequence of Theorem 2.4.9 in Berman and Plemmons [1] tells us that

$$\gamma(A) \leqslant m(n-1), \tag{1.5}$$

but here a bound is sought that is entirely dependent on m and that remains *competitive* with w_n.

We *conjecture* that $f(\cdot)$ is given by $f(x) = (x-1)^2 + 1$, that is,

$$\gamma(A) \leqslant (m-1)^2 + 1. \tag{1.6}$$

We shall show that in most cases the conjecture holds true and indicate in which instances we cannot prove the conjecture, but we shall also show that for the function $f(\cdot)$, where $f(x) = x(x-1)$, (1.4) is always true, so that

$$\gamma(A) \leqslant m(m-1). \tag{1.7}$$

Note that when $n > m$, we have $m(m-1) < w_n$, and also $m(m-1)$ is better than (1.5). Moreover, since it can happen that $n \gg m$, $m(m-1)$ can represent a good improvement on w_n.

Interestingly, the cases for which we cannot show (1.6) in general are:

(i) when A has m distinct eigenvalues, in which case A is diagonalizable, and $\Gamma(A)$ has no circuit of shorter length than $m-1$, but at least one of its vertices lies on a circuit whose length is not shorter than m, and

(ii) when A has $m-1$ distinct eigenvalues and $\Gamma(A)$ has no circuit of shorter length than $m-1$.

(The reader should note that it is a consequence from a paper of Fiedler [7] that *any* primitive matrix must have a principal submatrix of order m and of rank $m - 1$ or m). In Section 4 we shall prove that (1.6) is always true when $m \leqslant 4$. In Section 5, where we shall raise a number of open questions, we shall point out, though, that whether (1.6) is true or not for the cases (i) and (ii) just mentioned, interesting perturbation problems can be posed, which will become interesting facts should (1.6) turn out to be true.

Section 2 will be devoted to essential notation and preliminaries. To prove our main results here we shall prove in Section 3 a fact which is of interest in its own right, namely, that *the number of distinct eigenvalues of a nonnegative matrix A is an upper bound on the length of the shortest circuit in $\Gamma(A)$.* This could be interpreted as a sort of contribution of matrix graph theory to the theory of nonnegative matrices in the sense that by inspecting the circuits in the directed graph we can find a lower bound on the number of distinct eigenvalues of A.

2. NOTATION AND PRELIMINARIES

Throughout this paper we shall work with $n \times n$ real matrices. Whenever it will be clear from the context which matrix we are referring to, we shall let the letters m and s denote, respectively, the degree of the minimal polynomial and the number of distinct eigenvalues of the matrix. The characteristic and minimal polynomials of A will be denoted, respectively, by $\Delta(A)$ and $\psi(A)$. Recall (Gantmacher [9]) that for any positive integer $k \geqslant 1$, the degree of the minimal polynomial of A^k does not exceed the degree of the minimal polynomial of A.

If A and B are $n \times n$ matrices, we shall use the notation

$$A \overset{c}{=} B$$

to denote that A and B have nonzero entries in the same locations. On the other hand, if all entries which are zero in A are also zero in B, then we shall write that

$$A \overset{c}{\geqslant} B.$$

For a matrix A (a vector x) we shall use the notation $A \geqslant 0$ ($x \geqslant 0$), $A > 0$ ($x >$), and $A \gg 0$ ($x \gg 0$) to denote, respectively, that all the entries of A (x) are nonnegative, nonnegative with at least one positive entry, and positive. The usual kth unit vector will be denoted by $e^{(k)}$, $k = 1, \ldots, n$.

Associated with an $n \times n$ matrix $A = (a_{i,j})$ we shall consider its *directed graph* $\Gamma(A)$, which consists of a set V of n vertices, labeled conveniently

$1, \ldots, n$, and a set of directed edges E with a direct edge from vertex i to vertex j if and only if $a_{i,j} \neq 0$. We shall use the notation $i \to j$ to denote the fact that there is a direct edge from vertex i to vertex j, and we shall use the notation $i \nrightarrow j$ to denote the fact that there is no directed edge linking vertex i to vertex j. A *path* from vertex i to vertex j, if it exists, is a sequence of edges of the form $i \to i_1 \to i_2 \to \cdots \to i_r \to j$. If $i = j$, then the path is said to be *closed*, and if $i_1, \ldots, i_r$ are all distinct, then the closed path is called a *circuit*. The *length* of a path is the number of edges it consists of. If there are paths of length $d_1, \ldots, d_s$ connecting vertex i to vertex j, we shall write

$$i \overset{d_1, \ldots, d_s}{\longrightarrow} j.$$

Similarly, if there are no paths of length $d_1, \ldots, d_t$ connecting vertex i to vertex j, we shall write

$$i \overset{d_1, \ldots, d_t}{\nrightarrow} j.$$

The distance $d(i, j)$ from vertex i to vertex j is the minimal length of a path linking vertex i to vertex j. If $a_{i,i} \neq 0$, we shall say that $\Gamma(A)$ has a *loop* at vertex i and put $d(i, i) = 0$. Thus a loop is a circuit of length 1. Recall that A is called *irreducible* if for any $i, j \in V$ there exists a directed path in $\Gamma(A)$ from i to j, and it is well known (e.g., Varga [26]) that this is equivalent to the condition that for no permutation matrix P

$$P^T A P = \begin{pmatrix} \tilde{A}_{1,1} & \tilde{A}_{1,2} \\ 0 & \tilde{A}_{2,2} \end{pmatrix}.$$

Suppose now that $A = (a_{i,j}) \geqslant 0$. Some well-known facts concerning nonnegative matrices which we shall use are the following (cf. Berman and Plemmons [1] and Varga [26]). First, for some positive integer k,

$$(A^k)_{i,j} > 0 \quad \Leftrightarrow \quad \text{there is a path of length } k \text{ from vertex } i \text{ to vertex } j. \tag{2.1}$$

Thus, in particular, if $(A^k)_{i,i} > 0$, then the vertex i lies on a closed path of length k. Second, the spectral radius of A, given by

$$\rho(A) = \max\{|\lambda| : \det(\lambda I - A) = 0\},$$

is an eigenvalue of A, sometimes called the *Perron root* of A, to which there corresponds a nonnegative eigenvector. Third, A is primitive if and only if it is irreducible and its Perron root is simple and is the only eigenvalue of A with maximum modulus. Corresponding to $\rho(A)$, A has a positive eigenvector. In fact A is primitive if and only if A^k is primitive for all $k \geqslant 1$. Moreover, it is shown in Gantmacher [10] and in [1] that if $\lambda_1, \dots, \lambda_{s-1}$ are the distinct eigenvalues of (our primitive matrix) A other than $\rho(A)$ and their multiplicities in the minimal polynomial are, respectively, $k_1, \dots, k_{s-1}$, then

$$(A - \lambda_1 I)^{k_1}(A - \lambda_2)^{k_2} \cdots (A - \lambda_{s-1} I)^{k_{s-1}} \gg 0. \tag{2.2}$$

Finally, it follows immediately from (1.3) and (2.1) that if A is primitive, then for any two vertices $i, j \in V$,

$$d(i,j) \leqslant m - 1 + \delta_{i,j}. \tag{2.3}$$

3. MAIN RESULTS

We begin with the following lemma:

LEMMA 1. *Let A be an $n \times n$ nonnegative and primitive matrix and let m be the degree of its minimal polynomial. If $\Gamma(A)$ has a loop at vertex $k \in V$, then*

$$A^{m-1}e^{(k)} \gg 0. \tag{3.1}$$

In particular, if $\Gamma(A)$ has a loop at every vertex, then

$$A^{m-1} \gg 0. \tag{3.2}$$

Proof. Let $i \in V$ be any vertex. Then, by (1.3) and (2.3), $d(i,k) \leqslant m-1$. Since $A^{(m-1)-d(i,k)}e^{(k)} \geqslant 0$ and $[A^{(m-1)-d(i,k)}e^{(k)}]_k > 0$, it easily follows using (2.1) that

$$[A^{m-1}e^{(k)}]_i = \sum_{j=1}^{n} (A^{d(i,k)})_{i,j}[A^{(m-1)-d(i,k)}e^{(k)}]_j > 0.$$

Because of the arbitrariness of vertex $i \in V$, (3.1) now follows. ∎

The following corollary is a consequence of Lemma 1:

COROLLARY 1. *Suppose $A = (a_{i,j})$ is an $n \times n$ nonnegative and primitive matrix whose minimal polynomial is of degree m. If every vertex $k \in V$ lies on a circuit in $\Gamma(A)$ whose length is not greater than $m - 1$ or is one edge from a circuit whose length is at most $m - 1$, then* (1.6) *holds, that is,*

$$\gamma(A) \leqslant (m-1)^2 + 1.$$

Similarly, if every vertex lies on a circuit in $\Gamma(A)$ of length not greater than $m - 1$ or is one edge to a circuit whose length is at most $m - 1$, then (1.6) *holds.*

Proof. First, suppose that $k \in V$ is a vertex which lies on a circuit in $\Gamma(A)$ whose length is $j_k \leqslant m - 1$. Then $\Gamma(A^{j_k})$ has a loop at k. As A^{j_k} is a primitive matrix whose minimal polynomial has degree at most m, it follows by Lemma 1 that

$$(A^{j_k})^{m-1}e^{(k)} \gg 0. \tag{3.3}$$

Hence

$$A^{(m-1)^2}e^{(k)} = A^{[(m-1)-j_k](m-1)}\Big[(A^{j_k})^{m-1}e^{(k)}\Big] \gg 0.$$

Next suppose that k is one edge from a circuit of length at most $m - 1$. Then there is a vertex $p \in V$ such that $A^{(m-1)^2}e^{(p)} \gg 0$ and such that $a_{p,k} > 0$. It now readily follows that

$$A^{(m-1)^2+1}e^{(k)} = A^{(m-1)^2}(Ae^{(k)}) \gg 0.$$

The proof of the last part of the corollary follows by considering the transpose A^T of A. Now A^T is a primitive matrix and the vertices of its directed graph lie either on a circuit of length at most $m - 1$ or one edge from a circuit of length at most $m - 1$. ■

Lemma 1 also readily leads to the result indicated in (1.7):

THEOREM 1. *Let A be an $n \times n$ nonnegative and primitive matrix whose minimal polynomial has degree m. Then*

$$\gamma(A) \leqslant m(m-1). \tag{1.7}$$

Proof. The proof follows in a similar fashion to the proof of Corollary 1. By (1.3) for each $k \in V$ there is an exponent $1 \leqslant j_k \leqslant m$ such that $\Gamma(A^{j_k})$ has a loop at vertex k. Moreover, A^{j_k} is a nonnegative matrix whose minimal polynomial has degree at most m. Hence, by Lemma 1,

$$\left(A^{k_j}\right)^{m-1} e^{(k)} \gg 0.$$

Thus clearly

$$A^{m(m-1)} e^{(k)} = A^{(m-k_j)(m-1)}\left[\left(A^{k_j}\right)^{m-1} e^{(k)}\right] \gg 0.$$

The arbitrariness of $k \in V$ now leads to our conclusion. ■

We note that while the bound in (1.7) is not as good as the one conjectured in (1.6), it is sharp on the total class of all primitive matrices of all orders, since for $m = 2$, $m(m-1) = (m-1)^2 + 1$. Moreover, we note that on defining for an $n \times n$ nonnegative and primitive matrix A the class of $n \times n$ matrices given by

$$\mathscr{C}_A = \{B : B \text{ is nonnegative and primitive with } B \stackrel{c}{=} A\}, \tag{3.4}$$

we have that

$$\gamma(A) \leqslant \min_{B \in \mathscr{C}_A} \{m_B(m_B - 1)\}, \tag{3.5}$$

where m_B is the degree of the minimal polynomial of B. This raises the interesting perturbation question, to which we shall return later in some special circumstances, of when for an $n \times n$ primitive matrix A there is a matrix $B \in \mathscr{C}_A$ with a minimal polynomial such that $m_B < m_A$.

Another corollary of Lemma 1 is the following bound for the symmetric case, which improves on (1.2).

COROLLARY 2. *Let A be an $n \times n$ nonnegative primitive symmetric matrix whose minimal polynomial has degree m. Then*

$$\gamma(A) \leqslant 2(m-1).$$

Proof. Because of the symmetry of A, every diagonal entry of A^2 is positive, and hence $\Gamma(A^2)$ has a loop at each one of its vertices. Since the minimal polynomial of A^2 has degree not greater than m, it follows at once from Lemma 1 that for each $1 \leqslant k \leqslant m$,

$$(A^2)^{m-1} e^{(k)} \gg 0,$$

from which the result follows. ■

We turn now to investigate some quite general situations in which the bound in (1.6) is true. Recall that Sedláček [22] and Dulmage and Mendelsohn [5] proved that if the length of the shortest circuit in $\Gamma(A)$ is g, then

$$\gamma(A) \leqslant n + g(n-2). \tag{3.6}$$

Our next lemma is in the spirit of (3.6), but always improves on (3.6) if $n > m$. Recall that for any $n \times n$ matrix, the degree of its minimal polynomial is an upper bound on the degree of the minimal polynomial of any positive (integer) power of the matrix.

LEMMA 2. *Let A be an $n \times n$ nonnegative primitive whose minimal polynomial has degree m and suppose that $\Gamma(A)$ has a circuit of length k. Then*

$$\gamma(A) \leqslant (m-1) + k(m_{A^k} - 1), \tag{3.7}$$

where m_{A^k} is the degree of the minimal polynomial of A^k. In particular, if g is the minimal length of a circuit in $\Gamma(A)$, then

$$\gamma(A) \leqslant (m-1) + g(m_{A^g} - 1).$$

Proof. Our choice of wording of the proof follows closely the wording of the proof of Brualdi and Ryser [3, Theorem 3.5]. Now $\Gamma(A^k)$ has at least k loops. Let us denote by W the set of all vertices in $\Gamma(A^k)$ at which there is a loop. Because the degree of the minimal polynomial of A^k is m_{A^k}, any vertex in $\Gamma(A^k)$ can be reached from any vertex in W along a path of length

$m_{A^k} - 1$. Hence, in $\Gamma(A)$, any vertex can be reached from any vertex in W along a path of length $k(m_{A^k} - 1)$. On the other hand, from each vertex in $\Gamma(A)$ there is a path of length at most $m - 1$ to some vertex in W. It is now easy to ascertain that $(e^{(i)})^T A^{(m-1)+k(m_{A^k}-1)} \gg 0$ for all $1 \leqslant i \leqslant n$, from which (3.7) follows. ■

An immediate consequence of this lemma is the following corollary:

COROLLARY 3. *Let A be an $n \times n$ nonnegative and primitive matrix and let m be the degree of its minimal polynomial. If $\Gamma(A)$ contains a circuit of length at most $m - 2$, then* (1.6) *is valid. In fact,*

$$\gamma(A) \leqslant (m-1)^2.$$

Proof. The result follows from (3.7) upon noting that $m_{A^g} \leqslant m$ and $g \leqslant m - 2$. ■

COROLLARY 4. *Let A be an $n \times n$ nonnegative and primitive matrix. If $m \leqslant 4$ then* (1.6) *holds. Moreover, if $m \geqslant 5$ and A has real eigenvalues only, then*

$$\gamma(A) \leqslant (m-1) + 2(m_{A^2} - 1) \leqslant 3(m-1) \leqslant (m-1)^2.$$

Proof. We shall show that (1.6) is true whenever $m \leqslant 4$ in Theorem 4 in Section 4. Suppose now that $m \geqslant 4$ and A has real eigenvalues only. Since at least one of these eigenvalues is $\rho(A) > 0$, we have trace $A^2 > 0$, so that $\Gamma(A)$ must contain at least one circuit of length $2 \leqslant m - 2$. ■

The obvious question which Corollary 3 raises is: when does $\Gamma(A)$ contain a circuit of length not greater than $m - 2$? To this end let us prove the following lemma:

LEMMA 3. *Let A be an $n \times n$ nonnegative matrix with s distinct eigenvalues. Then $\Gamma(A)$ contains a circuit of length not greater than s.*[1]

Proof. If $\rho(A) = 0$, then A is nilpotent, $s = 1$, and $\Gamma(A)$ has no circuits. Suppose then that $\rho(A) > 0$, denote the distinct eigenvalues of A by $\lambda_1, \dots, \lambda_s$, and let their respective multiplicities in the characteristic polynomial be $l_1, \dots, l_s$.

[1] Seeing this lemma, Professor Hans Schneider observed that with few modifications, its conclusion nicely holds for general matrices also.

Suppose now that all circuits in $\Gamma(A)$ have length greater than s so that trace $A^k = 0$ for all $k = 1, \ldots, s$. But then

$$\begin{pmatrix} \lambda_1 & \cdots & \lambda_s \\ \vdots & & \vdots \\ \lambda_1^s & \cdots & \lambda_s^s \end{pmatrix} \begin{pmatrix} l_1 \\ \vdots \\ l_s \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.8}$$

This is not possible even if one of the eigenvalues of A is zero, as can be seen from the system of equations

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ \vdots & \vdots & & \vdots \\ \lambda_1^{s-1} & \lambda_2^{s-1} & \cdots & \lambda_s^{s-1} \end{pmatrix} \begin{pmatrix} \lambda_1 l_1 \\ \lambda_2 l_2 \\ \vdots \\ \lambda_s l_s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

in which the coefficient matrix is now nonsingular, being the Vandermonde matrix of order s on s mutually distinct symbols. ■

Lemma 3 and Corollary 2 have the following implication:

THEOREM 2. *Let A be an $n \times n$ nonnegative and primitive matrix whose minimal polynomial has degree m. If the number of distinct eigenvalues of A does not exceed $m - 2$, then* (1.6) *is true. In fact,*

$$\gamma(A) \leqslant (m-1)^2.$$

A corollary to Theorem 2 is the following:

COROLLARY 5. *Let A be an $n \times n$ nonnegative and primitive matrix whose minimal polynomial has degree m with a real root of multiplicity* 3 *or a nonreal root of multiplicity* 2. *Then*

$$\gamma(A) \leqslant (m-1)^2.$$

Another implication of Corollary 3 and of Lemmas 2 and 3 is the following:

COROLLARY 6. *Let A be an $n \times n$ nonnegative and primitive matrix with minimal polynomial $\psi(\lambda)$ of degree m and $s = m - 1$ distinct roots. If 0 is a root of $\psi(\lambda)$ of multiplicity* 2 *or $\psi(\lambda)$ has a pair of roots ν and μ such that $\nu^k = \mu^k$ for some integer $2 \leqslant k \leqslant s$ such that $k \mid s$, then*

$$\gamma(A) \leqslant (m-1)^2.$$

Proof. Let g be the length of the shortest circuit in $\Gamma(A)$. If $g \leqslant m-2$, the claim follows by Corollary 2. Suppose then that $g > m-2$. By Lemma 3 we now must have that $g = m-1$. The result now follows from (3.7) because $m_{A^{m-1}} \leqslant m-1$. ∎

We close this section with the following result, which differs in spirit from the preceding statements.

THEOREM 3. *Let A be an $n \times n$ nonnegative primitive matrix whose minimal polynomial has degree m. If there exists an integer $1 \leqslant p \leqslant m-1$ such that*

$$A^m \overset{c}{\geqslant} A^p, \tag{3.9}$$

then

$$\gamma(A) \leqslant p + (m-1)(m-p) \leqslant (m-1)^2 + 1, \tag{3.10}$$

with equality holding in the second inequality if and only if $p = 1$.

Proof. Consider the matrix $B := A^{m-p}$, which is primitive with a minimal polynomial of degree at most m. By (1.3)

$$1 + B + \cdots + B^{m-1} \gg 0. \tag{3.11}$$

Multiplying both sides of (3.11) by A^p and making use of the facts that $A^pB^k = A^mB^{k-1}$ for $k \geqslant 1$ and that, because of (3.9), $A^p + A^pB = A^p + A^m \overset{c}{=} A^m$, we get that

$$\begin{aligned} 0 &\ll \underbrace{A^p + A^pB}_{A^p + A^m \overset{c}{=} A^m} + BA^m + \cdots + B^{m-2}A^m \\ &\overset{c}{=} A^m(I + B + + \cdots + B^{m-2}) \\ &= B(A^p + A^pB + + \cdots + A^pB^{m-2}) \\ &\overset{c}{=} BA^m(I + \cdots + B^{m-3}) \\ &= \cdots \overset{c}{=} A^{(m-p)(m-1)+p}, \end{aligned}$$

establishing (3.10). That equality occurs in the second inequality in (3.10) when, and only when, $p = 1$ is straightforward to ascertain. ■

4. THE CASE $m \leqslant 4$

Corollaries 1 and 3 tell us that to examine to what extent the conjecture in (1.6) is true, it suffices to assume that A has *no* circuit s of length less than $m-1$ and that at least one vertex, say k, in V lies on a circuit of length m, but not shorter. Moreover, Lemma 2 tells us that in this case A must have at least $s = m - 1$ distinct eigenvalues $\lambda_1, \dots, \lambda_s$. If $s = m - 1$, then, without loss of generality, we shall suppose that λ_s is the (only) root of multiplicity 2 of the minimal polynomial. Note that if i is a vertex in $\Gamma(A)$ which lies on a circuit of length $m-1$, then i is a vertex in $\Gamma(A^T)$ which lies on a circuit of length $m-1$. Thus on inspecting the principal argument used in the proof of Corollary 1, viz. (3.3), we can deduce that $(e^{(i)})^T A^{(m-1)^2+1} \gg 0$.

Suppose, as usual, that A is an $n \times n$ primitive matrix whose minimal polynomial $\psi(A)$ has degree m. Then the substitution of A in the minimal polynomial results in the matrix equality

$$A^m = \alpha_{m-1} A^{m-1} + \cdots + \alpha_0 I \tag{4.1}$$

for some real scalars $\alpha_0, \dots, \alpha_{m-1}$. Because for some vertex $k \in V$ we must have that $(A^{m-1})_{k,k} = 0$ while $(A^m)_{k,k} > 0$, we see, using (2.2), that

$$\alpha_0 = (-1)^{m+1} \lambda_1 \cdots \lambda_m > 0. \tag{4.2}$$

Also, as

$$A^{m+1} = \alpha_{m-1} A^m + \alpha_{m-2} A^{m-1} + \cdots + \alpha_0 A, \tag{4.3}$$

we must have too that

$$\alpha_{m-1} = \lambda_1 + \cdots + \lambda_{s-1} + k_s \lambda_s \geqslant 0, \tag{4.4}$$

where k_s is 1 or 2 according as $s = m$ or $s = m - 1$. As in previous sections, we shall denote the multiplicities of $\lambda_1, \dots, \lambda_s$ as roots of the characteristic polynomial by $l_1, \dots, l_s$.

The goal of this section is to show that the bound (1.6) on the exponent of primitivity is true when $m \leqslant 4$.

THEOREM 4. *Suppose that A is an $n \times n$ nonnegative and primitive matrix whose minimal polynomial has degree $m \leqslant 4$. Then*

$$\gamma(A) \leqslant (m-1)^2 + 1.$$

Proof. As explained in the opening paragraph of this section, if $i \in V$ is a vertex which lies on a circuit of length $m-1$, then $(e^{(i)})^T A^{(m-1)^2+1} \gg 0$. Thus, we shall only need to show that $(e^{(i)})^T A^{(m-1)^2+1} \gg 0$ holds for any vertex in V which lies on a circuit of length m, but not shorter.

The case $m = 2$. For this case we have already found, following Theorem 1, that the bound $m(m-1)$ equals $(m-1)^2 + 1$.

The case $m = 3$. As mentioned at the start of this section, we can begin by supposing that $\Gamma(A)$ has no circuits of length smaller than $m - 1 = 2$. Without loss of generality, let us assume that the Perron root λ_1 of A is equal to 1. If the remaining roots of $\psi(\lambda)$ are denoted by λ_2 and λ_3, then (4.1) reduces to

$$A^3 = (1 + \lambda_2 + \lambda_3) A^2 - (\lambda_2 + \lambda_3 + \lambda_2 \lambda_3) A + \lambda_2 \lambda_3 I. \tag{4.5}$$

First suppose that λ_2 and λ_3 are real. Because $0 = \text{trace } A = 1 + l_1 \lambda_1 + l_2 \lambda_2$, then at least one of λ_2 or λ_3 must be negative. But then, by (4.2), both λ_2 and λ_3 must be negative. This shows that $\lambda_2 + \lambda_3 + \lambda_2 \lambda_3 = \lambda_2(1 + \lambda_3) + \lambda_3 < 0$. If now $\alpha_2 > 0$, then

$$A^3 = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$$

with all three coefficients positive, giving us that $A^3 \gg 0$ by (1.3) and $3 < (m-1)^2 + 1 = 5$. If, on the other hand, $\alpha_2 = 0$, then the conjunction of trace $A = 0$ and (1.4) gives that

$$(l_2 - 1)\lambda_2 = -(l_3 - 1)\lambda_3.$$

In view of λ_2 and λ_3 having the same sign, this is only possible when $l_2 = l_3 = 1$, in which case $m = n$. But then our assertion is true by (1.1).

Assume next that λ_2 and λ_3 are a conjugate pair, in which case $l_2 = l_3$. From trace $A = 0$ we must have that their real parts are negative. If $\alpha_2 = 0$, then taking account of (4.4) yields that

$$2(l_2 - 1)\Re(\lambda_2) = 0,$$

which is only possible when $l_2 = 1$, so that, as $l_2 = l_3$, we must have that

$m = n$, and the result follows by (1.1). Suppose now that $\alpha_2 > 0$. If $\alpha_1 > 0$, then $A^3 \overset{c}{=} A^2 + A + I \gg 0$ by (1.3). If $\alpha_1 = 0$, then $A^3 \overset{c}{=} A^2 + I$, so that $A^3 \overset{c}{\geqslant} A^2$, and the conclusion follows by Theorem 3. Suppose then that $\alpha_1 < 0$. Then

$$A^3 + A \overset{c}{=} A^2 + I.$$

But then (1.3) has the implication that $(A^2)_{i,j} > 0$ for all $i \neq j$, $i, j = 1, \ldots, n$. In this case we easily have that $A^4 \gg 0$ and $4 < 5 = (m-1)^2 + 1$.

The case $m = 4$. Again we can begin by supposing that $\Gamma(A)$ has no circuits of length smaller than $m - 1 = 3$. Here, as trace $A^2 = 0$, it is not possible for λ_2, λ_3, and λ_4 to all be real. Suppose, without loss of generality, that $\lambda_3 = \overline{\lambda}_4$, so that also $l_3 = l_4$. Because of (4.2), on continuing to take $\lambda_1 = 1$, we see that $-1 < \lambda_2 < 0$.

Suppose now that

$$0 = \alpha_3 = 1 + \lambda_2 + 2\Re(\lambda_3).$$

Then $\Re(\lambda_3) = \Re(\lambda_4) < 0$. But then we get, on using the fact that trace $A = 0$, that

$$(l_2 - 1)\lambda_2 + 2(l_3 - 1)\Re(\lambda_3) = 0.$$

In view of the negativeness of λ_2 and $\Re(\lambda_3)$, this equality is only possible if $l_2 = l_3 = 1$, showing that $m = n = 4$. The result now follows by (1.1).

Suppose next that $\alpha_3 > 0$. We require the following general observation, whose proof follows by simple verification:

OBSERVATION. *Suppose that a positive integer ν satisfies $\nu \geqslant 4$. Then for any integer $5 \leqslant e \leqslant \nu + 2$,*

$$e + (e-5)\nu + (\nu + 2 - e)(\nu + 1) = (\nu - 1)^2 + 1 =: \beta_\nu. \quad (4.6)$$

REMARK 1. What the condition (4.6) means is this. Suppose in $\Gamma(A)$ there is a vertex k which lies on circuits of length ν and $\nu + 1$, respectively. Suppose i is a vertex for which there is a path $i \xrightarrow{d} k$. If

$$d \in \mathscr{S}_\nu := \{1, 2, 5, 6, \ldots, \nu + 2\}, \quad (4.7)$$

then in $\Gamma(A)$ there is a path

$$i \xrightarrow{\beta_\nu} k. \tag{4.8}$$

We now return to the case $m = 4$ with $\alpha_3 > 0$. Putting $\nu = m = 4$ we see that if for a vertex $i \in V$, there is a path

$$i \xrightarrow{d} k \qquad \text{with} \quad d \in \mathscr{S}_m,$$

then there exists a path

$$i \xrightarrow{\beta_m} k.$$

Suppose then that

$$i \overset{d}{\nrightarrow} k \quad \text{in } \Gamma(A) \qquad \text{with} \quad d \in \mathscr{S}_m.$$

Then, by (4.7), $d = 3$ or $d = 4$. Clearly $i \neq k$; otherwise $i = k \xrightarrow{5} k$. Set

$$r := (A^3)_{i,k} \quad \text{and} \quad q := (A^4)_{i,k}.$$

At first let us suppose that

$$i \xrightarrow{3} k, \quad \text{but} \quad i \overset{4}{\nrightarrow} k.$$

Then from (4.1) we have that $0 = q = \alpha_3 r > 0$, which is not possible. Assume therefore that

$$i \overset{3}{\nrightarrow} k, \quad \text{but} \quad i \xrightarrow{4} k.$$

Then $0 < q = \alpha_3 r$, which is again not possible. Finally assume that $i \xrightarrow{3,4} k$, but recall that

$$i \overset{1,2,5,6}{\nrightarrow} k.$$

Then we have that $q = \alpha_3 r > 0$, $(A^5)_{i,k} = \alpha_3 q + \alpha_2 r = 0$, and $(A^6)_{i,k} = \alpha_2 q + \alpha_1 r = 0$. As $\alpha_3 > 0$, for these equalities to hold it is necessary that

$\alpha_2 < 0$ and $\alpha_1 > 0$. But then

$$A^4 + A^2 \overset{c}{=} A^3 + A + I \gg 0. \tag{4.9}$$

Consider now $i \xrightarrow{3} k$, say $i \to j \to t \to k$. It is not permissible that $j \xrightarrow{1} k$, for otherwise $i \xrightarrow{2} k$, which is not possible. Hence, from (4.9) we see that $j \xrightarrow{3} k$. Next, $k \overset{1}{\nrightarrow} j$, for otherwise vertex k would lie on a circuit of length 3, which is not possible. Thus, by (4.9), we see that $k \xrightarrow{3} j$, so that vertex k lies on a circuit of length 6. Hence

$$i \xrightarrow{4} k \xrightarrow{6} k \quad \Rightarrow \quad i \xrightarrow{\beta_4} k.$$

We have thus shown that every $i \in V$ has a path of length $\beta_4 = (4-1)^2 + 1 = 10$ to vertex k. Said otherwise, $(e^{(k)})^T A^{\beta_4} \gg 0$. ■

5. CLOSING REMARKS

As usual, let A be an $n \times n$ nonnegative and primitive matrix whose minimal polynomial $\psi(\lambda)$ has degree m. From Sections 3 and 4 we can conclude that the cases for which we were unable to prove the conjecture in (1.6) are when $n > m \geqslant 5$, $\psi(\lambda)$ has nonreal roots, the number of distinct roots of $\psi(\lambda)$ is $s = m - 1$ or $s = m$, and the length of the shortest circuit in $\Gamma(A)$ [the so-called *girth of* $\Gamma(A)$] is at least $m - 1$ with at least one of the vertices lying on a circuit of length no shorter than m.

On the basis of the above we remark the following:

(a) If $s = m - 1$, then by Lemma 3, the length of the shortest circuit in $\Gamma(A)$ is bounded by $m - 1$. We have not been able to show, by means of an example or otherwise, that (subject to the stipulation on A) if $s = m$, so that A is diagonalizable, then the length of the shortest circuit can attain m. This seems to us very unlikely to occur, at least not in many cases, because from

$$\operatorname{trace} A = \operatorname{trace} A^2 = \cdots = \operatorname{trace} A^{m-1} = 0$$

and using the notation of Lemma 3, we have that the multiplicities of λ_1 $[= \rho(A)]$, $\lambda_2, \ldots, \lambda_m$ in the characteristic polynomial have to satisfy

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \vdots & \vdots & & \vdots \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \cdots & \lambda_m^{m-1} \end{pmatrix} \begin{pmatrix} 1 \\ l_2 \\ \vdots \\ l_m \end{pmatrix} = \begin{pmatrix} n \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

It can be readily checked from Cramer's rule that the first component of the solution vector, namely 1, is given by

$$1 = \frac{n\prod_{i=2}^{m} \lambda_i}{\prod_{i=2}^{m}(\lambda_i - \lambda_1)}$$

or, equivalently,

$$\frac{1}{n} = \frac{\prod_{i=2}^{m} \lambda_i}{\prod_{i=2}^{m}(\lambda_i + \lambda_1)}. \tag{5.1}$$

Moreover, from (4.1) and (4.2), we also see that if such a minimum length for the shortest circuit is possible, then the diagonal entries of A^m are a constant given by

$$(A^m)_{k,k} = \alpha_0 = (-1)^{m+1} \prod_{i=1}^{m} \lambda_i, \qquad k = 1, \dots, n. \tag{5.2}$$

It is very doubtful in our minds that the conjunction of the algebraic and quantitative conditions of (5.1) and (5.2) can hold (at least in many many cases).[2]

(b) Suppose we continue with the assumption in (a) that $s = m$, $m < n$, and the minimal circuit in $\Gamma(A)$ has length m. Then for *no* matrix E such that $A + E$ is nonnegative and such that

$$A \stackrel{c}{=} A + E,$$

can $A + E$ have fewer than m distinct eigenvalues; otherwise $A + E$ would have a circuit of length at most $m - 1$. Thus it is not possible to perturb A in such a way that it remains primitive, its zero-nonzero structure is retained,

[2] If we forgo the assumptions on A made at the beginning of this section, then A can indeed satisfy $n > s = m$, yet have a shortest circuit of length m, as the following example illustrates:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Here the distinct eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$, and because of the symmetry of A, we have $s = m = 2$, but $\Gamma(A)$ has no circuits of length 1.

and the number of its distinct eigenvalues is decreased. Hence, subject to such perturbations, the first exceptional points which the trajectories of the eigenvalues of A can reach can only be points of bifurcation and not points of collapse. Is this possible for such matrices?

(c) Some of the results in this paper can be generalized by replacing m with any positive integer $t \geqslant 2$, which can depend on A, for which

$$I + A + \cdots + A^{t-1} \gg 0, \tag{5.3}$$

or in particular with the minimal positive integer $t = t_A$ for which (5.3) holds. Indeed, in a working paper Hartwig [11] conjectured that for the minimal such t (or for that matter for any such t),

$$\gamma(A) \leqslant (t-1)^2 + 1. \tag{5.4}$$

We have chosen to state here all our results in terms of $t = m = m_A$ because it is a very well recognizable function of the matrix and because of the connection between our bounds and other spectral properties of the matrix. It should be pointed, though, that computing the degree of minimal polynomial is not always a simple task.

During the Combinatorial Matrix Theory Workshop at the Institute of Mathematics and its Applications (IMA) at the University of Minnesota in November 1991, Professors David Gregory, Steve Kirkland, and Norm Pullman informed us that they are in the process of writing up a manuscript which contains a bound on $\gamma(A)$ which depends on the Boolean rank of A.

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