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Locally (soluble-by-finite) groups with all proper non-nilpotent subgroups of finite rank

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Abstract

A group G is said to have finite rank r if every finitely generated subgroup of G is at most r-generator. If c is a positive integer we let $\mathfrak{N}_c$ denote the class of nilpotent groups of class at most c, and $\mathfrak{N}_c^*$ the class of groups in which every proper non-$\mathfrak{N}_c$ subgroup has finite rank. Our main theorem shows that if G is a locally (soluble-by-finite) group in the class $\mathfrak{N}_c^*$ then either G is nilpotent of class at most c or G has finite rank. An analogous result holds for locally soluble $(\mathfrak{A}^2)^*$-groups, where $\mathfrak{A}^2$ denotes the class of metabelian groups. We give an example to show that locally finite $(\mathfrak{A}^2)^*$-groups need neither have finite rank nor be metabelian. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

It was shown in [3] that a locally (soluble-by-finite) group G has finite rank if every locally soluble subgroup of G has finite rank; here a group G is said to have (finite) rank r if every finitely generated subgroup of G is r-generated. The motivation for establishing results such as this arises from the classic theorems of Merzljakov [10] and Šunkov [16] concerning groups with rank restrictions on their abelian subgroups. Other results of this type are mentioned briefly in [3,14] provides an appropriate survey (in addition to some new results).

A locally soluble group with all abelian subgroups of finite rank need not have finite rank [10]. In the opposite direction, so to speak, one may consider groups in which every proper non-abelian subgroup has finite rank. The question as to which groups are

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"minimal non-finite rank" is far from having been answered satisfactorily. (A group G is minimal non-$\mathscr{P}$ if every proper subgroup of G has property $\mathscr{P}$ but G does not.) Our main purpose here is to show that in a locally (soluble-by-finite) group, finiteness of rank of the whole group can be inferred from that of the proper non-abelian subgroups. Indeed, considerably more than this may be shown. For a positive integer c let $\mathfrak{N}_c$ denote the class of nilpotent groups of class at most c, and $\mathfrak{N}_c^*$ the class of groups in which every proper non-$\mathfrak{N}_c$ subgroup has finite rank. We shall prove the following.

Theorem A. *Let G be a locally (soluble-by-finite) group in the class $\mathfrak{N}_c^*$ for some positive integer c. Then either $G \in \mathfrak{N}_c$ or G has finite rank.*

Note that there is no corresponding result if we replace $\mathfrak{N}_c$ by $\mathfrak{N}$, the class of all nilpotent groups; e.g., the Heineken–Mohamed groups [6] are locally nilpotent, non-nilpotent and of infinite rank but have all proper subgroups nilpotent. However, there is an analogue of Theorem A for the class of locally nilpotent groups, namely Theorem B below.

We now proceed to obtain a generalization of Theorem A. Recall first that a group G is locally graded if every finitely generated non-trivial subgroup of G has a finite non-trivial image. Denote by Λ the set of closure operations $\{\mathbf{L}, \mathbf{R}, \acute{\mathbf{P}}, \dot{\mathbf{P}}\}$; thus a class $\mathfrak{C}$ of groups is Λ-closed if it is closed under the formation of local systems, subcartesian products and both ascending and descending normal series. In [2], Černikov considers the class $\mathfrak{X}$ obtained by taking the Λ-closure of the class of locally graded periodic groups. It is routine to show that every $\mathfrak{X}$-group is locally graded. (We have been unable to locate an example of a locally graded group which is not in $\mathfrak{X}$. Certainly $\mathfrak{X}$ is a very extensive class of groups.) The main result of [2] is that an $\mathfrak{X}$-group of finite rank is almost locally soluble, i.e., has a locally soluble normal subgroup of finite index. Suppose that $G \in \mathfrak{X} \cap \mathfrak{N}_c^*$, and let F be an arbitrary finitely generated subgroup of G. From the $\mathfrak{N}_c^*$-property and [2], we see that F is soluble-by-finite. The following is then a consequence of Theorem A.

Theorem A′. *Let c be a positive integer and suppose that $G \in \mathfrak{X} \cap \mathfrak{N}_c^*$. Then either $G \in \mathfrak{N}_c$ or G has finite rank. In either case G is almost locally soluble.*

Now, let $\mathbf{L}\mathfrak{N}$ denote the class of locally nilpotent groups. The following result will be established, the notation being the obvious one.

Theorem B. *Let G be a locally (soluble-by-finite) group and suppose that $G \in (\mathbf{L}\mathfrak{N})^*$. Then either G is locally nilpotent or G has finite rank. In either case G is almost locally soluble.*

A similar observation to that above gives the following generalization.

Theorem B′. *Suppose that G belongs to $\mathfrak{X} \cap (\mathbf{L}\mathfrak{N})^*$. Then $G \in \mathbf{L}\mathfrak{N}$ or G has finite rank. In either case G is almost locally soluble.*

Having established some positive results with regard to the class $\mathfrak{R}_c^*$, one might direct attention to groups in the class $(\mathfrak{A}^d)^*$ for some d, where $\mathfrak{A}^d$ denotes the class of soluble groups of derived length at most d. However, as we shall see in Section 3, a locally finite group in $(\mathfrak{A}^2)^*$ need not be metabelian or of finite rank, and so we restrict our attention to locally soluble groups. Even here we have had only limited success; part of the problem seems to result from our lack of knowledge concerning (infinite) soluble groups all of whose proper subgroups have derived length at most $d(\geq 3)$.

Theorem C. *Let G be a locally soluble group in the class $(\mathfrak{A}^2)^*$. Then either G is metabelian or G has finite rank.*

It is natural to ask whether the conclusion of Theorem C holds for other classes of generalized soluble groups, for instance, the class of radical groups. In fact, just a little extra work shows that Theorem C may be extended so as to cover a very large class of groups, one which is obtained in a manner similar to that which gave the class $\mathfrak{X}$ mentioned above. We shall prove the following.

Theorem C′. *Let $\mathfrak{Y}$ denote the A-closure of the class of locally soluble groups. If $G \in \mathfrak{Y} \cap (\mathfrak{A}^2)^*$ then either G is metabelian or G has finite rank. In either case G is locally soluble.*

We have already remarked that Theorem C does not extend to locally (soluble-by-finite) groups. It is also possible to find groups in the class $\mathfrak{A}^*$ which are neither abelian nor of finite rank; some such groups are exhibited in Section 3.

We conclude this introduction with the rather trivial observation that the metabelian group $G = A \rtimes \langle x \rangle$, where A is an infinite elementary abelian 3-group and x is the inverting automorphism, has all of its proper subnormal subgroups abelian but is neither abelian nor of finite rank.

2. The proof of the theorems

Throughout this section, $\mathfrak{R}$ denotes the class of groups of finite rank. Our first aim is to establish Theorem B, and we approach the proof by means of a sequence of lemmas.

Lemma 1. *Let G be a locally soluble group and suppose that every proper subgroup of G has finite rank. Then G has finite rank.*

Proof. Let N be the product of all the proper normal subgroups of G. If $N \neq G$ then G/N is simple and hence of prime order, and the result follows. Suppose then that $N = G$. Since, a locally soluble group of finite rank has an ascending characteristic series with abelian factors [13, Lemma 10.39], G is hyperabelian. If G is abelian then the result is clear; otherwise every abelian subgroup of G has finite rank and therefore, by [1], so does G. □

Lemma 2. *Let G be a locally soluble group in the class $(\mathbf{L}\mathfrak{N})^*$. Then either $G \in \mathbf{L}\mathfrak{N} \cup \mathfrak{R}$ or $G' \in \mathbf{L}\mathfrak{N}$ and G/G' is a non-trivial cyclic p-group for some prime p.*

Proof. We shall assume that $G \notin \mathbf{L}\mathfrak{N} \cup \mathfrak{R}$, so that G has infinite rank and there exists a finitely generated non-nilpotent subgroup F of G. If G/G' has infinite rank, then there is a proper subgroup H of G such that $FG' \leq H$ and H/G' has infinite rank. Since $F \leq H$, H is not locally nilpotent and we have a contradiction. Thus, G' has infinite rank. Now FG' is neither locally nilpotent nor of finite rank so it follows that $G = FG'$. Suppose that the finitely generated abelian group G/G' is a direct product of non-trivial subgroups. Then there exist proper normal subgroups A, B of G such that G' is a proper subgroup of both A and B, and $G = AB$. However, A and B are locally nilpotent since they have infinite rank. This implies that $G = AB$ is locally nilpotent, a contradiction. Therefore, the result is clear if $G \neq G'$ and we may assume that G is perfect.

By Lemma 1 and Zorn's Lemma, G has a maximal locally nilpotent subgroup S of infinite rank and it follows that S is a maximal subgroup of G. If every proper normal subgroup of G is contained in S then all such subgroups are contained in $\mathrm{core}_G S$ and it follows that $G/\mathrm{core}_G S$ is simple and therefore cyclic. This contradicts our assumption that G is perfect, and consequently, $G = NS$ for some proper normal subgroup N. Now $G/N \cong S/S \cap N$ is locally nilpotent and perfect. Consequently, G/N is not finitely generated. It follows that if every proper subgroup of G which contains N is locally nilpotent, then G is also locally nilpotent, a contradiction. Thus, N is contained in a proper subgroup of G which is not locally nilpotent and so N has finite rank. Now, Lemma 1 implies that G/N has proper subgroups of infinite rank and so N is locally nilpotent. Therefore, we may assume that N is the Hirsch–Plotkin radical of G. Thus, every proper normal subgroup C/N of G/N has finite rank, otherwise C has infinite rank and is therefore locally nilpotent and then $C \leq N$. It follows (see [13, Section 6.3]) that G/N is hypercentral. However, G/N is perfect, and it is well known that the hypercentre of a perfect group is equal to its centre. Thus, $G = N$ and G is locally nilpotent. This contradiction completes the proof. □

We can now deal with the locally soluble case of Theorem B.

Lemma 3. *Let G be a locally soluble group in the class $(\mathbf{L}\mathfrak{N})^*$. Then $G \in \mathbf{L}\mathfrak{N} \cup \mathfrak{R}$.*

Proof. Assume the result false, so that, by Lemma 2, $G' \in \mathbf{L}\mathfrak{N}$ and G/G' is a non-trivial finite cyclic group. Observe that every nilpotent image of G is cyclic. Furthermore, since $G \notin \mathbf{L}\mathfrak{N}$, there exists a finitely generated non-nilpotent subgroup F of G. Let R be the finite residual of G.

Suppose first that G/R is infinite and let M denote an arbitrary normal subgroup of finite index in G. Clearly, M has infinite rank and, since FM contains the finitely generated non-nilpotent subgroup F, we deduce that $G = FM$. Thus, the derived length of $G/M \cong F/F \cap M$ is at most the derived length of F, and it follows that G/R is soluble. Therefore, G has an infinite soluble image and, since G/G' is finite, we deduce that there exists a G-invariant subgroup A of finite index in G' such that A/A' is infinite. Let D be any subgroup of A such that $A' \le D$ and A/D is periodic and of finite rank. Since A has finite index in G and A/D has finite rank we deduce that $A/\mathrm{core}_G D$ has finite rank. Thus, $G/\mathrm{core}_G D$ has finite rank and so $C = \mathrm{core}_G D$ has infinite rank. Therefore, $G = FC$. Now, A/C has finite index in the finitely generated group $G/C \cong F/F \cap C$ and so A/C is finitely generated. Thus, A/D is finitely generated and is isomorphic to a section of G/C. Therefore, A/D is a finite group that can be generated by s elements, where s is the rank of G/C. We have shown that all periodic images of A/A' that have finite rank are finite s-generator groups. It follows easily that A/A' is finitely generated, and hence, that G/A' is finitely generated but infinite.

Let N/A' be a normal subgroup of finite index in G/A'. Let X be any subset of G/A' such that $G/A' = \langle X, N/A' \rangle$. If $\langle X \rangle = G/A'$ for all such sets X, it follows that $N/A' \le \Phi(G/A')$, the Frattini subgroup of G/A', and so $\Phi(G/A')$ has finite index in G/A'. However, a finitely generated soluble group that is finite modulo its Frattini subgroup is itself finite [9] and since G/A' is infinite we deduce that $\langle X \rangle \ne G/A'$ for some $X \subseteq G/A'$ with $G/A' = \langle X, N/A' \rangle$. Now, G/A' is polycyclic so each proper subgroup of G/A' is contained in a proper subgroup of finite index. Since G/N is an image of $\langle X \rangle$ we see that every finite image of G/A' is a section of a proper subgroup of finite index in G/A'. If K/A' has finite index in G/A' it follows that K has infinite rank and so K is locally nilpotent. Thus, every finite image of G/A' is nilpotent. However, G/A' is polycyclic and hence residually finite. Therefore, G/A' is residually nilpotent and hence abelian. Thus, G/G' is infinite, a contradiction.

We deduce that G/R is finite. This implies that R has no non-trivial finite images and therefore, arguing as above with R/R' in place of A/A', that R has no non-trivial abelian images; that is, R is perfect. Let W be an arbitrary proper G-invariant subgroup of R. If W has infinite rank then $G = FW$ and R/W is isomorphic to a subgroup of the soluble group $F/F \cap W$, a contradiction, since R is perfect. Thus, W has finite rank and is therefore contained in the hypercentre $\bar{\zeta}(R)$ of R (as in Lemma 2). Since $\bar{\zeta}(R) \ne R$ we deduce that $R/\bar{\zeta}(R)$ is a chief factor of G, and hence is abelian [13, p. 154], contradicting $R = R'$. The lemma is proved. □

Corollary 4. *Let G be an almost locally soluble group. If $G \in (\mathbf{L}\mathfrak{N})^*$ then $G \in \mathbf{L}\mathfrak{N} \cup \mathfrak{R}$.*

Proof. Suppose that G has infinite rank and let H denote the locally soluble radical of G. Since H has finite index in G, H has infinite rank. It follows that every proper subgroup of G that contains H is locally nilpotent. Thus, every proper subgroup of

G/H is locally nilpotent, and hence, nilpotent. By [15], G/H is soluble and it follows that G/H is trivial. Thus, G is locally soluble and the result follows from Lemma 3. □

Our final requirement for the proof of Theorem B is the following.

Lemma 5. *If G is a locally finite $(\mathbf{L}\mathfrak{N})^*$-group then $G \in \mathbf{L}\mathfrak{N} \cup \mathfrak{R}$.*

Proof. Again we shall assume the result to be false.

We suppose first that $G' \neq G$. Suppose that there exists a proper subgroup N of G such that $G' \leq N$ and N has infinite rank. Then every proper subgroup of G that contains N is locally nilpotent. If G/N is not finitely generated then evidently G is locally nilpotent, a contradiction. If G/N is finitely generated then it is finite and so G is almost locally nilpotent, contradicting Corollary 4. Therefore, every proper subgroup of G that contains G' has finite rank. Since an abelian group with all its proper subgroups of finite rank itself has finite rank, we deduce that G/G' has finite rank. However, G' has finite rank and it follows that G has finite rank, a contradiction. Therefore, G is perfect.

A result of Šunkov [16] asserts that a locally finite group in which all abelian subgroups have finite rank is almost locally soluble. Thus, Corollary 4 implies that G contains an abelian subgroup of infinite rank and so Zorn's Lemma and the $(\mathbf{L}\mathfrak{N})^*$-hypothesis imply that G contains a maximal subgroup S which is locally nilpotent. If $G = NS$ for some proper normal subgroup N then we may argue as in the proof of Lemma 2 to obtain a contradiction (appealing to [16] in place of Lemma 1 at the appropriate stage). Thus, $G/\mathrm{core}_G S$ is simple, and since Corollary 4 implies that G is not almost locally soluble, we deduce that $H = G/\mathrm{core}_G S$ is an infinite simple group. Now, H is locally finite and every proper subgroup of H is locally nilpotent or of finite rank. Since locally finite groups of finite rank are almost locally soluble [16], it follows that every proper subgroup of H is almost locally soluble, and hence [8] shows that H is isomorphic either to $PSL(2, F)$ or $Sz(F)$ for some locally finite field F having no proper infinite subfields. But each of these groups has a proper non-$\mathbf{L}\mathfrak{N}$-subgroup of infinite rank [12], and the proof is complete. □

Proof of Theorem B. Let G be a locally (soluble-by-finite) group in the class $(\mathbf{L}\mathfrak{N})^*$ and suppose, for a contradiction, that G is neither locally nilpotent nor of finite rank. By Corollary 4, G is not almost locally soluble. Now, G contains a finitely generated non-nilpotent subgroup F and for each $i = 1, 2, \ldots$ a finitely generated subgroup T_i of rank i. Thus, $T = \langle T_1, T_2, \ldots \rangle$ is a countable group of infinite rank and so $G = \langle F, T \rangle$ is countable. Therefore, we may write $G = \bigcup_{i \geq 1} G_i$ where $G_i < G_{i+1}$ for all i, each G_i is finitely generated and $G_1 \notin \mathfrak{N}$. Let R_i be the soluble radical of G_i for each i and let $H = \langle G_1, R_2, R_3, \ldots \rangle$. Then H is non-$\mathbf{L}\mathfrak{N}$ and has the locally soluble normal subgroup $\langle R_1, R_2, \ldots \rangle$ of finite index. Hence, H has finite rank r, say, by Corollary 4. Since G is not almost locally soluble, there is no bound for the indices $|G_i : R_i|$, and hence, by Propositions 2.1 and 2.4 of [3] no bound for the ranks of the socles of the groups

G_i/R_i. By [13, Lemma 10.39] there is an integer d such that $R_i^{(d)}$ is periodic for all i and thus, by [3, Proposition 3.3] we may choose the G_i such that each has a normal locally finite subgroup A_i of rank n_i, where $n_i < n_{i+1}$ for all i. Write $A = \langle A_1, A_2, A_3, \ldots \rangle$. Then A is locally finite, of infinite rank and therefore locally nilpotent, by Lemma 5. But then AG_1 is neither locally nilpotent nor of finite rank so that $G = AG_1$, which is almost locally soluble. This is a contradiction, and the theorem is proved. $\square$

Proof of Theorem A. Let G be as stated and suppose that $G \notin \mathfrak{N}_c \cup \mathfrak{R}$. By Theorem B, G is locally nilpotent and then, by Lemma 1, G has a proper subgroup of infinite rank, and hence, has a maximal subgroup S which is nilpotent. Since G has no simple non-abelian images it is not perfect; arguing as in the proof of Lemma 2 we have $G' \in \mathfrak{N}_c$ and G/G' finite and cyclic. Let F be a finitely generated non-$\mathfrak{N}_c$ subgroup of G. If G'/G'' has infinite rank then it has a G-invariant subgroup H/G'' of infinite rank such that G'/H also has infinite rank. But then FH is a proper non-$\mathfrak{N}_c$ subgroup of infinite rank, a contradiction. It follows that G' has finite rank, by [13, Theorem 2.26], and we have the contradiction that $G \in \mathfrak{R}$. $\square$

Proof of Theorem C. Let G be a locally soluble group in the class $(\mathfrak{A}^2)^*$ and suppose, for a contradiction, that G is neither metabelian nor of finite rank.

We first show that G is not perfect. Now, by Lemma 1, G contains a proper subgroup of infinite rank which must be metabelian. It follows easily from Zorn's Lemma that there exists a maximal subgroup S of G that is metabelian and of infinite rank. If every proper normal subgroup of G is contained in S then $G/\mathrm{core}_G S$ is non-trivial, locally soluble and simple. Thus, G has a non-trivial abelian image. On the other hand, if $G = NS$ for some proper normal subgroup N of G, then $G/N \cong S/N \cap S$ is metabelian and non-trivial, and therefore, has a non-trivial abelian image. We deduce that $G \neq G'$.

Since G is not metabelian it contains a finitely generated subgroup F that is not metabelian. If G/G' has infinite rank then there exists a proper subgroup H of G such that $FG' \leq H$ and H/G' has infinite rank. Thus, H has infinite rank but is not metabelian. This contradiction shows that G/G' has finite rank, and hence, that G' has infinite rank. Now, FG' is neither metabelian nor of finite rank so $FG' = G$ and $G/G' \cong F/F \cap G'$ is finitely generated. Thus, G/G' has maximal subgroups. Let M be a maximal subgroup of G that contains G' and note that G/M is finite. Since G' is of infinite rank and $G' \neq G$ we deduce that M is metabelian and therefore G is metabelian-by-finite.

Suppose that G is finitely generated and let G/U be an arbitrary finite image of G. Arguing as in the proof of Lemma 3 we see that G/U is a homomorphic image of a proper subgroup X/Z of some abelian-by-finite image G/Z of G. Since G is finitely generated, X/Z is contained in a maximal subgroup K/Z of G/Z, and of course $|G:K|$ is finite. Thus, $\mathrm{core}_G K$ has finite index and is therefore of infinite rank. Thus, K has infinite rank and hence is metabelian. Therefore, every finite image of G is a section of a metabelian group and so is metabelian. But G is residually finite [5, Theorem 1] and so G is metabelian, a contradiction. Consequently, G is not finitely generated.

Let B denote an arbitrary finitely generated subgroup of G. Then $\langle F,B\rangle$ is a finitely generated (proper) subgroup of G that is not metabelian. Thus, $\langle F,B\rangle$ and in particular B, has finite rank. Hence, every finitely generated subgroup of G has finite rank.

Let D be a proper normal subgroup of finite index in G, so that $D\in\mathfrak{A}^2$ and $G=DF$. Suppose that D' has finite rank. Then so has $D'(D\cap F)=V$, say, which is normal in G, and G/V has infinite rank. Further, D/V is abelian and of finite index in G/V, which splits as $(D/V)\rtimes(FV/V)$. Choose J/V to be a proper subgroup of D/V such that D/J has finite rank and let $L=\mathrm{core}_G J$. Then L has infinite rank and $FL<G$, and we have a contradiction. Thus, D' has infinite rank and $G=FA$ for some normal abelian subgroup A. Since F is finitely generated and metabelian-by-finite it is residually finite. Then $\bigcap_{n=1}^{\infty}(F\cap A)^n=1$. Note that $(F\cap A)/(F\cap A)^n$ is finite since F has finite rank. It follows that there is a positive integer n such that $F/(F\cap A)^n\notin\mathfrak{A}^2$ since otherwise F is residually metabelian, and hence, metabelian. Now $(F\cap A)^n$ is normal in G, and it is clear that $F/(F\cap A)^n\neq G/(F\cap A)^n\neq A/(F\cap A)^n$. Furthermore, $F/(F\cap A)^n\notin\mathfrak{A}^2$ and $A/(F\cap A)^n$ has infinite rank since $(F\cap A)^n$ has finite rank, being a subgroup of F. Thus, $G/(F\cap A)^n$ is a locally soluble group in the class $(\mathfrak{A}^2)^*$ that is neither metabelian nor of finite rank. Therefore, there is no loss in assuming that $(F\cap A)^n=1$ and $F\cap A$ has finite order k, say.

Let Y be an arbitrary non-$\mathfrak{A}^2$ subgroup of F. Then $G=YA$ and so $F=Y(F\cap A)$, which shows that Y has index at most k in F. We may therefore choose Y to be a minimal non-$\mathfrak{A}^2$ subgroup of F. Certainly Y is finitely generated, and hence, residually finite. Arguing as before, if Y is infinite then it is metabelian. Consequently, Y is finite so G/A is finite. It now follows easily that A contains a G-invariant subgroup W of infinite rank such that $FW<G$. Since FW is neither metabelian nor of finite rank, we have a contradiction that completes the proof. □

Proof of Theorem C′. Let $\mathfrak{S}$ denote the class of locally soluble groups and let G belong to $\mathfrak{Y}\cap(\mathfrak{A}^2)^*$. By Theorem C it suffices to prove that G is locally soluble, and an obvious induction allows us to assume that $G\in\lambda\mathfrak{S}$ for some $\lambda\in\Lambda$. If $\lambda=\mathbf{L}$ there is nothing to prove.

Next, let $\lambda=\mathbf{R}$. Suppose that H is an arbitrary finitely generated subgroup of G. Then $H\in\mathbf{R}\mathfrak{S}$ and in fact H is residually soluble. If there is a normal subgroup N of H such that N has infinite rank and H/N is a non-trivial soluble group then H is soluble since N is metabelian by hypothesis. Thus, we may assume that every normal subgroup in a residual system for H whose factors are soluble has finite rank. Furthermore, if N is such a normal subgroup and if H/N has a proper subgroup of infinite rank then, again, N is metabelian and H is soluble. If every proper subgroup of H/N has finite rank then H/N has finite rank and therefore so does H. Thus, we may assume that H has finite rank and since $\mathbf{R}\mathfrak{S}\leq\mathfrak{X}$, Černikov's theorem [2] implies that then H is soluble-by-finite. Let $\{N_i\}_{i\in I}$ be a residual system for H such that H/N_i is soluble. If K is the soluble radical of H then H/KN_i is soluble of derived length at most $d=|H:K|$ and KN_i/N_i is soluble of derived length at most l, the derived length of K, for each i. Hence, H/N_i is soluble of derived length at most $d+l$ for each i, so

$H^{(d+l)} \leq N_i$ for each i and it follows that H is soluble in any case. Thus, G is locally soluble.

If $\lambda = \dot{\mathbf{P}}$ then we may suppose G is finitely generated and show that G is soluble. Then there exists $A \lhd G$ such that G/A is a non-trivial soluble group; as above, we may assume that A has finite rank and that every proper subgroup of G/A has finite rank so that G has finite rank, by Lemma 1. Let

$$H = R(G) = \bigcap \{N \lhd G \mid G/N \in \mathfrak{S}\}.$$

Then, $G/H \in \mathbf{R}\mathfrak{S}$. By the case $\lambda = \mathbf{R}$ it follows that $G/H \in \mathfrak{S}$. Suppose that $H \neq 1$. Since $H \in \dot{\mathbf{P}}\mathfrak{S}$, $H/R(H) \in \mathbf{R}\mathfrak{S}$ as above, so that $H/R(H)$ is hyperabelian of finite rank by [13, Lemma 10.39]. Hence, $G/R(H)$ is hyperabelian of finite rank, so is locally soluble and $H = R(H)$, a contradiction. Thus, G is locally soluble in the case when $\lambda = \dot{\mathbf{P}}$.

Finally, if $G \in \acute{\mathbf{P}}\mathfrak{S}$ we use transfinite induction on the length of an $\mathfrak{S}$-series $\{G_\alpha\}$ in G, and we may assume that the series terminates at some successor ordinal β. Let $G_{\beta-1} = N$ so that $1 \neq G/N \in \mathfrak{S}$. If N has infinite rank then G is metabelian-by-locally soluble, so G is locally soluble. If N has finite rank then we may assume that every proper subgroup of G/N has finite rank so Lemma 1 implies that G/N, and hence, G has finite rank. Then G is hyperabelian of finite rank by [13, Theorem 10.39], so G is locally soluble by [13, Theorem 10.38]. This completes the proof. $\square$

3. Some examples

In this section we present a few examples of groups in the class $\mathfrak{A}^*$ which are neither abelian nor of finite rank. In view of Theorem A′, some highly non-elementary construction is sure to be involved; we utilise the following remarkable result of Ol'šanskii [11].

Theorem. *Let $\{G_\lambda\}_{\lambda \in \Lambda}$ be a countable set of non-trivial finite or countably infinite groups G_λ without involutions. Suppose that $|\Lambda| \geq 2$ and that n is a sufficiently large odd number. Suppose further that $G_\lambda \cap G_\mu = 1$ for $\lambda \neq \mu$. Then, there is a countable simple group G which contains a copy of G_λ for each $\lambda \in \Lambda$ with the following properties:*

(i) *If $x, y \in G$ with $x \in G_\lambda \setminus \{1\}, y \notin G_\lambda$ for some $\lambda \in \Lambda$ then G is generated by x and y.*

(ii) *Every proper subgroup of G is either a cyclic group of order dividing n or is contained in a subgroup conjugate to some G_λ.*

Given this theorem it is rather easy to produce groups of the kind we want. We content ourselves with two examples.

Example 1. There is a 2-generator countably infinite simple group with every proper subgroup of finite rank, which does not have finite rank.

Proof. For each natural number i let G_i be a countable group of rank r_i and without involutions, and suppose that $r_{i+1} > r_i$ for all i. Then, the group G guaranteed by Ol'šanskii's theorem which is constructed using the G_i is easily seen to be of the desired type. □

Example 2. There is a 2-generator countably infinite simple group with all proper non-abelian subgroups of finite rank which has abelian subgroups of infinite rank.

Proof. Let G_1, G_2 be free abelian groups of infinite rank. Then the group G obtained from G_1 and G_2 via Ol'šanskii's construction clearly has abelian subgroups of infinite rank but it has no proper non-abelian subgroups and is therefore in the class $\mathfrak{A}^*$. □

Finally, we show that Theorem B cannot be extended so as to cover locally finite groups.

Example 3. There exists a locally finite simple group of infinite rank in the class $(\mathfrak{A}^2)^*$.

Proof. Let F be an infinite locally finite field all of whose proper subfields are finite, and let $G = PSL(2, F)$. Write F as the ascending union of finite subfields F_i, G the union of subgroups G_i, where $G_i \cong PSL(2, F_i)$ for each i. Let H be a non-metabelian subgroup of G. From [17, Theorem 6.25] we see that, for almost all i, $H \cap G_i$ is isomorphic either to $PSL(2, K_i)$ or $PGL(2, K_i)$ for some subfield K_i of F_i. It follows easily that either H is finite or $H = G$. Thus, every proper subgroup of G is either metabelian or finite, but of course G is non-metabelian and of infinite rank. □

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