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Existence of Multiple Positive Periodic Solutions of Delayed Predator-Prey Models with Functional Responses

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Abstract—In this paper, by applying the continuation theorem of coincidence degree theory, we establish some new criteria for the existence of multiple positive periodic solutions for the delayed predator-prey model.

$$x'(t) = x(t)\left(r(t) - a(t)x(t)\right) - b(t)f\left(x(t)\right)y(t),$$

$$y'(t) = y(t)\left(c(t)f\left(x(t-\tau)\right) - d(t)\right),$$

when functional response function f is monotonic or nonmonotonic.

Keywords—Predator-prey model, Positive periodic solution, Coincidence degree, Delay, Monotonic, Nonmonotonic.

1. INTRODUCTION

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance,see Berryman [1].

Recently, Beretta and Kuang [2], Hsu *et al.* [3], Jost *et al.* [4], and Kuang and Beretta [5] considered the system

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$$x' = rx\left(1 - \frac{x}{k}\right) - \frac{\alpha xy}{my + x},$$
$$y' = y\left(-d + \frac{fx}{my + x}\right) \tag{1}$$

Fan and Wang [6] and Fan *et al.* [7] considered more general delayed ratio-dependent predator-prey model with periodic coefficients and established some criteria for the existence of positive periodic solutions.

In general, the response function $f(u)$ is monotone. However, there is nonmonotonic responses occurrence, see Beretta and Kuang [2]. The so-called Modod-Haldane function $f(u) = cu/(m^2 + bu + u^2)$ has been proposed and used to model, see [8]. Sokol and Howell [9] proposed a simplified Monod-Haldane function of the form $f(u) = cu/(m^2 + u^2)$.

Ruan and Xiao [10] considered a special system with Monod-Haldane nonmonotonic functional response

$$x'(t) = rx(t)\left(1 - \frac{x(t)}{k}\right) - \frac{x(t)y(t)}{m^2 + x^2(t)},$$
$$y'(t) = y(t)\left(\frac{\mu x(t)}{m^2 + x^2(t)} - d\right), \tag{2}$$

with constant coefficients. Bush and Cook [11] have studied the system.

$$x'(t) = rx(t)(1 - \frac{x(t)}{k}) - \frac{x(t)y(t)}{m^2 + x^2(t)},$$
$$y'(t) = y(t)(\frac{\mu x(t-\tau)}{m^2 + x^2(t-\tau)} - d), \tag{3}$$

where r, k, μ, τ, and d are positive constants and m is a real constant. Xiao and Ruan [12] found that there is a Bogdanov-Takens singularity for any time delay value. Chen [13] considered a periodic predator-prey system with Type IV functional response.

In this paper, we consider the following more general delayed predator-prey system

$$x'(t) = x(t)(r(t) - a(t)x(t)) - b(t)f(x(t))y(t),$$
$$y'(t) = y(t)(c(t)f(x(t-\tau)) - d(t)), \tag{4}$$

where r, a, b, c, and d are all positive periodic continuous functions with period $\omega > 0$, τ is a positive constant.

The purpose of this paper is, by applying the coincidence degree theory developed by Gaines and Mawhin [14], to establish the existence of one positive ω-periodic solution of system (4) when the functional response function f is monotonic and two positive ω-periodic solutions of system (4) when the functional response function f is nonmonotonic.

For the work concerning the existence of periodic solution of delay differential equations which was done by using coincidence degree theory, see [6,7,15,16], but few papers have been published on the existence of multiple periodic solutions before by using this method, see Chen [13].

2. PRELIMINARIES

In order to use the continuation theorem of coincidence degree theory, we need to introduce a few notations.

Let X and Z be normed vector space. Let $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping and $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \ker L = \operatorname{co} \dim \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}\ (I - Q)$. It follows that $L \mid \operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible and its inverse is denoted by K_p. If Ω is a bounded open subset of X , the mapping N is called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact. Because $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

THEOREM A. CONTINUATION THEOREM. *(See [14].) Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}$. Suppose*

(a) *for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;*
(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$ and $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \bar{\Omega}$.

In this paper, we shall use the notation

$$\bar{u} = \frac{1}{\omega}\int_0^{\omega} u(t)\,dt,$$

where u is a continuous ω-periodic function.

3. MONOTONE CASE

In this section, we will study the existence of positive ω-periodic solution of system (4) when the functional response function f satisfies the following monotone condition (M):

(i) $f \in C^1(R,R)$ and $f(0) = 0$;
(ii) $f'(x) > 0$ for $x \in [0, +\infty)$;
(iii) $\lim_{x\to+\infty} f(x) = k > 0$, k is a constant.

LEMMA 1. *Assume that condition (M) holds. If system (4) has one positive ω-periodic solution, then $k\bar{c} \geq \bar{d}$.*

The proof is obvious and we omit it.

THEOREM 1. *Assume (M) and the following conditions hold:*

(H_1) *$k\bar{c} > \bar{d}$,*
(H_2) *$\bar{r} > \bar{a}e^A$, where $A = |\ln f^{-1}(\bar{d}/\bar{c})| + 2\bar{r}\omega$.*

Then system (4) has at least one positive ω-periodic solution.

PROOF. Consider the system

$$\begin{aligned} x_1'(t) &= r(t) - a(t)\exp\{x_1(t)\} - b(t)f(\exp\{x_1(t)\})\exp\{x_2(t) - x_1(t)\}, \\ x_2'(t) &= -d(t) + c(t)f(\exp\{x_1(t-\tau)\}). \end{aligned} \tag{5}$$

It is easy to see that if the system (5) has an ω-periodic solution $(x_1^*(t), x_2^*(t))^\top$, then $(\exp\{x_1^*(t)\}, \exp\{x_2^*(t)\})^\top$ is a positive ω-periodic solution of system (4). Therefore, for have (4) at least one positive ω-periodic solution, it is sufficient that (5) has at least one ω-periodic solution. In order to apply Theorem A to system (5), we take

$$X = Z = \left\{x(t) = (x_1(t), x_2(t))^\top \in C(R, R^2) : x(t+\omega) = x(t) \text{ for } t \in R\right\}$$

and denote

$$\|x\| = \left\|(x_1(t), x_2(t))^\top\right\| = \max_{t\in[0,\omega]} |x_1(t)| + \max t \in [0,\omega]\, |x_2(t)|.$$

Then X and Z are Banach space when they are endowed with the norm $\|\cdot\|$. Set

$$Nx(t) = \begin{bmatrix} r(t) - a(t)\exp\{x_1(t)\} - b(t)f(\exp\{x_1(t)\})\exp\{x_2(t) - x_1(t)\} \\ -d(t) + c(t)f(\exp\{x_1(t-\tau)\}) \end{bmatrix}$$

and

$$Lx = x', Px = \frac{1}{\omega}\int_0^{\omega} x(t)\, dt, \quad x \in X, \qquad Qz = \frac{1}{\omega}\int_0^{\omega} z(t)\, dt, \quad z \in Z.$$

Evidently, $\ker L = R^2, \operatorname{Im} L = \{z \mid \int_0^{\omega} z(t)\, dt = 0,\ z \in Z\}$ is closed in L and $\dim\ker L = \operatorname{co}\dim \operatorname{Im} L = 2$. Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse ,(to L)$K_p : \operatorname{Im} L \to \ker P \cap \operatorname{dom} L$ has the form

$$K_p(z) = \int_0^t z(s)\, ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t z(s)\, ds\, dt.$$

Thus,

$$QNx = \begin{bmatrix} \frac{1}{\omega}\int_0^{\omega} [r(t) - a(t)\exp\{x_1(t)\} - b(t)f(\exp\{x_1(t)\})\exp\{x_2(t) - x_1(t)\}]\, dt \\ \frac{1}{\omega}\int_0^{\omega} [-d(t) + c(t)f(\exp\{x_1(t-\tau)\})]\, dt \end{bmatrix}$$

and

$$K_p(I-Q)Nx = \begin{bmatrix} \int_0^t [r(s) - a(s)\exp\{x_1(s)\} - b(s)f(\exp\{x_1(s)\})\exp\{x_2(s) - x_1(s)\}]\, ds \\ \int_0^t [-d(s) + c(s)f(\exp\{x_1(s-\tau)\})]\, ds \end{bmatrix}$$
$$- \begin{bmatrix} \frac{1}{\omega}\int_0^{\omega}\int_0^t [r(s) - a(s)\exp\{x_1(s)\} - b(s)f(\exp\{x_1(s)\})\exp\{x_2(s) - x_1(s)\}]\, ds\, dt \\ \frac{1}{\omega}\int_0^{\omega}\int_0^t [-d(s) + c(s)f(\exp\{x_1(s-\tau)\})]\, ds\, dt \end{bmatrix}$$
$$- \begin{bmatrix} \left(\frac{t}{\omega} - \frac{1}{2}\right)\int_0^{\omega} [r(s) - a(s)\exp\{x_1(s)\} - b(s)f(\exp\{x_1(s)\})\exp\{x_2(s) - x_1(s)\}]\, ds \\ \left(\frac{t}{\omega} - \frac{1}{2}\right)\int_0^{\omega} [-d(s) + c(s)f(\exp\{x_1(s-\tau)\})]\, ds \end{bmatrix}.$$

Clearly, QN and $K_p(I-Q)N$ are continuous and, moreover, applying the Arzela-Ascoli theorem, it is easy to show that $QN(\bar{\Omega})$, $K_p(I-Q)(\bar{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Hence, N is L-compact on $\bar{\Omega}$, where Ω is open bounded set in X. Corresponding to equation $Lx = \lambda Nx, \lambda \in (0,1)$, we have

$$\begin{aligned} x_1'(t) &= \lambda\left[r(t) - a(t)\exp\{x_1(t)\} - b(t)f(\exp\{x_1(t)\})\exp\{x_2(t) - x_1(t)\}\right], \\ x_2'(t) &= \lambda\left[-d(t) + c(t)f(\exp\{x_1(t-\tau)\})\right]. \end{aligned} \tag{6}$$

Suppose that $x(t) = (x_1, x_2)^{\top} \in X$ is a solution of system (6) for a certain $\lambda \in (0,1)$. By integrating (6) over the interval $[0,\omega]$, we obtain

$$\begin{aligned} \int_0^{\omega} [r(t) - a(t)\exp\{x_1(t)\} - b(t)f(\exp\{x_1(t)\})\exp\{x_2(t) - x_1(t)\}]\, dt &= 0, \\ \int_0^{\omega} [-d(t) + c(t)f(\exp\{x_1(t-\tau)\})]\, dt &= 0. \end{aligned} \tag{7}$$

Hence,

$$\int_0^{\omega} [a(t)\exp\{x_1(t)\} + b(t)f(\exp\{x_1(t)\})\exp\{x_2(t) - x_1(t)\}]\, dt = \bar{r}\omega \tag{8}$$

and

$$\int_0^{\omega} [c(t)f(\exp\{x_1(t-\tau)\})]\, dt = \bar{d}\omega. \tag{9}$$

From (6)–(9), we obtain

$$\begin{aligned}\int_0^\omega |x_1'(t)|\,dt &\le \int_0^\omega r(t)\,dt \\ +\int_0^\omega [a(t)\exp\{x_1(t)\} &+ b(t)f(\exp\{x_1(t)\})\exp\{x_2(t)-x_1(t)\}]\,dt = 2\bar{r}\omega\end{aligned} \tag{10}$$

and

$$\int_0^\omega |x_2'(t)|\,dt \le \int_0^\omega d(t)\,dt + \int_0^\omega c(t)f(\exp\{x_1(t-\tau)\})\,dt = 2\bar{d}\omega. \tag{11}$$

Note that $(x_1(t), x_2(t))^\top \in X$, then there exist $\zeta, \xi, \eta \in [0,\omega]$ such that

$$\bar{c}f(\exp\{x_1(\zeta\}) = \bar{d}, x_2(\xi) = \min_{t\in[0,\omega]} x_2(t), x_2(\eta) = \max_{t\in[0,\omega]} x_2(t). \tag{12}$$

By (12) and (M), we have

$$x_1(\zeta) = \ln f^{-1}\left(\frac{\bar{d}}{\bar{c}}\right). \tag{13}$$

Hence,

$$\begin{aligned}|x_1(t)| &\le |x_1(\zeta)| + \int_0^\omega |x_1'(t)|\,dt \\ &\le \left|\ln f^{-1}\left(\frac{\bar{d}}{\bar{c}}\right)\right| + 2\bar{r}\omega =: A.\end{aligned} \tag{14}$$

By (8), (12), and (14), we have

$$\begin{aligned}\bar{r}\omega &\ge \bar{b}\omega f(\exp\{-A\})\exp\{x_2(\xi)-A\} \\ &= \bar{b}\omega f(e^{-A})e^{-A}\exp\{x_2(\xi)\}\end{aligned}$$

and

$$\begin{aligned}\bar{r}\omega &\le \bar{a}\omega\exp\{A\} + \bar{b}\omega f(\exp\{A\})\exp\{x_2(\eta)-A\} \\ &= \bar{a}\omega e^A + \bar{b}\omega f(e^A)e^A\exp\{x_2(\eta)\}.\end{aligned}$$

Hence,

$$x_2(\xi) \le \ln\bar{r} - \ln\bar{b}f(e^{-A})e^{-A}$$

and

$$x_2(\eta) \ge \ln(\bar{r} - \bar{a}e^A) - \ln bf(e^A)e^A.$$

Therefore,

$$\begin{aligned}x_2(t) &\le x_2(\xi) + \int_0^\omega |x_2'(t)|\,dt \\ &\le \ln\bar{r} - \ln\bar{b}f(e^{-A})e^{-A} + 2\bar{d}\omega =: B_1\end{aligned} \tag{15}$$

and

$$\begin{aligned}x_2(t) &\ge x_2(\eta) - \int_0^\omega |x_2'(t)|\,dt \\ &\ge \ln(\bar{r} - \bar{a}e^A) - \ln\bar{b}f(e^A)e^A - 2\bar{d}\omega =: B_2.\end{aligned} \tag{16}$$

Let

$$B = \max\{|B_1|, |B_2|\},$$

then

$$|x_2(t)| \le B.$$

Clearly, A and B_i $(i = 1, 2)$ are independent of λ. Under the assumption in Theorem 1, it is easy to show that the equations

$$\begin{aligned} \bar{r} - \bar{a}u - \bar{b}f(u)\frac{v}{u} = 0, \\ \bar{d} - \bar{c}f(u) = 0, \end{aligned} \tag{18}$$

has a unique solution

$$(u^*, v^*)^\top \in \int R_+^2 := \{(u, v) \mid u > 0,\ v > 0\}.$$

Denote $H = A + B + G$, where $G > 0$ is taken sufficiently large such that

$$\left\| (\ln\{u^*\}, \ln\{v^*\})^\top \right\| = |\ln\{u^*\}| + |\ln\{v^*\}| < G$$

and define $\Omega = \{x(t) \in X : \|x\| < H\}$. It is clear that Ω satisfies Condition (a) of Theorem A. When $x = (x_1, x_2)^\top \in \partial\Omega \cap \ker L = \partial\Omega \cap R^2$, x is a constant vector in R^2 with $\|x\| = H$. Then

$$QNx = \begin{bmatrix} \bar{r} - \bar{a}\exp\{x_1\} - \bar{b}f(\exp\{x_1\})\exp\{x_2 - x_1\} \\ -\bar{d} + \bar{c}f(\exp\{x_1\}) \end{bmatrix} \neq 0.$$

Furthermore,

$$\begin{aligned} &\deg\{JQN, \ker L \cap \Omega, 0\} \\ &= \operatorname{sgn}\det \begin{bmatrix} -\bar{a}\exp\{x_1\} + \bar{b}f(\exp\{x_1\})\exp\{x_2 - x_1\} - \bar{b}f'(\exp\{x_1\})\exp\{x_2\} & -\bar{b}f(\exp\{x_1\})\exp\{x_2 - x_1\} \\ \bar{c}f'(\exp\{x_1\})\exp\{x_1\} & 0 \end{bmatrix} \\ &= \operatorname{sgn}\left(\bar{b}\bar{c}f(\exp\{x_1\})f'(\exp\{x_1\})\exp\{x_2\}\right) \neq 0. \end{aligned}$$

According to the Theorem A we know that the system (5) has at least one ω-periodic solution. The proof of Theorem 1 is complete. ∎

4. NONMONOTONE CASE

In this section, we will study solution of system (4) when the functional response function f satisfies the following nonmonotone condition (N):

(i) $f \in C^1(R, R)$ and $f(0) = 0$;
(ii) there exists a constant $l > 0$ such that $(x - l)f'(x) < 0$ for $x \neq l$;
(iii) $\lim_{x\to\infty} f(x) = 0$.

LEMMA 2. *Assume that condition (N) holds. If system (4) has one positive ω-periodic solution, then $\bar{d} \leq \bar{c}f(l)$.*

The proof is obvious and we omit it.

From condition (N), we can easily know that, if $\bar{d} < \bar{c}f(l)$, then the equation $f(u) = \bar{d}/\bar{c}$ has two positive solutions, namely r_1, r_2. Without loss of generality, we suppose that $r_1 < r_2$, and we have the following conclusion.

THEOREM 2. *Assume (N) and the following conditions hold:*

(H_3) $\bar{d} < \bar{c}f(l)$;
(H_4) $r_2 > r_1 e^{4\bar{r}\omega}$;
(H_5) $\bar{r} > \bar{a}e^L$, *where* $L = \max\{|\ln r_1|, |\ln r_2|\} + 2\bar{r}\omega$.

Then system (4) has at least two positive ω-periodic solutions.

PROOF. By the similar analysis as that of Theorem 1, we have

$$\bar{c}f(\exp\{x_1(\zeta)\}) = \bar{d}.$$

Hence

$$x_1(\zeta) = \ln r_1 \qquad \text{or} \qquad x_1(\zeta) = \ln r_2.$$

By (10), we have

$$\begin{aligned} |x_1(t)| &\le |x_1(\zeta)| + \int_0^{\omega} |x_1'(t)|\, dt \\ &\le \max\{|\ln r_1|, |\ln r_2|\} + 2\bar{r}\omega = L. \end{aligned} \tag{19}$$

From (8), (12), and (19), we obtain

$$\bar{r}\omega \ge \bar{b}\omega f(e^{-L})e^{-L}\exp\{x_2(\xi)\}$$

and

$$\bar{r}\omega \le \bar{a}\omega e^{L} + \bar{b}\omega f(e^{L})e^{L}\exp\{x_2(\eta)\}.$$

Hence,

$$\begin{aligned} x_2(t) &\le x_2(\xi) + \int_0^{\omega} |x_2'(t)|\, dt \\ &\le \ln\bar{r} - \ln\bar{b}f\left(e^{-L}\right)e^{-L} + 2\bar{d}\omega =: B_3 \end{aligned} \tag{20}$$

and

$$\begin{aligned} x_2(t) &\ge x_2(\eta) - \int_0^{\omega} |x_2'(t)|\, dt \\ &\ge \ln\left(\bar{r} - \bar{a}e^{L}\right) - \ln\bar{b}f\left(e^{L}\right)e^{L} - 2\bar{d}\omega =: B_4. \end{aligned} \tag{21}$$

Let $C = \max\{|B_3|, |B_4|\}$, then $|x_2(t)| \le C$. Clearly, L and B_i $(i = 3, 4)$ are independent of λ and equation (18) has two solutions $(u_i, v_i)^{\top} \in intR_+^2$, $i = 1, 2$.

Choose $E > 0$ such that $E > \max\{|\ln u_1|, |\ln u_2|, |\ln v_1|, |\ln v_2|\}$ and choose a sufficiently small $\varepsilon > 0$, let $l_1 = \ln r_1 - 2\bar{r}\omega - \varepsilon$, $l_2 = \ln r_1 + 2\bar{r}\omega + \varepsilon$, $l_3 = \ln r_2 - 2\bar{r}\omega - \varepsilon$, $l_4 = \ln r_2 + 2\bar{r}\omega + \varepsilon$, such that $l_2 < l_3$.

Set

$$\begin{aligned} \Omega_1 &= \{x = (x_1, x_2) \in X : x_1(t) \in (l_1, l_2), |x_2(t)| < C + E\}, \\ \Omega_2 &= \{x = (x_1, x_2) \in X : x_1(t) \in (l_3, l_4), |x_2(t)| < C + E\}. \end{aligned}$$

Then both Ω_1 and Ω_2 are bounded open subsets of X. It is easy to see that $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \emptyset$ and Ω_i satisfies requirement (a) in Theorem A for $i = 1, 2$. Moreover, $QNx \ne 0$ for $x \in \partial\Omega_i \cap R^2$ and $\deg(JQN, \Omega_i \cap \operatorname{Ker} L, 0) = (-1)^{i+1} \ne 0$, $i = 1, 2$.

So far we have proved that Ω_i $(i = 1, 2)$ satisfies all the assumptions in Theorem A. Hence, system (5) has at least two ω-periodic solutions x^* and $\hat{x}$. Then, $\exp\{x^*\}$ and $\exp\{\hat{x}\}$ are two different positive ω-periodic solutions of system (4). The proof of Theorem 2 is complete. ∎

From Theorem 2, we can easily obtain the following results.

COROLLARY 1. *Assume* (H_3),(H_4), (N) *and the following condition hold:*

(H_6) $\bar{r} > \bar{a}r_2e^{2\bar{r}\omega}$ *for* $r_1 \ge 1$.

Then system (4) *has at least two positive* ω*-periodic solutions.*

COROLLARY 2. *Assume* (H_3),(H_4), (N) *and the following condition hold:*

(H_7) $\bar{r} > (\bar{a}/r_1)e^{2\bar{r}\omega}$ *for* $r_2 \le 1$.

Then system (4) has at least two positive ω*-periodic solutions.*

We also remark that our above results can be extended to the following system

$$x'(t) = x(t)(r(t) - a(t)x(t - \tau_1)) - b(t)f(x(t))y(t - \sigma),$$

$$y'(t) = y(t)(c(t)f(x(t - \tau_2)) - d(t)).$$

5. APPLICATIONS

In this section, we apply our results in Sections 3 and 4 to the Monod-Haldane models.

Consider the following system

$$
\begin{aligned}
x'(t) &= x(t)\left[x(t) - a(t)x(t)\right] - \frac{b(t)x(t)y(t)}{m + x(t)}, \\
y'(t) &= y(t)\left[\frac{c(t)x(t-\tau)}{m + x(t-\tau)} - d(t)\right],
\end{aligned}
\tag{22}
$$

which can be obtained by letting $f(u) = u/(m+u)$ in system (4), where all the functions are defined as that of system (4).

By Theorem 1, we have the following result.

THEOREM 3. *Assume that*

(i) $\bar{c} > \bar{d}$,

(ii) $\bar{r} > m\bar{a}\bar{d}e^{2\bar{r}\omega}/(\bar{c}-\bar{d})$ *for* $(m+1)\bar{d} > \bar{c}$
or $\bar{r} > \bar{a}(\bar{c}-\bar{d})e^{2\bar{r}\omega}/m\bar{d}$ *for* $(m+1)\bar{d} < \bar{c}$
or $\bar{r} > \bar{a}e^{2\bar{r}\omega}$ *for* $(m+1)\bar{d} = \bar{c}$.

Then (22) has at least one positive ω*-periodic solution.*

Consider the following system with Holling III-type functional response function

$$
\begin{aligned}
x'(t) &= x(t)\left[r(t) - a(t)x(t)\right] - \frac{b(t)x^2(t)y(t)}{m + x^2(t)}, \\
y'(t) &= y(t)\left[c(t)\frac{x^2(t-\tau)}{m + x^2(t-\tau)} - d(t)\right],
\end{aligned}
\tag{23}
$$

which is a special case of (4) by letting $f(u) = u^2/(m+u^2)$, where all the functions are defined as above.

By Theorem 1, we have the following result.

THEOREM 4. *Assume that*

(i) $\bar{c} > \bar{d}$,

(ii) $\bar{r} > \bar{a}e^{2\bar{r}\omega}\sqrt{m\bar{d}/(\bar{c}-\bar{d})}$ *for* $(m+1)\bar{d} > \bar{c}$
or $\bar{r} > \bar{a}e^{2\bar{r}\omega}\sqrt{(\bar{c}-\bar{d})/m\bar{d}}$ *for* $(m+1)\bar{d} < \bar{c}$
or $\bar{r} > \bar{a}e^{2\bar{r}\omega}$ *for* $(m+1)\bar{d} = \bar{c}$.

Then (23) has at least one positive ω*-periodic solution.*

Consider the following system with Holling III-type functional response function

$$
\begin{aligned}
x'(t) &= x(t)\left[r(t) - a(t)x(t)\right] - \frac{b(t)x^2(t)y(t)}{m + ex(t) + x^2(t)}, \\
y'(t) &= y(t)\left[c(t)\frac{x^2(t-\tau)}{m + ex(t-\tau) + x^2(t-\tau)} - d(t)\right],
\end{aligned}
\tag{24}
$$

which is a special case of (4) by letting $f(u) = u^2/(m+eu+u^2)$, where all the functions are defined as above.

By Theorem1, we have the following result.

THEOREM 5. *Assume that*

(i) $\bar{c} > \bar{d}$,

(ii) $\bar{r} > \bar{a}e^{2\bar{r}\omega}(\bar{d}e + \sqrt{\bar{d}^2e^2 + 4\bar{d}m(\bar{c}-\bar{d})})/2(\bar{c}-\bar{d})$ *for* $\bar{d}e + \sqrt{\bar{d}^2e^2 + 4\bar{d}m(\bar{c}-\bar{d})} > 2(\bar{c}-\bar{d})$
or $\bar{r} > 2(\bar{c}-\bar{d})\bar{a}e^{2\bar{r}\omega}/(\bar{d}e + \sqrt{\bar{d}^2e^2 + 4\bar{d}m(\bar{c}-\bar{d})})$ *for* $\bar{d}e + \sqrt{\bar{d}^2e^2 + 4\bar{d}m(\bar{c}-\bar{d})} < 2(\bar{c}-\bar{d})$
or $\bar{r} > \bar{a}e^{2\bar{r}\omega}$ *for* $\bar{d}e + \sqrt{\bar{d}^2e^2 + 4\bar{d}m(\bar{c}-\bar{d})} = 2(\bar{c}-\bar{d})$.

Then (24) has at least one positive ω*-periodic solution.*

Consider the following system with the Monod-Haldne nonmonotonic functional response function

$$\begin{aligned} x'(t) &= x(t)\left[r(t) - a(t)x(t)\right] - \frac{b(t)x(t)y(t)}{m + x^2(t)}, \\ y'(t) &= y(t)\left[c(t)\frac{x(t-\tau)}{m + x^2(t-\tau)} - d(t)\right], \end{aligned} \tag{25}$$

which is a special case of (4) by letting $f(u) = u/(m + u^2)$, where all the functions are defined as above.

By Corollaries 1 and 2, we have the following result.

THEOREM 6. *Assume that*

(i) $\bar{c} > 2\sqrt{m}\bar{d}$;
(ii) $(e^{4\bar{r}\omega} + 1)\sqrt{\bar{c}^2 - 4m\bar{d}^2} > (e^{4\bar{r}\omega} - 1)\bar{c}$;
(iii) $r > \bar{a}e^{2\bar{r}\omega}(\bar{c} + \sqrt{\bar{c}^2 - 4m\bar{d}^2})/2\bar{d}$ *for* $(m+1)\bar{d} \geq \bar{c}$ *and* $\bar{c} > 2\bar{d}$ *or* $r > 2\bar{d}\bar{a}e^{2\bar{r}\omega}/(\bar{c} - \sqrt{\bar{c}^2 - 4m\bar{d}^2})$ *for* $(m+1)\bar{d} \geq \bar{c}$ *and* $\bar{c} < 2\bar{d}$.

Then (25) has at least two positive ω-periodic solutions.

Consider the following system with the Monod-Haldne nonmonotonic functional response function.

$$\begin{aligned} x'(t) &= x(t)\left[r(t) - a(t)x(t)\right] - \frac{b(t)x(t)y(t)}{m + ex(t) + x^2(t)}, \\ y'(t) &= y(t)\left[c(t)\frac{x(t-\tau)}{m + ex(t-\tau) + x^2(t-\tau)} - d(t)\right], \end{aligned} \tag{26}$$

which is a special case of (4) by letting $f(u) = u/(m + eu + u^2)$, where all the functions are defined as above.

By Corollary 1 and Corollary 2, we have the following result.

THEOREM 7. *Assume that*

(i) $\bar{c} > (2\sqrt{m} + e)\bar{d}$;
(ii) $(e^{4\bar{r}\omega} + 1)\sqrt{(\bar{c} - e\bar{d})^2 - 4m\bar{d}^2} > (e^{4\bar{r}\omega} - 1)(\bar{c} - e\bar{d})$;
(iii) $r > \bar{a}e^{2\bar{r}\omega}(\bar{c} - e\bar{d} + \sqrt{(\bar{c} - e\bar{d})^2 - 4m\bar{d}^2})/2\bar{d}$ *for* $(m + e + 1)\bar{d} \geq \bar{c}$ *and* $\bar{c} > (2+e)\bar{d}$ *or* $r > 2\bar{d}\bar{a}e^{2\bar{r}\omega}/(\bar{c} - \bar{e} - \sqrt{(\bar{c} - e\bar{d})^2 - 4m\bar{d}^2})$ *for* $(m + e + 1)\bar{d} \geq \bar{c}$ *and* $\bar{c} < (2+e)\bar{d}$.

Then (26) has at least two positive ω-periodic solutions.

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