Mathematical Modelling, Vol. 2, pp. 213–225, 1981

0270–0255/81/030213–13$02.00/0

MARKOV DECISIONS IN URBAN MODELLING

E. J. BELTRAMI

Department of Applied Mathematics
State University of New York
Stony Brook, New York 11794, USA

(Received February 1981)

Abstract—In urban fire departments, fire companies are dispatched to respond to alarms which occur spatially and temporally in a generally unpredictable way. Also, the time during which one of the responding units is busy on a call, and hence unavailable to other alarms, is itself a random variable. This variability in demand and service time makes it difficult to maintain a balance between the need for effective response to alarms which occur now and those which may arrive in the future. In this paper, we discuss two specific mathematical models, based on work done by the Rand Institute in New York City, for determining not only how many but also which of the available fire-fighting units to deploy to any given alarm. Each is a Markovian decision model in which the conflicting objectives of adequate response to present or future incidents are explicitly accounted for. Similar considerations are applicable to other municipal emergency services.

1. INTRODUCTION

The optimal deployment over time of a set of limited resources under conditions of uncertainty can often be posed as Markov decision problems. In urban fire departments, for example, engine and ladder companies are dispatched to respond to alarms which occur spatially and temporally in a generally unpredictable way. Also, the time required to arrive at the scene of an incident and to extinguish a fire is itself only known probabilistically. This variability in demand and service time makes it difficult to maintain a balance between the need for effective response to alarms which occur now and those which may arrive in the future. Because of this random nature of both demand and length of service and in view of the usual restrictions that the fire-fighting force is of limited size, the number of units currently busy at alarms may be large enough to have saturated the ability of the system to handle new incidents which arrive in the interim. It is therefore of considerable tactical interest to determine not only how many, but which units to deploy to any given alarm so as to maintain an effective fire supression capability over time. Similar considerations occur in other municipal emergency services (police, ambulances, repair) and even in military combat, but we will illustrate the use of Markov decision theory by considering two specific problems concerning fire department operations. Our treatment is adapted from mathematical models developed by the Rand Institute in New York City about a decade ago (see [1] for an overview of the Rand work). An interesting feature of the models discussed below is that they explicitly take into account the conflicting goals of reducing response time to alarms which occur now and to those which arrive later, thereby providing examples of stochastic multiobjective optimization.

This work was supported in part by the National Science Foundation under Grant SED 80-17652.

2. MARKOV DECISIONS

We briefly review here some of the elementary concepts of decision theory that will be needed later. For a more comprehensive treatment of the relevant background in Markov theory see, for example, the book by Ross [2].

Suppose we have a Markov process with a finite state space. Just after each transition, a decision rule is employed resulting in one of a finite number of actions. It is assumed that the rule is stationary in the sense that the action taken depends only on the state of the system and not on the chosen instant of time. A decision rule or, as it is sometimes called, a policy R is defined by assigning a specific action to each state. With n states and m actions to choose from in each state, there are n^m possible policies R. A choice of R affects successive moves from state to state and so the transition probability from states i to j is written $P_{ij}(R)$.

By viewing the process only at the moments of transition, we obtain a finite Markov chain which is assumed throughout to be aperiodic and irreducible. Let the state at instant $l = 0, 1 \ldots$ be described by the random variable x_l and suppose that a cost $C_R(x_l)$ is incurred in state x_l as a result of policy R. When $x_l = j$ for $j = 1 \ldots n$, we write $C_R(x_l) = C_R(j)$.

The expected average cost over an infinite time horizon under policy R, given that the system is initially in state i, is defined by

$$V_R(i) = \lim_{s\to\infty} \frac{1}{s+1} E\left(\sum_{l=0}^{s} C_R(x_l) \mid x_0 = i\right). \tag{1}$$

The expected value in (1) can also be written as

$$\frac{1}{s+1}\sum_{l=0}^{s} E(C_R(x_l) \mid x_0 = i) = \frac{1}{s+1}\sum_{l=0}^{s}\sum_{j=1}^{n} C_R(j)P_{ij}^{l}(R)$$
$$= \sum_{j=1}^{n} C_R(j)\frac{1}{s+1}\sum_{l=0}^{s} P_{ij}^{l}(R),$$

where $P_{ij}^{l}(R)$ is the transition probability into state j in l steps. By the assumptions made, the equilibrium probabilities $\pi_R(j)$ of being in state j under policy R exist and are given by

$$\pi_R(j) = \lim_{l\to\infty} P_{ij}^{l}(R). \tag{2}$$

It follows that the limit in (1) can be obtained as

$$V_R = \sum_{j=1}^{n} C_R(j)\pi_R(j), \tag{3}$$

independent of the initial state. The Markov decision problem is to minimize V_R over all possible policies R. This may be shown to be equivalent to solving a particular linear program but in the two urban models to be discussed below the problem will be treated more directly.

It may happen that a policy R determines two different costs $C_{R,1}(j)$ and $C_{R,2}(j)$ for each state j. This results in an expected average costs $V_{R,1}$ and $V_{R,2}$ similar to (3). For example,

$$V_{R,1} = \sum_{j=1}^{n} C_{R,1}(j)\pi_j(R). \tag{4}$$

A change in decision rule R may decrease either $V_{R,1}$ or $V_{R,2}$ while increasing the other and so we seek a set of Pareto optimal solutions in the sense of policies R which cannot be altered to improve one cost without degrading the other. The usual procedure (see [3]) is to form a new scalar valued function

$$V_R = \alpha V_{R,1} + (1-\alpha) V_{R,2} \tag{5}$$

with $0 \leq \alpha \leq 1$ and to minimize this linear combination. The scalar α acts as a trade-off parameter between the separate goals of minimizing $V_{R,1}$ or $V_{R,2}$, and by varying α one generates the possible compromise (Pareto) solutions.

3. ADAPTIVE RESPONSE

We now turn our attention to an application of the foregoing by reviewing a simple model which is designed to answer the question of how many fire companies to dispatch to any given incident, a problem of interest in the actual day to day operations of many urban fire departments. There are M types of incidents, each of which occurs as an independent Poisson process at rate λ_i, arranged in order of increasing probability of seriousness. Normal deployment to serious alarms, such as structural fires, is initially two ladder companies (as well as some engine or "pumper" companies which will not be considered here). Only one is sent when the fire is not serious, as in the case of a small brush fire. Because of the uncertainty of the incident, the dispatcher is faced with a decision on how many to dispatch. There are trade-offs. Sending two when the event is not serious reduces temporarily the number of companies which are available to a future incident, whereas if less are sent and the alarm is serious, this introduces the risk of a delay until an additional unit is dispatched. The delay may allow escalation and even loss of life. The decision to call for a second unit or to release one of those initially dispatched is made upon arrival at the scene of the fire.

A compromise between a strong fire supression capability now and improved readiness for incidents which occur in the future can be formulated as a Markov decision process, as we now show.

Let (m, n) be the state of the fire response system, where m is the number of first arriving fire engines which are presently busy at alarms and n is the number of busy second arriving units. If

$$\lambda = \sum_{i=1}^{M} \lambda_i,$$

then the probability that a given alarm is of type i is given by λ_i/λ. Consider the following options for dispatch policy in state (m, n):

$$k_i(m,n) = \begin{cases} 1, & \text{if we send one unit to alarm-type } i \\ 2, & \text{if we send two units to alarm-type } i. \end{cases} \tag{6}$$

It is apparent that there are 2^M different possible actions which can be taken in each state and that a choice of one of these constitutes a policy R in the sense of Markov decision theory. With each such policy R, define the sets S_1, S_2 for each (m, n) by

$$\begin{aligned} S_1 &= \{i \mid k_i(m,n) = 1\} \\ S_2 &= \{i \mid k_i(m,n) = 2\}. \end{aligned} \tag{7}$$

Then,

$$\text{prob(send one unit when in state } (m, n)) = \frac{1}{\lambda} \sum_{i \text{ in } S_1} \lambda_i \equiv \beta(m, n)$$

$$\text{prob(send two units when in state } (m, n)) = \frac{1}{\lambda} \sum_{i \text{ in } S_2} \lambda_i \equiv 1 - \beta(m, n).$$

The equilibrium state transition diagram is shown in Fig. 1, in which $1/\mu_1$ is the average time that first arriving units are busy at alarms and $1/\mu_2$ is the average time for second arriving units, with service time assumed to be exponentially distributed. We assume there is a total of N fire companies in the region of interest.

The steady-state balance equations can be found from inspection of the transition

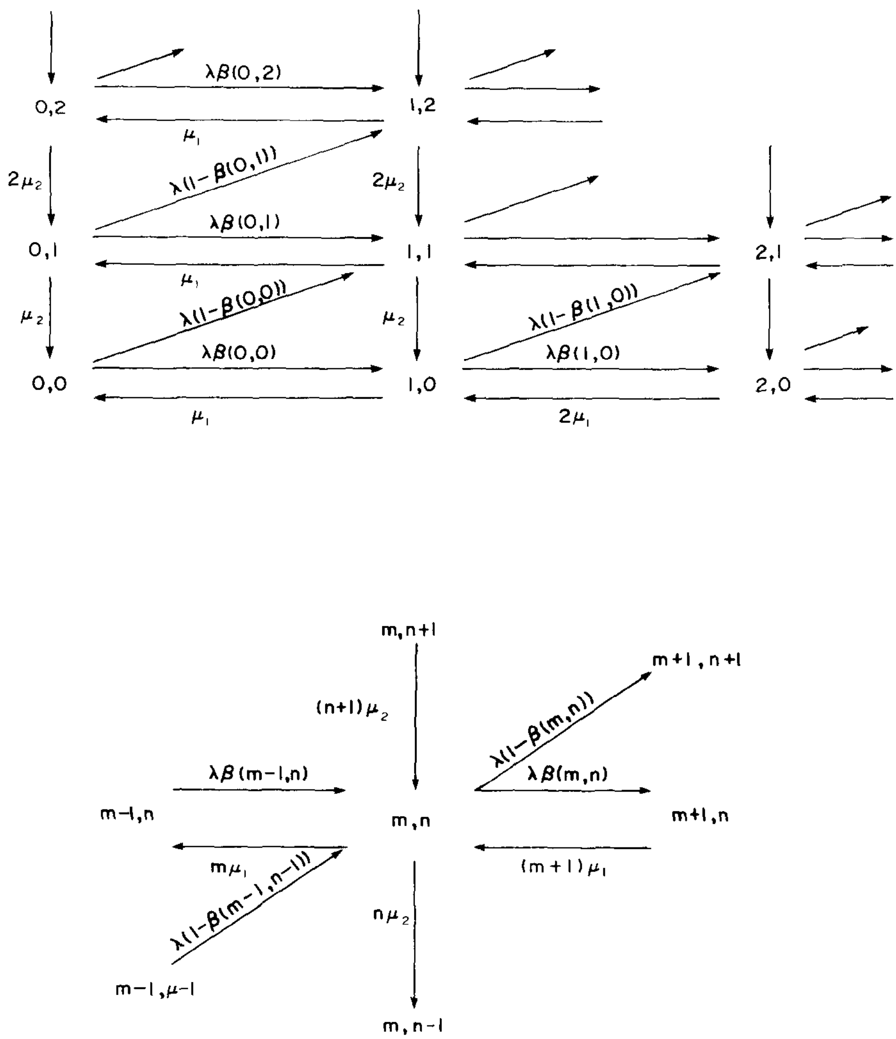

Fig. 1. State transition diagrams.

diagrams. For instance, when $m, n \geq 1$, one has

$$(\lambda + n\mu_2 + m\mu_1)\pi(m, n) = \lambda\beta(m-1, n)\pi(m-1, n) + (m+1)\mu_1\pi(m+1, n)$$
$$+ (n+1)\mu_2\pi(m, n+1) + \lambda(1-\beta(m-1, m-1))\pi(m-1, n-1). \tag{8}$$

We now wish to construct a cost function. Let $T_j(m, n)$ be the average response time of the first ($j = 1$) and second ($j = 2$) arriving fire engines to a new alarm when the system is in state (m, n). We assume that these units are the closest and next closest to the given incident. For the exposition which follows it suffices to know that T_j are determined as functions of $j = N - (m+n)$ by the relation

$$T_j(m, n) = K_j\gamma^{-1/2} \tag{9}$$

for suitable constants K_j which depend on the size of the region being serviced and on the impediments to travel. Relation (9) is known as the inverse square root law [1, 4]. The quantity γ represents the number of fire-fighting units still available to deploy (that is, which are not busy at other alarms). Therefore, the average *response delay* $U_j(m, n)$ to any alarm for the first and second arriving units ($j = 1, 2$) is given by

$$U_1 = T_1$$

$$U_2 = \begin{cases} T_2, \text{ if both units are initially dispatched} \\ T_1 + T_2, \text{ if initially only one unit is sent.} \end{cases} \tag{10}$$

Relation (10) can also be expressed as

$$U_j(m, n) = \begin{cases} T_1(m, n) & \text{for } j = 1 \\ T_2(m, n) + (2 - k_i(m, n))T_1(m, n) & \text{for } j = 2. \end{cases} \tag{11}$$

Given the dispatch policy R, the expected cost (delay) in state (m, n) is

$$C_r(m, n) = \sum_{i=1}^{M} E(\text{delay} \mid \text{alarm-type } i)\text{prob(alarm type-}i). \tag{12}$$

Now $E(\text{delay} \mid \text{alarm-type } i)$ is independent of i and can be written as a linear combination,

$$\alpha U_1(m, n) + (1-\alpha)U_2(m, n),$$

for $0 \leq \alpha \leq 1$, and so (12) becomes

$$C_R(m, n) = \frac{1}{\lambda}\sum_{I=1}^{M} \lambda_I(\alpha U_1(m, n) + (1-\alpha)U_2(m, n))$$
$$\equiv \alpha C_{R,1}(m, n) + (1-\alpha)C_{R,2}(m, n). \tag{13}$$

Using the relation

$$\beta = \frac{1}{\lambda}\sum_{i \text{ in } S_1} \lambda_i,$$

expression (13) may be rewritten as

$$C_R(m, n) = \alpha T_1(m, n) + (1-\alpha)\beta T_1(m, n) + (1-\alpha)T_2(m, n). \tag{14}$$

From the form of (14) it is seen that the cost in state (m, n) (response delay) depends on the policy R only through the corresponding value of β as determined by (6) and (7). The expected average cost is then given by [see also (3)]

$$V_R = \sum_{(m,n)} C_R(m, n)\pi_R(m, n),$$

where $\pi_R(m, n)$ are determined from (8) for each choice of R. In view of (13), V_R can be decomposed into a form which displays a linear combination of different costs as in relation (5). Let us consider what happens in the two extreme cases of $\alpha = 0,1$. When $\alpha = 1$, then $C_R(m, n) = T_1(m, n)$. Let $\pi_R(m, n)$ correspond to an optimal policy and suppose $\tilde{\pi}(m, n)$ corresponds to the policy defined by $\beta = 1$ (that is, only one unit is sent to all alarms). When $\beta = 1$, the state transition diagram shows that the only possible states are $(m, 0)$ for $m \geq 0$. Therefore, for $\alpha = 1$, we obtain from (14) that

$$V_R = \sum_{(m,n)} T_1(m, n)\pi_R(m, n)$$

and

$$\tilde{V} = \sum_{m} T_1(m, n)\tilde{\pi}(m, n).$$

Let the difference between state probabilities be given by

$$\delta(m, n) = \pi_R(m, n) - \tilde{\pi}(m, 0).$$

As relation (9) shows, $T_1(m, n) \geq T_1(m, 0)$ for $n \geq 0$. Also $T_1(m, 0) \geq T_1(0, 0)$, and so we find that

$$\begin{aligned} V_R - \tilde{V} &= \sum_{(m,n)} [T_1(m, n) - T_1(m, 0)]\pi_R(m, n) + \sum_{(m,n)} T_1(m, 0)\delta(m, n) \\ &\geq T_1(0, 0) \sum_{(m,n)} \delta(m, n). \end{aligned} \tag{15}$$

Since the sum of equilibrium probabilities over all states must equal unity, it follows that the last sum in (15) is zero. This shows that when $\alpha = 1$ an optimal policy is obtained by letting $\beta = 1$ for all states. This is an entirely reasonable conclusion in view of the fact that $\alpha = 1$ is tantamount to stating that only the response delays of the first arriving fire company are of any significance and that having two companies at a fire is of no importance under any circumstances. In the contrary case of $\alpha = 0$, the cost C_R becomes $\beta T_1 + T_2$, and an argument completely analogous to the one given above shows that an optimal policy is achieved by setting $\beta = 0$ in all states. This means, of course, that one always sends two units to any alarm. We therefore see that the extremes of α correspond to either minimizing response time to all current alarms, with no concern for future risk due to the possible unavailability of units, or minimizing response time to alarms which may occur in the future, achieved by sacrificing serious alarms which

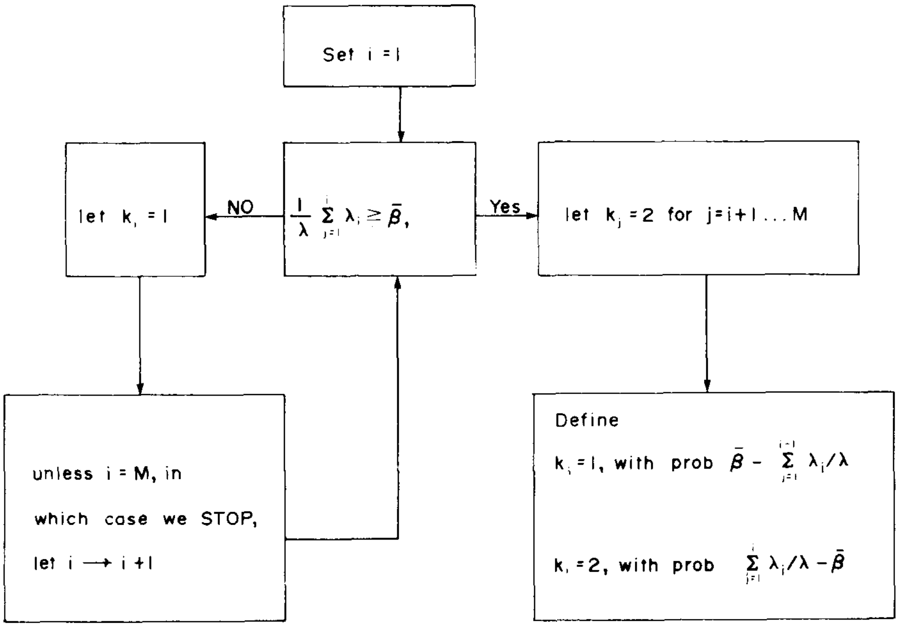

Fig. 2

arrive now. In effect, the trade-off parameter is an indicator of how much one wished to discount future costs. There will be no attempt here to solve the general minimization problem (see Swersey [5]) but we do want to show that the size of the problem can be drastically reduced by proving that the optimal policy can be chosen to have a special form. In order to do this, suppose that $\bar{R}$ is an optimal decision rule defined by some $\bar{k}_i(m, n)$ for $i = 1, 2 \ldots M$. Now construct a new decision rule in the following way. Suppose, first, that for an alarm of type 1, $\lambda_1/\lambda \geq \bar{\beta}(m, n)$, where $\bar{\beta}$ is the sum over i in S_1 of λ_i/λ and S_1 is determined by $\bar{k}_1$. Then send one unit to this alarm with probability $\bar{\beta}$ and send two with probability $\lambda_1/\lambda - \bar{\beta}$. For all other alarms of type $i > 1$, send two. Then continue as indicated in the flow diagram of Fig. 2.

It is clear that there is a smallest integer i^*, $1 \leq i^* \leq M$, for which

$$\sum_{j=1}^{i^*} \lambda_j/\lambda \geq \bar{\beta}.$$

Moreover, under this new policy one has

$$\begin{aligned}\text{prob(send one unit)} &= \sum_{j=1}^{i^*-1} \lambda_j/\lambda + \bar{\beta} - \sum_{j=1}^{i^*-1} \lambda_j/\lambda = \bar{\beta} \\ \text{prob(send two units)} &= \sum_{j=i^*+1}^{M} \lambda_j/\lambda + \sum_{j=1}^{i^*} \lambda_j/\lambda - \bar{\beta} = 1 - \bar{\beta}.\end{aligned} \tag{16}$$

Therefore, the transition probabilities for this policy are the same as for the optimal policy $\bar{R}$ so that the steady-state probabilities $\pi(m, n)$ as determined by (8) are unchanged. Moreover, the expected average cost, (15), also remains unaltered in value since the terms

$$\sum_{i \text{ in } S_1} \lambda_i/\lambda$$

which determines C_R in (14) are the same for the new policy as for $\bar{R}$. Thus, we have obtained a new optimal decision rule which is defined by

$$k_i(m, n) = \begin{cases} 1, \text{ if } i \leq i^*(m, n) \\ 2, \text{ if } i \geq i^*(m, n). \end{cases} \tag{17}$$

At the cutoff value $i = i^*$, we randomize by sending one unit with probability

$$\sum_{i=1}^{i^*} \lambda_i/\lambda - \bar{\beta}$$

and two with probability

$$\bar{\beta} - \sum_{i=1}^{i^*-1} \lambda_i/\lambda.$$

That is, if

$i < i^*$, send one

$i > i^*$, send two

$i = i^*$, send one or two according to the randomized scheme given above.

Roughly, this policy (known as the "Swersey cutoff theorem" [1, 5] states that one fire company is sent to all less serious alarms and two to all the more serious incidents, which is certainly plausible. The value of the optimal decision rule in this form is that it is simple to implement in practice when $i^*(m, n)$ is known. Moreover, the search for $i^*(m, n)$ has now been reduced from a total of 2^M possible actions in each state (m, n) to the much smaller number of $M + 1$ possibilities (ignoring randomization) enumerated by $k_i(m, n) = 1$ for $i \leq j$ and $k_i(m, n) = 2$ for $i > j$, $j = 0 \ldots M$.

4. SPATIAL RESPONSE

It is not uncommon in practice for the vehicles dispatched from facilities stationed in two adjoining sectors to cooperate by providing mutual aid in the event that one or the other is not able to respond to a call. The interdistrict response to calls outside the primary area of responsibility introduces an important consideration in the design of district boundaries. In an effort to minimize the maximum response distance (or time, as the case may be) one would be inclined to draw the boundary between two services areas A and B, so that all points in A are closer to the facility located in A than in B, and vice versa for B. However, if the average alarm rate λ_B is much larger than the rate λ_A, then when all units in B are saturated because of the heavy demand, the next arriving call in B is answered by a unit from A. The average response time, however, would be lower if the district boundaries had been closer to the facility B. This can be demonstrated mathematically (which will be done below) and is illustrated in Fig. 3. This represents a somewhat idealized situation in which each district has a facility located at its center with one responding unit. In effect what the figure suggests is that under

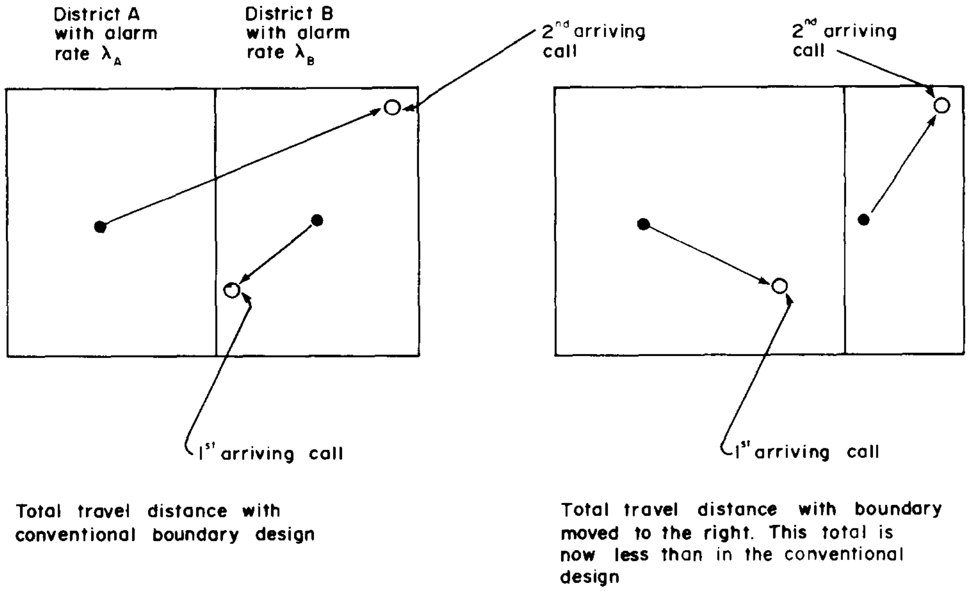

Fig. 3. The effect of changing district boundaries on overall response time.

interdistrict dispatching there is a trade-off between the two conflicting objectives of rapid response to a call now and the ability to respond effectively to calls in the future. This is not too dissimilar to the decision of *how many* units to send to a given alarm which we considered earlier. The question now is *which* units to send, which is a problem of districting. This can be turned into a Markov decision problem, as we will see.

It is best to begin by considering the case of two fixed service areas (or "response neighborhoods") A_1, A_2 (as shown in Fig. 4) which are the primary sectors of respon-

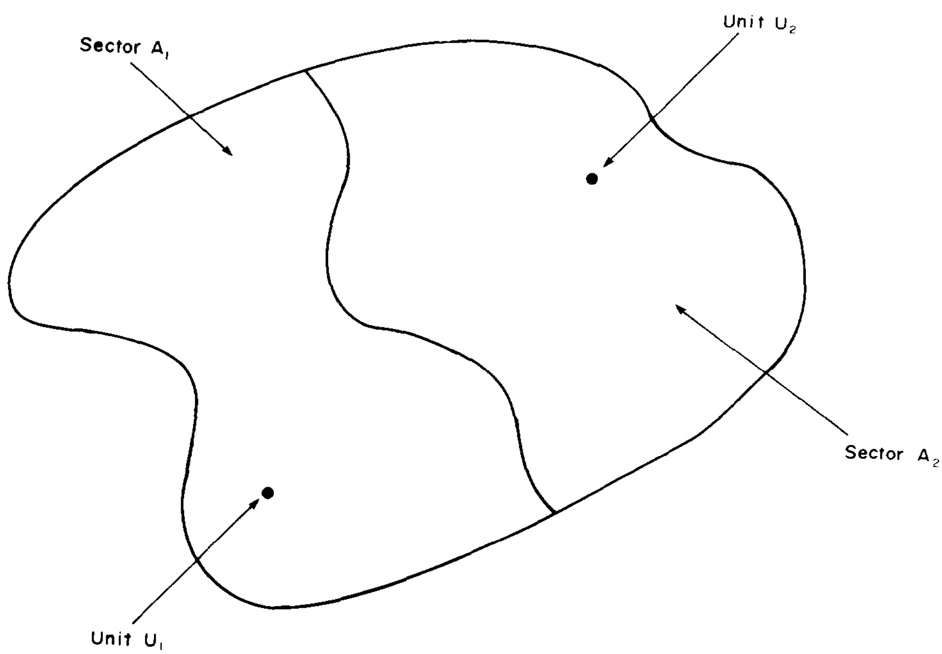

Fig. 4

sibility for two fire companies U_1, U_2. Alarms occur as independent Poisson processes at rates λ_1, λ_2 and service time for each unit is exponential at the same rate μ.

When a call arrives in A_1, it is serviced by U_1, if available. Otherwise, it is serviced by U_2. If both units are busy, then the call is "lost" (that is, it is handled by a backup unit from outside the region). Each server can either be busy or free (label these possibilities by a one or zero, respectively. Then the system has four states:

$$\begin{aligned}&(0,0)\text{—both free}\\&(1,0)\text{—}U_1 \text{ busy, } U_2 \text{ free}\\&(0,1)\text{—}U_1 \text{ free, } U_2 \text{ busy}\\&(1,1)\text{—both busy.}\end{aligned}$$

The equilibrium equations are easily obtained for this system, where $\lambda = \lambda_1 + \lambda_2$:

$$\begin{aligned}(\lambda + \mu)\pi_{10} &= \lambda_1\pi_{00} + \mu\pi_{11}\\ 2\mu\pi_{11} &= \lambda(\pi_{10} + \pi_{01})\\ \lambda\pi_{00} &= \mu(\pi_{10} + \pi_{01})\\ (\lambda + \mu)\pi_{01} &= \lambda_2\pi_{00} + \mu\pi_{11},\end{aligned}$$

which, together with $\pi_{00} + \pi_{10} + \pi_{01} + \pi_{11} = 1$, can be solved to give

$$\begin{aligned}\pi_{00} &= \frac{1}{1 + \rho + \rho^2/2}\\ \pi_{10} &= \frac{\rho_1 + \rho^2/2}{1+\rho}\,\pi_{00}\\ \pi_{01} &= \frac{\rho_2 + \rho^2/2}{1+\rho}\,\pi_{00}\\ \pi_{11} &= \rho^2/2\;\pi_{00},\end{aligned} \tag{18}$$

where $\rho_1 = \lambda_1/\mu$, $\rho_2 = \lambda_2/\mu$. If one does not distinguish between servers, then $\pi_{01} + \pi_{10}$ represents the probability of one unit being busy, without caring which. In this case, the formulas (18) are identical with the well-known Erlang formulas [2], as is easily verified. Moreover, since the Erlang formulas remain valid for an arbitrary service time distribution having finite mean $1/\mu$ [6], one might suppose that (18) also holds true for arbitrary service time distributions. That this is so was shown by Chaiken and Ignall [7]. Thus, (18) represents an extension of the Erlang case.

We now pose a decision problem. Suppose the given area actually consists of M contiguous but disjoint subregions (or "atoms") in which the calls arrive at Poisson rates $\tilde{\lambda}_i$, $i = 1 \ldots M$. The two areas A_1, A_2 of primary responsibility are to formed as clusters of these atoms. There are 2^M ways of accomplishing this and each of these affects overall response time since they alter district boundaries. A policy R consists in choosing a specific partition of the M atoms into sets A_1, A_2 and so we can write $S_i(R) = \{j \mid \text{atom } j \text{ belongs to } A_i \text{ as a result of policy } R\}$. Then let

$$\lambda_i = \sum_{j \text{ in } S_i} \tilde{\lambda}_j \quad \text{for} \quad i = 1, 2. \tag{19}$$

Our system has four states (m, n), $0 \le m, n \le 1$, whose probabilities $\pi_{m,n}$ depend on the choice of R through the values of λ_1, λ_2. These are given by (18). Then, following Eq. (3), the total *average response time* to any alarm, conditional on the call not being lost (the probability of which is $1 - \pi_{11}$) is given by

$$V_R = \sum_{0 \le m, n \le 1} C_R(m, n)\pi_{m,n}, \tag{20}$$

where the "cost" $C_R(m, n)$ is average travel time under policy R to an alarm when the system is in state (m, n). If t_{ik} is the mean travel time of unit U_i to a call originating in atom $k(1 \le k \le M)$ then, since $\tilde{\lambda}_k/\lambda$ is the probability of an alarm from atom k,

$$C_R(0,0) = \left(\frac{1}{\lambda}\sum_{k \text{ in } s_1} t_{1k}\tilde{\lambda}_k + \frac{1}{\lambda}\sum_{k \text{ in } S_2} t_{2k}\tilde{\lambda}_k\right)/(1 - \pi_{11})$$

$$C_R(0, 1) = \left(\frac{1}{\lambda}\sum_{k=1}^{M} t_{1k}\tilde{\lambda}_k\right)/(1 - \pi_{11})$$

$$C_R(1, 0) = \left(\frac{1}{\lambda}\sum_{k=1}^{M} t_{2k}\tilde{\lambda}_k\right)/(1 - \pi_{11})$$

$$C_R(1, 1) = 0.$$

Apparently there is only one opportunity to exercise a decision rule, and that is in state $(0, 0)$, where, as we have seen, there are 2^M choices. In the other states, response time to a call is independent of how the region is partitioned into the two districts.

In order to display V_R as the linear combination $V_R = \alpha V_{R,1} + (1 - \alpha)\, V_{R,2}$ as in Eq. (5), simply define $C_{R,1}(0, 0) = C_R(0, 0)$ and $C_{R,2}(0, 0) = 0$, $C_{R,1}(1, 0) = 0$ and $C_{R,2}(1, 0) = C_R(1, 0)$, $C_{R,1}(0, 1) = 0$ and $C_{R,2}(0, 1) = C_R(0, 1)$. Then

$$V_R = \alpha C_R(0, 0)\pi_{0,0} + (1 - \alpha)[C_R(1, 0)\pi_{1,0} + C_R(0, 1)\pi_{0,1}] \tag{21}$$

Now

$$\sum_{S_1} t_{ik}\tilde{\lambda}_k = \sum_{k=1}^{M} t_{ik}\tilde{\lambda}_k - \sum_{S_2} t_{ik}\tilde{\lambda}_k$$

and so if we subtract and add

$$\left(\frac{1}{\lambda}\sum_{k \text{ in } S_1} t_{2k}\tilde{\lambda}_k\right)\frac{\pi_{00}}{1 - \pi_{11}}$$

to (20) or (21) it follows that

$$V_R = \frac{1}{\lambda(1 - \pi_{11})}\left[\pi_{00}\sum_{k \text{ in } S_1}(t_{1k} - t_{2k})\tilde{\lambda}_k + \sum_{k=1}^{M} t_{1k}\tilde{\lambda}_k + (\pi_{10} + \pi_{00})\sum_{k=1}^{M} t_{2k}\tilde{\lambda}_k\right].$$

Now substitute in the values of π_{01} and π_{10} from (18):

$$V_R = \frac{\pi_{00}}{\lambda(1 - \pi_{11})}\left[\sum_{k \text{ in } S_1}(t_{1k} - t_{2k})\tilde{\lambda}_k - \frac{\rho_1}{1 + \rho}\sum_{k=1}^{M} t_{1k}\tilde{\lambda}_k + \frac{\rho_1}{1 + \rho}\sum_{k=1}^{M} t_{2k}\tilde{\lambda}_k\right] + \alpha,$$

where

$$\alpha = \left(\frac{\rho+\rho^2/2}{1+\rho}\sum_{k=1}^{M} t_{1k}\tilde{\lambda}_k + \frac{1+\rho+\rho^2/2}{1+\rho}\sum_{k=1}^{M} t_{2k}\tilde{\lambda}_k\right)\frac{\pi_{00}}{\lambda(1-\pi_{11})}.$$

Observe that α is a constant independent of S_1, S_2 and hence independent of the choice of policy R. Now define a constant S by

$$S = \frac{\rho}{1+\rho}\sum_{k=1}^{M}(t_{1k}-t_{2k})\tilde{\lambda}_k/\lambda. \tag{22}$$

Then, since $\rho_1 = (\rho/\lambda)\lambda_1$,

$$V_R = \frac{\pi_{00}}{\lambda(1-\pi_{11})}\left(\sum_{k \text{ in } S_1}(t_{1k}-t_{2k}-S)\tilde{\lambda}_k\right)+\alpha, \tag{23}$$

which is clearly minimized by making the sum as negative as possible. This is achieved by choosing A_i to be the set of atoms defined by

$$A_1 = \{k \mid t_{1k}-t_{2k} \le S\}$$
$$A_2 = \{k \mid t_{1k}-t_{2k} > S\}.$$

The Markov decision problem is completely resolved in this case, a result which was established by Carter, Chaiken, and Ignall [8]. Note that if $S = 0$, then the optimal policy describes a boundary in which the travel times of U_1, U_2 to any incident are equal. In this case, one follows the rule "dispatch the closest unit" to any alarm, a policy which is nonoptimal when $S \neq 0$ even though it is the one generally observed in most public services. In essence, S represents the amount of shift from the equal time boundary. It is clear that (23) applies to any geographic configuration of atoms.

It is useful to consider the extreme cases of α. When $\alpha = 1$, then $V_R = C_R(0,0)\ \pi_{0,0}$ where $\pi_{0,0}$ is independent of the choice of R as we see from (18). Proceeding as above, $C_R(0,0)$ may be written as

$$C_R(0,0) = \frac{1}{\lambda}\sum_{k=1}^{M} t_{1k}\tilde{\lambda}_k + \sum_{k \text{ in } S_2}(t_{2k}-t_{1k})\tilde{\lambda}_k, \tag{24}$$

in which the first sum in the right is also clearly independent of the decision rule R. Therefore, the minimum of V_R is achieved by letting the sum of terms $t_{2k}-t_{1k}$ over k in S_2 be as negative as possible. This means that S_2 is defined by $t_{2k} \le t_{1k}$ or, to put it another way, the region is partitioned so as to always send the closest unit. Thus, $\alpha = 1$ corresponds to a policy which ignores future degradation of service under the optimistic assumption that there is always a sufficient number of fire-fighting units available to respond to new calls. The choice $\alpha = 0$ in (21) considers the opposite situation in which one pessimistically believes that at least some fire companies are always busy on other calls and hence unavailable to respond to current alarms. Under these circumstances, the closest unit is never sent except for a most fortuitous choice of alarm rates $\tilde{\lambda}_j$.

The above considerations may be extended to the case of more than two sectors in which r response vehicles are located. Each unit may or may not be busy so that there is now 2^r possible states in the system instead of just four. The corresponding model is called the "Hypercube Model" [9] since the states are described as vertices of a hypercube in r-dimensional space. A discussion of the case $r = 3$ is in [10] and [11]. A treatment of the Markov decision process in this more general setting is also provided by Jarvis [12].

REFERENCES

1. W. Walker, J. Chaiken, and E. Ignall, eds., *Fire Department Deployment Analysis*, Elsevier, New York (1980).
2. S. Ross, *Applied Probability Models with Optimization Applications*, Holden-Day, San Francisco (1970).
3. L. Zadeh, Optimality and Non-Scalar-Valued Performance Criteria. *IEEE Trans. Autom. Control*, 59–60 (1963).
4. P. Kolesar and E. Blum, Square Root Laws for Fire Engine Response Distances. *Manage. Sci.* **19**, 1365–1378 (1973).
5. A. Swersey, A Markovian Decision Model for Deciding How Many Fire Companies to Dispatch, Technical Report 23, Yale School of Organization and Management (1978).
6. L. Takacs, On Erlang's Formula. *Ann. Math. Stat.* **40**, 71–78 (1969).
7. J. Chaiken and E. Ignall, An Extension of Erlang's Formulas Which Distinguish Individual Servers. *J. Appl. Probability* **9**, 192–197 (1972).
8. G. Carter, J. Chaiken, and E. Ignall, Response Areas for Two Emergency Units. *Oper. Res.* **20**, 571–597 (1974).
9. R. Larson, A Hypercube Queueing Model for Facility Location and Redistricting in Urban Emergency Services. *Comput. Oper. Res.*, 67–95 (1974).
10. E. Beltrami, *Models for Public Systems Analysis*, Academic, New York (1977).
11. R. Larson and A. Odoni, *Urban Operations Research*, Prentice-Hall, Englewood Cliffs, NJ (1981).
12. J. Jarvis, Optimal Dispatch Policies for Urban Service Systems, Operations Research Center Report 02-73, Massachusetts Institute of Technology (1973).