DISCRETE APPLIED MATHEMATICS

ELSEVIER Discrete Applied Mathematics 95 (1999) 321–330

Recognizing graphs of acyclic cubical complexes

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Received 21 October 1997; accepted 23 December 1998

Abstract

Acyclic cubical complexes have first been introduced by Bandelt and Chepoi in analogy to acyclic simplicial complexes. They characterized them by cube contraction and elimination schemes and showed that the graphs of acyclic cubical complexes are retracts of cubes characterized by certain forbidden convex subgraphs. In this paper we present an algorithm of time complexity $O(m \log n)$ which recognizes whether a given graph G on n vertices with m edges is the graph of an acyclic cubical complex. This is significantly better than the complexity $O(m\sqrt{n})$ of the fastest currently known algorithm for recognizing retracts of cubes in general.

Keywords: Median graph; Acyclic cubical complex; Algorithm; Complexity

1. Introduction

A *cubical complex* $\mathscr{K}$ is a finite set of cubes of any dimension which is closed under taking subcubes and nonempty intersections. The vertices of $\mathscr{K}$ are the zero-dimensional cubes of $\mathscr{K}$ and in the *graph G of* $\mathscr{K}$ two vertices are adjacent if they constitute a one-dimensional cube. Suppose the complex $\mathscr{K}$ contains a maximal subcube Q which can be subdivided into two complementary subcubes Q_1 and Q_2 such that no cubes of $\mathscr{K}$ other than subcubes of Q intersect both Q_1 and Q_2. Then contraction of all edges between Q_1 and Q_2 gives rise to a new cubical complex $\mathscr{K}'$. Such an operation is called a *cube contraction* and a cubical complex is called *acyclic* if there exists a sequence of cube contractions eventually transforming it into the trivial one-vertex complex. One says that such a complex has a *cube contraction scheme*.

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[1] Supported by the Ministry of Science and Technology of Slovenia under the grant J1-0498-0101.

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PII: S0166-218X(99)00084-0

This is one of several possible definitions of acyclic cubical complexes which have been introduced by Bandelt and Chepoi [5]. In the cited paper they also showed that the graphs of acyclic cubical complexes are retracts of cubes characterized by certain forbidden convex subgraphs. This characterization does not seem to give rise to an efficient algorithm for checking whether a given graph is the graph of an acyclic cubical complex. It is the aim of this paper to derive such an algorithm. Its complexity is $O(m \log n)$, where n is the number of vertices and m the number of edges of the graph G underlying the given cubical complex $\mathscr{K}$ to be checked for acyclicity. This compares favorably with the complexity $O(m\sqrt{n})$ of the presently fastest known algorithm for recognizing retracts of cubes but still falls short of the complexity $O(m)$ with which cubes can be recognized, cf. [6,10].

Retracts of cubes constitute a rich class of graphs which coincides with the class of *median graphs*, see [11] for a recent survey. Median graphs have originally been introduced by Avann [3], Nebeský [14] and Mulder [12] as graphs in which there exists a unique vertex x to every triple of vertices u, v, w such that x lies on shortest paths between any pair of vertices from u, v, w. The vertex x is called the *median* of the triple u, v, w. It is easy to see that median graphs are bipartite and that trees and cubes are median graphs. That this class coincides with the class of retracts of cubes is due to Bandelt [4].

2. Preliminaries

In this paper we shall consider finite simple graphs, i.e. finite undirected graphs without loops or multiple edges. To fix notation, let $G=(V,E)$ be such a graph, where V denotes the vertex set of G and E its edge set. Also, for $X \subseteq V$, let $\langle X \rangle$ denote the subgraph induced by X.

A subgraph H of a graph G is an *isometric* subgraph, if the distance $d_G(u,v)$ in G between any pair of vertices u and v of H is equal to the distance $d_H(u,v)$ between u and v in H. If there is no danger of confusion we will often write $d(u,v)$ instead of $d_G(u,v)$. Isometric subgraphs of cubes are called *partial cubes* and are also known as partial binary Hamming graphs. Furthermore, we say H is a *convex* subgraph of G if all shortest paths between vertices of H lie in H.

We have already mentioned that median graphs are retracts of cubes and vice versa. Although we shall not use this fact as a definition of median graphs we wish to mention that it implies that they are isometric subgraphs of cubes and also bipartite. For every edge ab of a bipartite graph G it is useful to define the following sets:

$$W_{ab} := \{w \in V \mid d(w,a) < d(w,b)\},$$

$$W_{ba} := \{w \in V \mid d(w,b) < d(w,a)\},$$

$$U_{ab} := \{u \in W_{ab} \mid u \text{ is adjacent to a vertex in } W_{ba}\},$$

$$U_{ba} := \{u \in W_{ba} \mid u \text{ is adjacent to a vertex in } W_{ab}\},$$

$$F_{ab} := \{uv \mid u \in U_{ab},\ v \in U_{ba}\}.$$

It is well-known, cf. [12,13], that G is a median graph if and only if it is bipartite and if the subgraphs $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$ are convex for every edge ab. It is easy to see that in this case F_{ab} is a matching inducing an isomorphism between $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$. Also, in partial cubes any two sets F_{ab} are either identical or distinct.

For a partial cube, and thus also for a median graph, finding the sets F_{ab} is actually equivalent to isometrically embedding G into a cube. Suppose the number of distinct sets F_{ab} is k. Then we color the edges of G with k colors, choosing one color for every set F_{ab}, and note that one can show that in any shortest path in G no two edges have the same color. Moreover, the set of colors of the edges of any two shortest paths P and P' between two vertices v and w of G are always the same. We then fix a vertex v. For every vertex w we then consider a shortest path P from v to w and assign a vector of length k to w as follows: If there is an edge of P colored with color i we set the ith component of this vector 1, otherwise we set it 0. The vertices of G are then embedded via the canonical labeling of the cube Q_k with 0–1 vectors of length k.

If both $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$ are cubes, then they must be convex in G. If G is the graph of a cubical complex then contraction of the edges of F_{ab} to single vertices corresponds to a cube contraction as defined in the introduction. Of particular interest are those contractions, where either $\langle U_{ab}\rangle = \langle W_{ab}\rangle$ or $\langle U_{ba}\rangle = \langle W_{ba}\rangle$. In such a case we call this operation a *contraction of a pendant cube*, because if G is the graph of a cubical complex this corresponds to the contraction of a so-called pendant cube of the cubical complex. The proof of Proposition 1 of [5] makes use of the fact that an acyclic cubical complex can be contracted to the one-vertex complex by contractions performed only on pendant cubes and, moreover, such a sequence can be started from any pendant cube.

In order to decide whether a given median graph G is the graph of an acyclic cubical complex it thus suffices to check whether there is a sequence of contractions of pendant cubes reducing G to the one-vertex graph. This is not too difficult once all the sets F_{ab} have been found. Our main difficulty will be to avoid having to check initially whether the given graph G is a median graph, because currently the fastest known algorithm for this task has complexity $O(m\sqrt{n})$. It is due to Hagauer et al. [8] and uses induction based on the following theorem of Mulder [12,13]:

Theorem 1 (Mulder's Convex Expansion Theorem). *Let ab be an edge of a connected bipartite graph G. Then G is a median graph if and only if the following three conditions hold*:

(i) *F_{ab} is a matching defining an isomorphism between $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$.*
(ii) *$\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$ are convex in $\langle W_{ab}\rangle$ and $\langle W_{ba}\rangle$, respectively.*
(iii) *Both $\langle W_{ab}\rangle$ and $\langle W_{ba}\rangle$ are median graphs.*

The algorithm uses (iii) as the induction hypothesis, finds F_{ab} and checks the validity of (i) in $O(m\log n)$ steps, but needs $O(m\sqrt{n})$ steps to check condition (ii).

The idea of the algorithm in this paper for investigating whether a given graph G is the graph of an acyclic cubical complex is based on the observation that every induced subgraph H of a median graph which is a cube is convex and that checking whether the $\langle U_{ab}\rangle$-s are cubes can be reduced to counting their vertices and edges.

However, instead of modifying the rather involved proof in [8] we shall present a new, self-contained proof which is shorter and more transparent. It is also based on Mulder's convex expansion theorem and on several basic facts about median graphs and isometric subgraphs of cubes in general. These facts and the algorithm of complexity $O(m \log n)$ for checking whether a given graph G is a median graph with a cube elimination scheme will be the subject of the next sections. By the results of Bandelt mentioned in the introduction the algorithm checks whether the given graph G is the graph of an acyclic cubical complex.

3. Median graphs

We have already mentioned that median graphs are partial cubes. Such graphs can be characterized by the transitivity of a relation Θ introduced by Djoković [7]. Given a graph G the relation Θ is defined on $E(G)$ as follows: one says two edges $e = xy \in E(G)$ and $f = uv \in E(G)$ are in relation Θ, in symbols $e\Theta f$, if

$$d(x,u) + d(y,v) \neq d(x,v) + d(y,u).$$

Θ is reflexive and symmetric but need not be transitive. We denote its transitive closure by Θ^*. It is easy to see that opposite edges of a four-cycle without diagonals are in relation Θ and that adjacent edges of a bipartite graph cannot be in relation Θ.

For us the importance of this relation is due to a result of Winkler [15], who showed that a bipartite graph is a partial cube if and only if $\Theta = \Theta^*$. Moreover, the sets F_{ab} are the equivalence classes of the edges ab with respect to Θ^*, and these are the sets which we wish to determine. Unfortunately the fastest known algorithm for finding Θ^* in a bipartite graph has complexity $O(mn)$ and cannot be of much use for our purpose, see [2,9,10].

However, if one already knows that G is a median graph (without knowing its embedding) one can determine Θ^* more efficiently by making use of a relation δ. One says two edges e, f of a graph are in relation δ if they are either identical or if they are opposite edges of a square without diagonals. Clearly $\delta \subseteq \Theta$ and thus $\delta^* \subseteq \Theta^*$. By (ii) of Theorem 1 and the fact that convex subgraphs are connected we immediately infer that δ^* must be equal to Θ^* and thus to Θ for median graphs. As we shall see δ and δ^* can be determined in $O(m \log n)$ steps for a class of graphs containing all median graphs.

In our algorithms we shall always assume that G is connected and bipartite and that the vertices of G are arranged in a breadth first search (BFS) order with respect to a root v_0. For a graph G given by its adjacency list this can be done in $O(m)$ time and thus is no restriction for our complexity considerations. Moreover, considering the edges of G as oriented pairs of vertices, we call ab an *up-edge* if $d(v_0, a) < d(v_0, b)$

and a *down-edge* if $d(v_0,a) > d(v_0,b)$. (Since G is bipartite $d(v_0,a)$ cannot be equal to $d(v_0,b)$.) Also, we call the set

$$L_i = \{v \mid d(v_0,v) = i\}$$

the ith *level* of G with respect to the root v_0 and denote the index of the level of a vertex v by $l(v)$. Thus, if $v \in L_i$, then $l(v) = i$.

It is clear what is meant by the *down-degree* $\rho_d(v)$ of a vertex v in G, it is the number of neighbors of v in $L_{l(v)-1}$. For us it is important that the down degree of a vertex in a subgraph G of a cube can never exceed $\log|V(G)|$, see [8]. Clearly every (unoriented) edge is a down-edge of exactly one of its endpoints. Thus, $\sum_{v\in V(G)} \rho_d(v) = m$ and $m \leqslant n\log n$ for all subgraphs of cubes. Also, if G is a median graph and v a vertex of down-degree k, then v is contained in a cube of dimension k which meets the levels $L_{l(v)}, L_{l(v)-1}, L_{l(v)-2}, \ldots, L_{l(v)-k}$. This has also been shown in [8]. For us the main consequence of this fact is, that to any two down-edges ab, ac of a median graph there exists a vertex d in level $L_{l(a)-2}$ which is adjacent to both b and c. We call the property that such a vertex exists *down-closure*.

Lemma 2. *A median graph G contains at most $m\log n$ squares and they can be found in $O(m\log n)$ steps.*

Proof. We first note that every square of a median graph meets three levels. For, suppose the square $abcd$ just meets levels L_k and L_{k-1}, where $a,c \in L_k$ and $b,d \in L_{k-1}$. By down-closure we infer the existence of a vertex $v \in L_{k-2}$ which is adjacent to both b and d. But then both b and d are medians of a,c and v, which is not possible. (We observe that we have actually shown that the vertices a,c,v, together with b,d form an induced subgraph $K_{2,3}$ of G, which is not possible in a median graph.) □

Given a vertex x we shall now find all squares containing x which meet the levels $L_{l(x)}, L_{l(x)-1}$ and $L_{l(x)-2}$ by the following procedure:

Algorithm 1. Let x be a vertex of G. Perform the following steps:

(i) List the down-neighbors of x in some order.

(ii) Let a be the first down-neighbor in this order. Label every down-neighbor of a with $[a]$.

(iii) Let b be the successor of a. Process the down-neighbors of b. If they are unlabeled, label them $[b]$. If they are labeled (with a string of length 1) we have found a square and we add b to the string with which this vertex is labeled.

(iv) Let z be a down-neighbor of x processed after a and b. Processing the down-neighbors of z we attach the label $[z]$ if they are unlabeled, notice that we have found a square if they already have a label of length 1 and attach z to the label.

We remark that in a median graph labels of length 2 can be assigned only once. If two identical labels of length two are found G cannot be a median graph. It is easy to look for such labels. If we detect them, we reject G.

If we find a label of length 3 then G contains a $K_{2,3}$ and is not a median graph. Again G is rejected as a non-median graph.

(v) Having processed all down-neighbors of x we erase the labels of the at most $\log^2 n$ vertices we have considered and store the information of the (at most $\log^2 n$) squares we have found.

Clearly the complexity of performing steps (i)–(v) is $O(\log^2 n)$. Doing this for every x we see that there are at most $n \log^2 n = m \log n$ squares in G and that the complexity of finding them all is $O(m \log n)$.

Lemma 3. *The sets F_{ab} of a median graph can be found in* $O(m \log n)$ *time.*

Proof. By Lemma 2 there are at most $m \log n$ squares in G. Hence, the relation δ has at most $2m \log n$ elements. To compute δ^* within the same time we proceed as follows. Let Γ be the graph defined on $V(\Gamma) = E(G)$, where e, f are adjacent if $e \delta f$. Clearly, δ^* is defined by the connected components of Γ. They can be found in $|E(\Gamma)| = O(m \log n)$ steps. The proof is then completed by the observation that the δ^*-classes of G are exactly the sets F_{ab}. □

As we already mentioned, for a partial cube finding the sets F_{ab} is equivalent to isometrically embedding G into a cube. Hence we conclude:

Corollary 4. *A median graph can be isometrically embedded into a cube in* $O(m \log n)$ *time.*

Corollary 4 was first obtained in [8] using a different, more complicated approach.

4. Recognizing acyclic cubical complexes

In the previous section we showed that a median graph G can be isometrically embedded into a cube with complexity $O(m \log n)$. However, although we can detect some non-median graphs and reject them, Algorithm 1 already assumes that the graph G is a median graph and accepts some non-median graphs. In particular, it may accept graphs with squares meeting only two levels. We wish to reject such graphs.

Algorithm 2. Let G be a simple graph given by its adjacency list.

(1) Check the number of vertices and edges. If $m > n \log n$ reject.

(2) Check connectedness, arrange vertices in BFS order with respect to an arbitrarily chosen root and determine levels. If G is disconnected, reject.

(3) Check bipartiteness. If not bipartite, reject.

(4) Check the down-degree for every vertex. If a down-degree larger than $\log n$ is found, reject G.

(5) Find squares that meet three levels by Algorithm 1.

(6) Check down-closure. Reject G when down-closure is violated.

(7) Find δ and δ^* with respect to these squares.

(8) Check whether any two distinct edges incident with the same vertex are in different δ^*-classes. If this condition is violated, reject.

We observe that Algorithm 2 clearly accepts all median graphs. Also, with the exception of steps (5) and (6), all steps of the algorithm are straightforward and clearly have at most complexity $O(m \log n)$.

For step (5) we turn to the proof of Lemma 2 and the algorithm presented there. The proof starts with the remark that a median graph has no squares that meet just two levels. This condition need not be satisfied for our graph G being investigated and we will eventually have to detect and reject graphs with such squares. But we can use this algorithm as it is to find all squares that meet three levels. Of course we wish to reject all graphs in which a $K_{2,3}$ is detected in step (iv). Clearly the number of squares meeting three levels is at most $m \log n$ and the complexity of finding them at most $O(m \log n)$.

Down-closure of step (6) is easy to check. We first note that the algorithm in the proof of Lemma 2 already rejects all graphs in which three down-neighbors of a vertex x have a common down-neighbor. Thus, any two distinct pairs of down-edges of x have distinct neighbors in level $L_{l(x)-2}$. However, two distinct down-edges of x might have more than one common neighbor in $L_{l(x)-2}$. But then a label of length two in step (iv) would be assigned twice. We can detect this within the same time complexity and reject such graphs.

The rest is easy, because now we only have to count whether there are as many squares with top vertex x as there are distinct pairs of down-edges from x.

Note that we still have not dealt with graphs containing squares meeting two levels only. Let $abcd$ be such a square and let $l(a) = l(c)$ and $l(a) + 1 = l(b) = l(d)$. By down-closure there must be a vertex v in level $L_{l(a)-1}$ which is adjacent to both a and c. But then the edges ab and ad are in relation δ to the edge vc and thus in the same δ^*-class. Such graphs are detected and rejected in step (8).

In summary we can state:

Lemma 5. *Algorithm* 2 *admits all median graphs and has time complexity* $O(m \log n)$. *Moreover, for all admitted graphs it determines* δ *and* δ^* *correctly.*

Since $\delta \subseteq \Theta$ we infer that $\delta^* \subseteq \Theta^*$. We would like to show $\Theta = \delta^* = \theta^*$. In general, however, it is not even easy to show that for all graphs admitted $\Theta \subseteq \delta^*$ holds. Also, the graphs admitted need not be isometrically embeddable into a cube. To see this, consider a Q_3, label the vertices by 0–1 vectors of length 3, remove the edge $(0\,1\,1)(1\,1\,1)$ and choose $v_0 = (0\,0\,0)$, as depicted in the left graph in Fig. 1. Also note that this graph would not be admitted if we had chosen $(0\,1\,1)$ as a base. One of the problems with this graph is that edges of the same color do not induce isomorphisms. We can check this, of course.

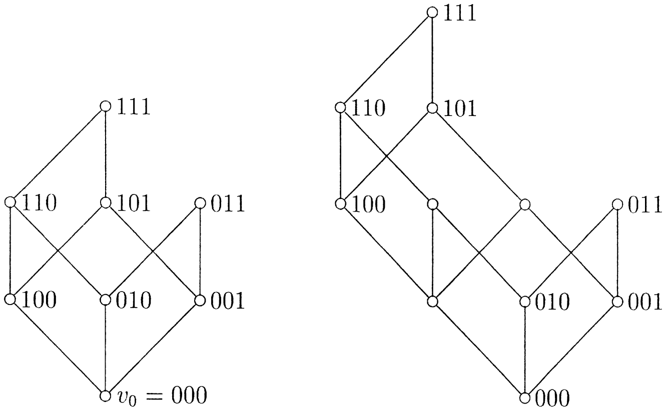

Fig. 1. Two examples.

To do this we define F^*_{ab} as the set of edges in relation δ^* to ab. Suppose that ab is an up-edge. Then we set

$$U^*_{ab} = \{x \mid xy \in F^*_{ab} \text{ and } xy \text{ is an up-edge}\}$$

as the set of vertices and define U^*_{ba} analogously. Since every vertex x is in $d(x)$ sets F^*_{ab} the total number of vertices in the U^*_{ab}'s and U^*_{ba}'s equals $2m$. Also, since every edge is a down-edge with respect to one endpoint and an up-edge with respect to the other, the total number of edges in the subgraphs $\langle U^*_{ab}\rangle$ and $\langle U^*_{ba}\rangle$ is thus bounded by the total number of vertices times the bound $\log n$ for the down degree, i.e. these graphs contain at most $m \log n$ edges. We can thus check within this complexity whether F^*_{ab} induces an isomorphism from $\langle U^*_{ab}\rangle$ onto $\langle U^*_{ba}\rangle$.

Lemma 6. *Algorithm 2 can be modified such that an isomorphism induced by F^*_{ab} between $\langle U^*_{ab}\rangle$ and $\langle U^*_{ba}\rangle$ can be checked within time complexity* $\mathrm{O}(m \log n)$.

Unfortunately even then the graphs admitted need not be partial cubes. To see this it suffices to introduce two new edges from the midpoint of the edge $(0\,0\,0)(1\,0\,0)$ to the midpoints of the edges $(0\,0\,1)(1\,0\,1)$ and $(0\,1\,0)(1\,1\,0)$ and start the modified algorithm from vertex $(0\,0\,0)$ (see the second graph of Fig. 1).

What we are interested in, however, are graphs with a cube elimination scheme. We thus wish to recognize subgraphs $\langle U_{ab}\rangle$ which are cubes, and in particular those which are pendant cubes. Since one can check in $\mathrm{O}(m)$ time whether a given graph is a cube and since $\sum_{ab} |\langle U_{ab}\rangle| \leqslant 2m \log n$ we can determine all $\langle U^*_{ab}\rangle$ which are cubes in $\mathrm{O}(m \log n)$ time. We can even check whether they have been properly colored by our algorithm.

Lemma 7. *Suppose $\langle U^*_{ab}\rangle$ is isomorphic to the cube Q_k and suppose the degree $d_G(x)$ of every vertex $x \in U^*_{ab}$ equals $k+1$. Then G is the graph of an acyclic cubical complex if and only if $G \backslash\langle U^*_{ab}\rangle$ is.*

Proof. If G is an acyclic cubical complex then $\langle U^*_{ab}\rangle = \langle U_{ab}\rangle$ and by Theorem 1 we conclude that $G\backslash\langle U^*_{ab}\rangle$ is an acyclic cubical complex too.

If $G\backslash\langle U^*_{ab}\rangle$ is an acyclic cubical complex it is a median graph. As a median graph it has been properly colored by our algorithm. This implies that $\langle U^*_{ba}\rangle$ is convex. Using Mulder's convex expansion theorem again G is a median graph, and since $\langle U^*_{ab}\rangle$ is a pendant cube G is the graph of an acyclic cubical complex. □

Lemma 7 suggests a recursive procedure for checking whether G is the graph of an acyclic cubical complex. In fact, since we have determined all possible candidates for the U_{xy} in G it would suffice to be able to determine the graphs $\langle U^*_{xy}\rangle\backslash\langle U^*_{ab}\rangle$ in time proportional to the number of edges in $\langle U^*_{ab}\rangle$.

This we can do by setting up our data structure accordingly [1]. We recall that we assume that G is given by an adjacency list. This list should also include the name (address) of the edges corresponding to the endpoints in every line. We add an incidence list, i.e. to every edge e we list the two vertices incident with it, color and orientation (up-edge or down-edge). Recall that color refers to the set F^*_{ab} to which the edge e belongs.

Also, for every color i we get two graphs $\langle U^*_{xy}\rangle$ and $\langle U^*_{yx}\rangle$, where xy denotes any edge of color i. For these graphs we prepare adjacency lists as for G. Also, we shall keep track of the degrees (in the subgraphs considered) of the vertices and update the numbers of edges and vertices for these graphs as well as the sum of the degrees (in G).

We now remove the vertices of $\langle U^*_{ab}\rangle$ from (a copy of) G and from the $\langle U^*_{xy}\rangle$ and $\langle U^*_{yx}\rangle$. We first observe that v belongs to U_{xy} if the color of xy is a color occurring in $\langle U^*_{ab}\rangle$ and that every vertex v is adjacent to all colors. However, we still screen the edges incident with v to determine whether they are up- or down-edges in order to go to the proper $\langle U^*_{xy}\rangle$ or $\langle U^*_{yx}\rangle$.

We can go immediately to the proper vertex in, say, $\langle U^*_{xy}\rangle$ and delete the line of that vertex. Before doing so we go through the list of neighbors of v though and for every edge we go to the lines in which it is listed and remove it from these lines. For every vertex and edge removed we adjust the edge- and vertex-counter and list the new degree. Also, for every vertex removed the total of the sum of the degrees in $G\backslash\langle U^*_{ab}\rangle$ is reduced by the dimension, say k of $\langle U^*_{ab}\rangle$.

To remove a vertex v we have to search for it in k lists, where k is the degree of v reduced by 1. As the time for such a search is at most $\log n$ and since every vertex is removed but once from G we stay within the time complexity of $O(m\log n)$. To remove an edge we go through the vertices in the line to be removed. Every such vertex contains the address of the edge to which it corresponds, this edge can be deleted in constant time. Since the $\langle U_{ab}\rangle$ contain altogether at most $2m\log n$ edges, we stay within the required time complexity.

Following this procedure our degree count enables us to keep an updated list of all pendant cubes. If this list becomes empty before we end up at a one-vertex graph we reject G, otherwise we find that G is the graph of an acyclic cubical complex.

Theorem 8. *Graphs of acyclic cubical complexes can be recognized within time complexity* $\mathrm{O}(m \log n)$.

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