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Regular Steinhaus graphs of odd degree

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ABSTRACT

A Steinhaus matrix is a binary square matrix of size n which is symmetric, with a diagonal of zeros, and whose upper-triangular coefficients satisfy $a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$ for all $2 \leqslant i < j \leqslant n$. Steinhaus matrices are determined by their first row. A Steinhaus graph is a simple graph whose adjacency matrix is a Steinhaus matrix. We give a short new proof of a theorem, due to Dymacek, which states that even Steinhaus graphs, i.e. those with all vertex degrees even, have doubly-symmetric Steinhaus matrices. In 1979 Dymacek conjectured that the complete graph on two vertices K_2 is the only regular Steinhaus graph of odd degree. Using Dymacek's theorem, we prove that if $(a_{i,j})_{1\leqslant i,j\leqslant n}$ is a Steinhaus matrix associated with a regular Steinhaus graph of odd degree then its sub-matrix $(a_{i,j})_{2\leqslant i,j\leqslant n-1}$ is a multi-symmetric matrix, that is a doubly-symmetric matrix where each row of its upper-triangular part is a symmetric sequence. We prove that the multi-symmetric Steinhaus matrices of size n whose Steinhaus graphs are regular modulo 4, i.e. where all vertex degrees are equal modulo 4, only depend on $\left\lceil \frac{n}{24} \right\rceil$ parameters for all even numbers n, and on $\left\lceil \frac{n}{30} \right\rceil$ parameters in the odd case. This result permits us to verify Dymacek's conjecture up to 1500 vertices in the odd case.

1. Introduction

Let $s = (a_1, a_2, \ldots, a_{n-1})$ be a binary sequence of length $n-1 \geqslant 1$ with entries a_j in $\mathbb{F}_2 = \{0, 1\}$. The *Steinhaus matrix* associated with s is the square matrix $M(s) = (a_{i,j})$ of size n, defined as follows:

- $a_{i,i} = 0$ for all $1 \leqslant i \leqslant n$,
- $a_{1,j} = a_{j-1}$ for all $2 \leqslant j \leqslant n$,
- $a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$ for all $2 \leqslant i < j \leqslant n$,
- $a_{i,j} = a_{j,i}$ for all $1 \leqslant i, j \leqslant n$.

By convention $M(\emptyset) = (0)$ is the Steinhaus matrix of size $n = 1$ associated with the empty sequence. For example, the following matrix $M(s)$ in $\mathcal{M}_5(\mathbb{F}_2)$ is the Steinhaus matrix associated with the binary sequence $s = (1, 1, 0, 0)$ of length 4.

$$M(s) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The set of all Steinhaus matrices of size $n \geqslant 2$ will be denoted by $\mathcal{SM}_n(\mathbb{F}_2)$. It is clear that, for every positive integer n, the set $\mathcal{SM}_n(\mathbb{F}_2)$ has a cardinality of 2^{n-1}.

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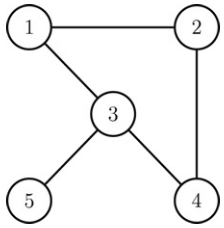
The *Steinhaus triangle* associated with s is the upper-triangular part of the Steinhaus matrix $M(s)$. It was introduced by Hugo Steinhaus in 1963 [14], who asked whether there exists a Steinhaus triangle containing as many 0's as 1's for each admissible size. Solutions of this problem appeared in [12,11]. A generalization of this problem to all finite cyclic groups was posed in [13] and was partially solved in [4].

The *Steinhaus graph* associated with s is the simple graph $G(s)$ on n vertices whose adjacency matrix is the Steinhaus matrix $M(s)$. For graph theory terminology we refer to [5]. A vertex of a Steinhaus graph $G(s)$ is usually labelled by its corresponding row number in $M(s)$ and the ith vertex of $G(s)$ will be denoted by V_i. For instance, the following graph is the Steinhaus graph $G(s)$ associated with the sequence $s = (1, 1, 0, 0)$.



For every positive integer n, the zero-edge graph on n vertices is the Steinhaus graph associated with the sequence of zeros of length $n-1$.

Steinhaus graphs were introduced by Molluzzo in 1978 [13]. A general problem on Steinhaus graphs is that of characterizing those satisfying a given graph property. The bipartite Steinhaus graphs were characterized in [3,6,10] and the planar ones in [9]. In [7], the following conjectures were made:

Conjecture 1. *The regular Steinhaus graphs of even degree are the zero-edge graph on n vertices, for all positive integers n, and the Steinhaus graph $G(s)$ on $n = 3m + 1$ vertices generated by the periodic sequence $s = (1, 1, 0, \dots, 1, 1, 0)$ of length $3m$, for all positive integers m.*

Conjecture 2. *The complete graph on two vertices K_2 is the only regular Steinhaus graph of odd degree.*

These conjectures were verified up to $n \leqslant 25$ in 1988 by exhaustive search [2]. More recently [1], Augier and Eliahou extended the verification up to $n \leqslant 117$ vertices by considering the weaker notion of *parity-regular* Steinhaus graphs, i.e. Steinhaus graphs where all vertex degrees have the same parity. They searched regular graphs in the set of parity-regular Steinhaus graphs. This has enabled them to perform the verification because it is known that Steinhaus matrices associated with parity-regular Steinhaus graphs on n vertices depend on approximately $n/3$ parameters [2,1]. This result is based on a theorem, due to Dymacek, which states that Steinhaus matrices associated with parity-regular Steinhaus graphs of even type are *doubly-symmetric* matrices, i.e. where all the entries are symmetric with respect to the diagonal and the anti-diagonal of the matrices. A short new proof of this theorem is given in Section 2. Using Dymacek's theorem, Bailey and Dymacek showed [2] that binary sequences associated with regular Steinhaus graphs of odd degree are of the form $(x_1, x_2, \dots, x_k, x_k, \dots, x_2, x_1, 1)$. In Section 3, we refine this result and, more precisely, we prove that if $(a_{i,j})_{1\leqslant i,j\leqslant n}$ is a Steinhaus matrix associated with a regular Steinhaus graph of odd degree, then its sub-matrix $(a_{i,j})_{2\leqslant i,j\leqslant n-1}$ is a *multi-symmetric* Steinhaus matrix, i.e. a doubly-symmetric matrix where each row of the upper-triangular part is a symmetric sequence. A parametrization and a counting of multi-symmetric Steinhaus matrices of size n are also given in Section 3 for all $n \geqslant 1$. In Section 4, we show that, for a Steinhaus graph whose Steinhaus matrix is multi-symmetric, the knowledge of the vertex degrees modulo 4 leads to a system of binary equations on the entries of its Steinhaus matrix. In Section 5, we study the special case of multi-symmetric Steinhaus matrices whose Steinhaus graphs are *regular modulo* 4, i.e. where all vertex degrees are equal modulo 4. We show that such a matrix of size n only depends on $\left\lceil \frac{n}{24} \right\rceil$ parameters for all n even, and on $\left\lceil \frac{n}{30} \right\rceil$ parameters in the odd case. Using these parametrizations, we obtain, by computer search, that for all positive integers $n \leqslant 1500$, the zero-edge graph on n vertices is the only Steinhaus graph on n vertices with a multi-symmetric matrix and which is regular modulo 4. This permits us to extend the verification of Conjecture 2 up to 1500 vertices.

2. A new proof of Dymacek's theorem

Recall that a square matrix $M = (a_{i,j})$ of size $n \geqslant 1$ is said to be *doubly-symmetric* if the entries of M are symmetric with respect to the diagonal and to the anti-diagonal of M, that is

$$a_{i,j} = a_{j,i} = a_{n-j+1,n-i+1}, \quad \text{for all } 1 \leqslant i, j \leqslant n.$$

In [7], Dymacek characterized the parity-regular Steinhaus graphs. These results are based on the following theorem on parity-regular Steinhaus graphs of even type, where all vertex degrees are even.

Theorem 2.1 (*Dymacek's Theorem*). *The Steinhaus matrix of a parity-regular Steinhaus graph of even type is doubly-symmetric.*

In this section we give a new easier proof of Dymacek's theorem. The main idea of our proof is that the anti-diagonal entries of a Steinhaus matrix are determined by the vertex degrees of its associated Steinhaus graph.

Theorem 2.2. *Let G be a Steinhaus graph on $n \geqslant 2$ vertices and $M = (a_{i,j})$ its associated Steinhaus matrix. Then every anti-diagonal entry of M can be expressed by means of the vertex degrees of G. If we denote by $\deg(V_i)$ the degree of the vertex V_i in G, then for all $1 \leqslant i \leqslant \left\lfloor \frac{n}{2} \right\rfloor$, we have*

$$a_{i,n-i+1} \equiv \sum_{k=0}^{i-1} \binom{i-1}{k} \deg\left(V_{i+k+1}\right) \equiv \sum_{k=0}^{i-1} \binom{i-1}{k} \deg\left(V_{n-i-k}\right) \pmod 2.$$

The proof is based on the following lemma which shows that each entry of the upper-triangular part of a Steinhaus matrix $M = (a_{i,j})$ can be expressed by means of the entries of the first row $\{a_{1,2}, \ldots, a_{1,n}\}$, the last column $\{a_{1,n}, \ldots, a_{n-1,n}\}$ or the over-diagonal $\{a_{1,2}, \ldots, a_{n-1,n}\}$ of M.

Lemma 2.3. *Let $M = (a_{i,j})$ be a Steinhaus matrix of size $n \geqslant 2$. Then, for all $1 \leqslant i < j \leqslant n$, we have*

$$a_{i,j} = \sum_{k=0}^{i-1} \binom{i-1}{k} a_{1,j-k} = \sum_{k=0}^{n-j} \binom{n-j}{k} a_{i+k,n} = \sum_{k=0}^{j-i-1} \binom{j-i-1}{k} a_{i+k,i+k+1}.$$

Proof. Easily follows from the relation: $a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$ for all $2 \leqslant i < j \leqslant n$. $\square$

Proof of Theorem 2.2. We begin by expressing each vertex degree of the Steinhaus graph G by means of the entries of the first row, the last column and the over-diagonal of M. Here we view the entries $a_{i,j}$ as 0, 1 integers. For all $2 \leqslant i \leqslant n-1$, we obtain

$$\begin{aligned}
\deg(V_i) &= \sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{i-1} a_{j,i} + \sum_{j=i+1}^{n} a_{i,j} \\
&\equiv \sum_{j=1}^{i-1} (a_{j,i+1} + a_{j+1,i+1}) + \sum_{j=i+1}^{n} (a_{i-1,j-1} + a_{i-1,j}) \\
&\equiv \sum_{j=1}^{i-1} a_{j,i+1} + \sum_{j=2}^{i} a_{j,i+1} + \sum_{j=i}^{n-1} a_{i-1,j} + \sum_{j=i+1}^{n} a_{i-1,j} \\
&\equiv a_{1,i+1} + a_{i,i+1} + a_{i-1,i} + a_{i-1,n} \pmod 2.
\end{aligned}$$

By Lemma 2.3, it follows that

$$\begin{aligned}
\sum_{k=0}^{i-1} \binom{i-1}{k} \deg\left(V_{i+k+1}\right) &\equiv \sum_{k=0}^{i-1} \binom{i-1}{k} \left(a_{1,i+k+2} + a_{i+k+1,i+k+2} + a_{i+k,i+k+1} + a_{i+k,n}\right) \\
&\equiv \sum_{k=0}^{i-1} \binom{i-1}{k} a_{1,2i-k+1} + \sum_{k=0}^{i-1} \binom{i-1}{k} a_{i+k+1,i+k+2} \\
&\quad + \sum_{k=0}^{i-1} \binom{i-1}{k} a_{i+k,i+k+1} + \sum_{k=0}^{i-1} \binom{i-1}{k} a_{i+k,n} \\
&\equiv a_{i,2i+1} + a_{i+1,2i+1} + a_{i,2i} + a_{i,n-i+1} \equiv a_{i,n-i+1} \pmod 2,
\end{aligned}$$

for all $1 \leqslant i \leqslant \left\lfloor \frac{n}{2} \right\rfloor$. The second congruence can be treated in the same way. $\square$

Remark. We deduce from Theorem 2.2 a necessary condition on the vertex degrees of a given labelled graph to be a Steinhaus graph. Indeed, vertex degrees of a Steinhaus graph on n vertices must satisfy the following binary equations:

$$\sum_{k=0}^{i-1} \binom{i-1}{k} \deg\left(V_{i+k+1}\right) \equiv \sum_{k=0}^{i-1} \binom{i-1}{k} \deg\left(V_{n-i-k}\right) \pmod 2, \quad \text{for all } 1 \leqslant i \leqslant \left\lfloor \frac{n}{2} \right\rfloor.$$

More generally, an open problem, corresponding to Question 3 in [8], is to determine if an arbitrary graph, not necessary labelled, is isomorphic to a Steinhaus graph.

Now, we characterize doubly-symmetric Steinhaus matrices.

Proposition 2.4. *Let $M = (a_{i,j})$ be a Steinhaus matrix of size $n \geqslant 3$. Then the following assertions are equivalent:*

(i) *the matrix M is doubly-symmetric,*
(ii) *the over-diagonal of M is a symmetric sequence,*
(iii) *the entries $a_{i,n-i+1}$ of the anti-diagonal of M vanish for all $1 \leqslant i \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor$.*

Proof. (i) $\Longrightarrow$ (ii): Trivial.
(ii) $\Longrightarrow$ (iii): Suppose that the over-diagonal of M is a symmetric sequence, that is

$$a_{i,i+1} = a_{n-i,n-i+1},$$

for all $1 \leqslant i \leqslant n-1$. If n is odd, then we have

$$a_{i,n-i+1} = \sum_{k=0}^{n-2i} \binom{n-2i}{k} a_{i+k,i+k+1} = \sum_{k=0}^{\frac{n-2i+1}{2}} \binom{n-2i}{k} (a_{i+k,i+k+1} + a_{n-i-k,n-i-k+1}) = 0,$$

for all $1 \leqslant i \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor$. Otherwise, if n is even, then we obtain

$$a_{i,n-i+1} = \sum_{k=0}^{\frac{n}{2}-i-1} \binom{n-2i}{k} (a_{i+k,i+k+1} + a_{n-i-k,n-i-k+1}) + 2\binom{n-2i-1}{\frac{n}{2}-i} a_{\frac{n}{2},\frac{n}{2}+1} = 0,$$

for all $1 \leqslant i \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor$.

(iii) $\Longrightarrow$ (i): By induction on n. Consider the sub-matrix $N = (a_{i,j})_{2 \leqslant i,j \leqslant n-1}$ that is a Steinhaus matrix of size $n-2$. By induction hypothesis, the matrix N is doubly-symmetric. Then it remains to prove that $a_{1,j} = a_{n-j+1,n}$ for all $2 \leqslant j \leqslant n$. First, since $a_{1,n} = 0$, it follows that $a_{1,n-1} = a_{1,n} + a_{2,n} = a_{2,n}$ and for all $2 \leqslant j \leqslant n-2$, we have

$$a_{1,j} = \sum_{k=j+1}^{n-1} a_{2,k} + a_{1,n-1} = \sum_{k=2}^{n-j} a_{k,n-1} + a_{2,n} = a_{n-j+1,n}. \quad \square$$

We are now ready to prove Dymacek's theorem.

Proof of Theorem 2.1. Let G be a parity-regular Steinhaus graph of even type on n vertices and $M = (a_{i,j})$ its Steinhaus matrix. If $n = 1$, then $M = (0)$ which is trivially doubly-symmetric. Otherwise, for $n \geqslant 2$, Theorem 2.2 implies that

$$a_{i,n-i+1} \equiv \sum_{k=0}^{i-1} \binom{i-1}{k} \deg(V_{i+k+1}) \equiv 0 \pmod 2,$$

for all $1 \leqslant i \leqslant \left\lfloor \frac{n}{2} \right\rfloor$. Finally, the matrix M is doubly-symmetric by Proposition 2.4. $\square$

3. Multi-symmetric Steinhaus matrices

In this section, we will study in detail the structure of Steinhaus matrices associated with regular Steinhaus graphs of odd degree.

Let G be a Steinhaus graph on $n \geqslant 1$ vertices. Then, for every integer $1 \leqslant i \leqslant n$, we denote by $G \setminus \{V_i\}$ the graph obtained from G by deleting its ith vertex V_i and its incident edges in G. Since the adjacency matrix of the graph $G \setminus \{V_1\}$ (resp. $G \setminus \{V_n\}$) is the Steinhaus matrix obtained by removing the first row (resp. the last column) in the adjacency matrix of G, it follows that the graph $G \setminus \{V_1\}$ (resp. $G \setminus \{V_n\}$) is a Steinhaus graph on $n-1$ vertices.

Bailey and Dymacek studied the regular Steinhaus graphs of odd degree in [2], where the following theorem is stated, using Dymacek's theorem.

Theorem 3.1 (*[2]*). *Let G be a regular Steinhaus graph of odd degree d on $2n \geqslant 4$ vertices. Then $d = n$, the Steinhaus graph $G \setminus \{V_1, V_{2n}\}$ is regular of even degree $n-1$, and $a_{1,j} = a_{1,2n-j+1}$ for all $2 \leqslant j \leqslant 2n-1$.*

Remark. In every simple graph, there are an even number of vertices of odd degree. Therefore parity-regular Steinhaus graphs of odd type and thus regular Steinhaus graphs of odd degree have an even number of vertices.

In their theorem, the authors studied the form of the sequence associated with G. We are more interested in the Steinhaus matrix of $G \setminus \{V_1, V_{2n}\}$ in what follows.

Recall that a square matrix of size $n \geqslant 1$ is said to be *multi-symmetric* if M is doubly-symmetric and each row of the upper-triangular part of M is a symmetric sequence, that is

$$a_{i,j} = a_{i,n-j+i+1}, \quad \text{for all } 1 \leqslant i < j \leqslant n.$$

First, it is easy to see that each column of the upper-triangular part of a multi-symmetric matrix is also a symmetric sequence.

Proposition 3.2. *Let $M = (a_{i,j})$ be a multi-symmetric matrix of size n. Then, each column of the upper-triangular part of M is a symmetric sequence, that is $a_{i,j} = a_{j-i,j}$ for all $1 \leqslant i < j \leqslant n$.*

Proof. Easily follows from the relation: $a_{i,j} = a_{i,n-j+i+1} = a_{j-i,n-i+1} = a_{j-i,j}$ for all $1 \leqslant i < j \leqslant n$. □

As for doubly-symmetric Steinhaus matrices, multi-symmetric Steinhaus matrices can be characterized as follows.

Proposition 3.3. *Let $M = (a_{i,j})$ be a Steinhaus matrix of size $n \geqslant 3$. Then the following assertions are equivalent:*

(i) *the matrix M is multi-symmetric,*
(ii) *the first row, the last column and the over-diagonal of M are symmetric sequences,*
(iii) *the entries $a_{i,n-i+1}$, $a_{n-2i+1,n-i+1}$ and $a_{i,2i}$ vanish for all $1 \leqslant i \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor$.*

Proof. Similar to the proof of Proposition 2.4 and by using Lemma 2.3 and Proposition 3.2. □

We now refine Theorem 3.1.

Theorem 3.4. *Let G be a regular Steinhaus graph of odd degree n on $2n \geqslant 4$ vertices. Then $G \setminus \{V_1, V_{2n}\}$ is a regular Steinhaus graph of even degree $n-1$ whose associated Steinhaus matrix is multi-symmetric.*

Proof. Let $M = (a_{i,j})$ be the Steinhaus matrix associated with G. Theorem 3.1 implies that the Steinhaus graph $G \setminus \{V_1, V_{2n}\}$ is regular of even degree $n-1$ and that we have

$$a_{1,j} = a_{1,2n-j+1},$$

for all $2 \leqslant j \leqslant 2n-1$. Therefore, for all $3 \leqslant j \leqslant 2n-1$, we have

$$a_{2,j} + a_{2,2n-j+2} = (a_{1,j-1} + a_{1,j}) + (a_{1,2n-j+1} + a_{1,2n-j+2}) = (a_{1,j-1} + a_{1,2n-j+2}) + (a_{1,j} + a_{1,2n-j+1}) = 0.$$

Then the first row of the matrix $B = (a_{i,j})_{2 \leqslant i,j \leqslant 2n-1}$, the Steinhaus matrix of the graph $G \setminus \{V_1, V_{2n}\}$, is a symmetric sequence. Moreover, by Dymacek's theorem, the matrix B is doubly-symmetric. Finally, by Proposition 3.3, the matrix B is multi-symmetric. □

Remark. By Theorem 3.4, it is easy to show that Conjecture 1 implies Conjecture 2. Indeed, if Conjecture 1 is true, then the zero-edge graph on n vertices is the only regular Steinhaus graph of even degree whose Steinhaus matrix is multi-symmetric. It follows, by Theorem 3.4, that if $G(s)$ is a regular Steinhaus graph of odd degree on $n+2$ vertices then $s = (0, \ldots, 0, 1)$ or $s = (1, \ldots, 1)$. Therefore the Steinhaus graph $G(s)$ is the star graph on $n+2$ vertices which is not a regular Steinhaus graph.

In the rest of this section we will study in detail the multi-symmetric Steinhaus matrices. First, in order to determine a parametrization of these matrices, we introduce the following operator

$$T : \mathcal{SM}_n(\mathbb{F}_2) \longrightarrow \mathcal{SM}_{n-3}(\mathbb{F}_2),$$

which assigns to each matrix $M = (a_{i,j})$ in $\mathcal{SM}_n(\mathbb{F}_2)$ the Steinhaus matrix $T(M) = (b_{i,j})$ in $\mathcal{SM}_{n-3}(\mathbb{F}_2)$ defined by $b_{i,j} = a_{i-1,j-2}$, for all $1 \leqslant i < j \leqslant n-3$. As depicted in the following matrix, the upper-triangular part of M is an extension of the upper-triangular part of $T(M)$.

$$\begin{pmatrix}
0 & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & \cdots & \cdots & a_{1,n-4} & a_{1,n-3} & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\
 & 0 & a_{2,3} & \mathbf{b_{1,2}} & \mathbf{b_{1,3}} & \mathbf{b_{1,4}} & \cdots & \cdots & \cdots & \mathbf{b_{1,n-5}} & \mathbf{b_{1,n-4}} & \mathbf{b_{1,n-3}} & a_{2,n} \\
 & & 0 & a_{3,4} & \mathbf{b_{2,3}} & \mathbf{b_{2,4}} & & & & & \mathbf{b_{2,n-4}} & \mathbf{b_{2,n-3}} & a_{3,n} \\
 & & & 0 & a_{4,5} & \mathbf{b_{3,4}} & & & & & & \mathbf{b_{3,n-3}} & a_{4,n} \\
 & & & & 0 & a_{5,6} & \ddots & & & & & \vdots & a_{5,n} \\
 & & & & & 0 & \ddots & \ddots & & & & \vdots & \vdots \\
 & & & & & & \ddots & \ddots & \ddots & & & \vdots & \vdots \\
 & & & & & & & 0 & a_{n-5,n-4} & \mathbf{b_{n-6,n-5}} & \mathbf{b_{n-6,n-4}} & \mathbf{b_{n-6,n-3}} & a_{n-5,n} \\
 & & & & & & & & 0 & a_{n-4,n-3} & \mathbf{b_{n-5,n-4}} & \mathbf{b_{n-5,n-3}} & a_{n-4,n} \\
 & & & & & & & & & 0 & a_{n-3,n-2} & \mathbf{b_{n-4,n-3}} & a_{n-3,n} \\
 & & & & & & & & & & 0 & a_{n-2,n-1} & a_{n-2,n} \\
 & & & & & & & & & & & 0 & a_{n-1,n} \\
 & & & & & & & & & & & & 0
\end{pmatrix}.$$

Proposition 3.5. *Let $M = (a_{i,j})$ be a Steinhaus matrix of size $n \geqslant 4$. Then the extension M of $T(M)$ only depends on the parameters $a_{1,2}$, a_{1,j_0} and $a_{1,n}$, with j_0 in $\{3, \ldots, n-1\}$.*

Proof. Let $3 \leqslant j_0 \leqslant n-1$. Each entry $a_{1,j}$, for $3 \leqslant j \leqslant n-1$, can be expressed by means of a_{1,j_0} and the entries of $T(M) = (b_{i,j})$. Indeed, we have

$$a_{1,j} = a_{1,j_0} + \sum_{k=j-1}^{j_0-2} b_{1,k}, \quad \text{for all } 3 \leqslant j < j_0,$$

$$a_{1,j} = a_{1,j_0} + \sum_{k=j_0-1}^{j-2} b_{1,k}, \quad \text{for all } j_0 < j \leqslant n-1.$$

Then the entries $a_{1,2}$, a_{1,j_0} and $a_{1,n}$ determine the extension M of $T(M)$. □

Therefore, for every Steinhaus matrix N of size $n-3$, there exist 8 distinct Steinhaus matrices M of size n such that $T(M) = N$. We can also use this operator to determine parametrizations of multi-symmetric Steinhaus matrices.

Proposition 3.6. *Let $M = (a_{i,j})$ be a multi-symmetric Steinhaus matrix of size n. Let j_i be an element of the set $\{2i+1, \ldots, n-i\}$ for all $1 \leqslant i \leqslant \left\lfloor \frac{n-1}{3} \right\rfloor$. Then the matrix M depends on the following parameters:*

- a_{1,j_1} *and* $\left\{a_{2i,j_{2i}} \mid 1 \leqslant i \leqslant \left\lceil \frac{n}{6} \right\rceil - 1\right\}$, *for n even,*
- $\left\{a_{2i+1,j_{2i+1}} \mid 0 \leqslant i \leqslant \left\lceil \frac{n-3}{6} \right\rceil - 1\right\}$, *for n odd.*

Proof. Let $M = (a_{i,j})$ be a multi-symmetric matrix of size n. We consider the sub-matrices $T(M)$, $T^2(M) = T(T(M))$, $T^3(M)$, $T^4(M)$, ... By successive application of Proposition 3.5 on the extension $T^{i-1}(M)$ of $T^i(M)$ and since the entries $a_{i,n-i+1}$, $a_{n-2i+1,n-i+1}$ and $a_{i,2i}$ vanish for all $1 \leqslant i \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor$ by Proposition 3.3, the parametrizations of the multi-symmetric matrix M follow. □

For all positive integers n, the number of multi-symmetric Steinhaus matrices of size n immediately follows.

Theorem 3.7. *Let n be a positive integer. If we denote by $MS(n)$ the number of multi-symmetric Steinhaus matrices of size n, then we have*

$$MS(n) = \begin{cases} 2^{\left\lceil \frac{n}{6} \right\rceil}, & \text{for } n \text{ even}, \\ 2^{\left\lceil \frac{n-3}{6} \right\rceil}, & \text{for } n \text{ odd}. \end{cases}$$

4. Vertex degrees of Steinhaus graphs associated with multi-symmetric Steinhaus matrices

In this section, we analyse the vertex degrees of a Steinhaus graph associated with a multi-symmetric Steinhaus matrix of size n. We begin with the case of doubly-symmetric Steinhaus matrices.

Proposition 4.1. *Let n be a positive integer and G be a Steinhaus graph on n vertices whose Steinhaus matrix is doubly-symmetric. Then, for all $1 \leqslant i \leqslant n$, we have*

$$\deg(V_i) = \deg(V_{n-i+1}).$$

Proof. If we denote by $M = (a_{i,j})$ the Steinhaus matrix associated with the graph G, then, for all $1 \leqslant i \leqslant n$, we have

$$\deg(V_i) = \sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n} a_{n-j+1,n-i+1} = \sum_{j=1}^{n} a_{j,n-i+1} = \deg(V_{n-i+1}). \quad \square$$

We shall now see that, for a Steinhaus graph associated with a multi-symmetric Steinhaus matrix, the knowledge of the vertex degrees modulo 4 imposes strong conditions on the entries of its Steinhaus matrix. In order to prove this result, we distinguish different cases depending on the parity of n.

Proposition 4.2. *Let n be an even number and G be a Steinhaus graph on n vertices whose Steinhaus matrix $M = (a_{i,j})$ is multi-symmetric. Then, we have*

$$\deg(V_1) = \deg(V_n) \equiv a_{1,\frac{n}{2}+1} \pmod 2,$$
$$\deg(V_2) = \deg(V_{n-1}) \equiv 2a_{1,\frac{n}{2}+1} \pmod 4,$$
$$\deg(V_3) = \deg(V_{n-2}) \equiv 2a_{2,\frac{n}{2}+1} \pmod 4,$$
$$\deg(V_{2i}) = \deg(V_{n-2i+1}) \equiv 2a_{2,2i+1} + 2a_{i,2i+1} \pmod 4, \quad \text{for all } 2 \leqslant i \leqslant \frac{n}{2} - 2.$$

Proof. First, Proposition 3.3 implies that the entries $a_{i,2i}$ and $a_{2i+1,\frac{n}{2}+i+1}$ vanish for all $1 \leqslant i \leqslant \frac{n}{2}-1$. This leads to

$$\deg(V_1) = \sum_{j=2}^{n} a_{1,j} = \sum_{j=2}^{\frac{n}{2}} (a_{1,j} + a_{1,n-j+2}) + a_{1,\frac{n}{2}+1} \equiv a_{1,\frac{n}{2}+1} \pmod 2,$$

$$\deg(V_2) = a_{1,2} + \sum_{j=3}^{\frac{n}{2}+1} (a_{2,j} + a_{2,n-j+3}) = 2\sum_{j=3}^{\frac{n}{2}+1} a_{2,j} \equiv 2a_{1,2} + 2a_{1,\frac{n}{2}+1} \equiv 2a_{1,\frac{n}{2}+1} \pmod 4,$$

$$\deg(V_3) = (a_{1,3} + a_{2,3}) + \sum_{j=4}^{\frac{n}{2}+1} (a_{3,j} + a_{3,n-j+4}) + a_{3,\frac{n}{2}+2} = 2a_{2,3} + 2\sum_{j=4}^{\frac{n}{2}+1} a_{3,j} \equiv 2a_{2,\frac{n}{2}+1} \pmod 4,$$

and, for all $2 \leqslant i \leqslant \frac{n}{2}-2$, we have

$$\begin{aligned}
\deg(V_{2i}) &= \sum_{j=i+1}^{2i-1} (a_{j,2i} + a_{2i-j,2i}) + a_{i,2i} + \sum_{j=2i+1}^{\frac{n}{2}+i} (a_{2i,j} + a_{2i,n-j+2i+1}) \\
&= 2\sum_{j=i+1}^{2i-1} a_{j,2i} + 2\sum_{j=2i+1}^{\frac{n}{2}+i} a_{2i,j} \\
&\equiv 2\sum_{j=i+1}^{2i-1} a_{j,2i+1} + 2\sum_{j=i+2}^{2i} a_{j,2i+1} + 2\sum_{j=2i}^{\frac{n}{2}+i-1} a_{2i-1,j} + 2\sum_{j=2i+1}^{\frac{n}{2}+i} a_{2i-1,j} \\
&\equiv 2a_{i+1,2i+1} + 2a_{2i,2i+1} + 2a_{2i-1,2i} + 2a_{2i-1,\frac{n}{2}+i} \\
&\equiv 2a_{i+1,2i+1} + 2a_{2i-1,2i+1} \equiv 2a_{2,2i+1} + 2a_{i,2i+1} \pmod 4.
\end{aligned}$$

Finally, we complete the proof by Proposition 4.1. □

Remark. Let n be an even number. In every Steinhaus graph on n vertices whose Steinhaus matrix is multi-symmetric the fourth vertex V_4 has a degree divisible by 4.

Proposition 4.3. *Let n be an odd number and G be a Steinhaus graph on n vertices whose Steinhaus matrix $M = (a_{i,j})$ is multi-symmetric. Then, we have*

$$\begin{aligned}
&\deg(V_1) = \deg(V_n) \equiv 0 \pmod 2, \\
&\deg(V_2) = \deg(V_{n-1}) \equiv 2a_{1,\frac{n+1}{2}} \pmod 4, \\
&\deg(V_{2i}) \equiv 2a_{i+1,2i+1} + 2a_{2i-1,2i+1} + 2a_{2i-1,\frac{n-1}{2}+i} \pmod 4, \quad \textit{for all } 2 \leqslant i \leqslant \frac{n-3}{2}, \\
&\deg(V_{2i+1}) \equiv 2a_{2,2i+2} \pmod 4, \quad \textit{for all } 1 \leqslant i \leqslant \frac{n-3}{2}.
\end{aligned}$$

Proof. Proposition 3.3 implies that the entries $a_{i,2i}$ and $a_{2i,(n+1)/2+i}$ vanish for all $1 \leqslant i \leqslant \frac{n-3}{2}$. Since each row and each column of the upper-triangular part of M is symmetric, we can use the relation

$$\sum_{k=1}^{m} a_{i,j+k} \equiv a_{i-1,j} + a_{i-1,j+m} \pmod 2, \quad \text{for all } 2 \leqslant i < j \leqslant n-m+1$$

as in the proof of Proposition 4.2, and the results follow. □

Remark. Let n be an odd number. In every Steinhaus graph on n vertices whose Steinhaus matrix is multi-symmetric the third vertex V_3 has a degree divisible by 4.

5. Multi-symmetric Steinhaus matrices of Steinhaus graphs with regularity modulo 4

In this section, we consider the multi-symmetric Steinhaus matrices associated with Steinhaus graphs which are regular modulo 4, i.e. where all vertex degrees are equal modulo 4. First, we determine an upper bound of the number of these matrices. Two cases are distinguished, according to the parity of n.

Theorem 5.1. *For all odd numbers n, there are at most $2^{\lceil \frac{n}{30} \rceil}$ multi-symmetric Steinhaus matrices of size n whose associated Steinhaus graphs are regular modulo 4.*

Proof. Let n be an odd number and $M=(a_{i,j})$ a multi-symmetric Steinhaus matrix of size n. By Proposition 3.6, the matrix M depends on the parameters $a_{2i+1,\frac{n+1}{2}+i}$ for $0\leqslant i\leqslant\left\lceil\frac{n-3}{6}\right\rceil-1$. If the Steinhaus graph associated with M is regular modulo 4, then Proposition 4.3 implies that $a_{2,2j}=0$ for all $2\leqslant j\leqslant\frac{n-1}{2}$ and thus

$$a_{2i,2j}=\sum_{k=0}^{i-1}a_{2,2j-2k}=0,$$

for all $1\leqslant i<j\leqslant\frac{n-1}{2}$.

If $n\equiv 1 \pmod 4$, then $\frac{n+1}{2}$ is odd and

$$a_{4i+1,\frac{n+1}{2}+2i}=a_{4i,\frac{n-1}{2}+2i}+a_{4i,\frac{n+1}{2}+2i}=0,$$

for all $0\leqslant i\leqslant\left\lfloor\frac{\left\lceil\frac{n-3}{6}\right\rceil-1}{2}\right\rfloor$. Therefore the matrix M can be parametrized by

$$\left\{a_{4i+3,\frac{n+3}{2}+2i}\mid 0\leqslant i\leqslant m-1\right\},$$

with

$$m=\left\lceil\frac{\left\lceil\frac{n-3}{6}\right\rceil-1}{2}\right\rceil.$$

Suppose that we know the p parameters in

$$P=\left\{a_{4i+3,\frac{n+3}{2}+2i}\mid m-p\leqslant i\leqslant m-1\right\}.$$

Then, by Proposition 3.6 again, the multi-symmetric matrix $T^{4(m-p)-1}(M)$ can be parametrized by P. Therefore the entries

$$\left\{a_{i,2i+1}\mid 4(m-p)\leqslant i\leqslant\frac{n-1}{2}-2(m-p)\right\}$$

in $T^{4(m-p)-1}(M)$ depend on the parameters in P. Moreover, if the Steinhaus graph associated with M is regular modulo 4, then Proposition 4.3 implies that

$$a_{2,2i+1}=a_{2i-1,2i+1}\equiv a_{i+1,2i+1}+a_{2i-1,\frac{n-1}{2}+i}\equiv a_{i+1,2i+1}+a_{\left(\frac{n+1}{2}-i\right)+1,2\left(\frac{n+1}{2}-i\right)+1}\pmod 2,$$

for all $1\leqslant i\leqslant\frac{n-1}{2}$. If the inequality

$$\frac{n+1}{2}-4(m-p)\geqslant 4(m-p)$$

holds, then the entries $a_{2,2i+1}$ depend on the parameters in P for all $4(m-p)\leqslant i\leqslant\frac{n+1}{2}-4(m-p)$. Since we have $a_{2,2i}=0$ for all $4(m-p)\leqslant i\leqslant\frac{n+3}{2}-4(m-p)$, it follows that the entries

$$\left\{a_{i,j}\mid \begin{array}{c}2\leqslant i\leqslant n+5-16(m-p)\\ 8(m-p)+i-1\leqslant j\leqslant n+3-8(m-p)\end{array}\right\}$$

depend on the parameters in P. Suppose now that p is a solution of the following inequality

$$n+5-16(m-p)\geqslant 4(m-p)-1.$$

Therefore the extension M of $T^{(4(m-p)-1)}(M)$ depends on the entries $a_{i,n+3-8(m-p)}$ for $2\leqslant i\leqslant 4(m-p)-1$ and $a_{1,\frac{n+1}{2}}$ which vanishes by Proposition 4.3. Thus, all the entries of the matrix M depend on the p parameters in P. Finally, a solution of this inequality can be obtained when

$$p=\left\lceil\frac{n}{30}\right\rceil\geqslant\left\lceil\frac{\left\lceil\frac{n-3}{6}\right\rceil-1}{2}\right\rceil-\frac{n+6}{20}.$$

If $n\equiv 3\pmod 4$, then $\frac{n+1}{2}$ is even and

$$a_{4i+3,\frac{n+3}{2}+2i}=a_{4i+2,\frac{n+1}{2}+2i}+a_{4i+2,\frac{n+3}{2}+2i}=0,$$

for all $0\leqslant i\leqslant\left\lceil\frac{\left\lceil\frac{n-3}{6}\right\rceil-1}{2}\right\rceil-1$. Therefore the matrix M can be parametrized by

$$\left\{a_{4i+1,\frac{n+1}{2}+2i} \mid 0 \leqslant i \leqslant m\right\}$$

with

$$m = \left\lfloor \frac{\left\lceil \frac{n-3}{6} \right\rceil - 1}{2} \right\rfloor.$$

As above, in the case $n \equiv 1 \pmod 4$, we can prove that all the entries of the matrix M depend on the p parameters in

$$\left\{a_{4i+1,\frac{n+1}{2}+2i} \mid m-p+1 \leqslant i \leqslant m\right\}$$

if p is a solution of the following inequality

$$n - 16(m-p) - 4 \geqslant 4(m-p) + 1.$$

A solution is obtained when

$$p = \left\lceil \frac{n}{30} \right\rceil \geqslant \left\lfloor \frac{\left\lceil \frac{n-3}{6} \right\rceil - 1}{2} \right\rfloor - \frac{n-5}{20}. \quad \square$$

Theorem 5.2. *For all even numbers n, there are at most $2^{\lceil \frac{n}{24} \rceil}$ multi-symmetric Steinhaus matrices of size n whose associated Steinhaus graphs are regular modulo 4.*

Sketch of proof. Similar to the proof of Theorem 5.1. Let $M = (a_{i,j})$ be a multi-symmetric Steinhaus matrix of even size n. First, by Proposition 3.6, for all positive integers $p < m - 1$ with $m = \lceil \frac{n}{6} \rceil$, the multi-symmetric Steinhaus matrix $T^{2(m-p-1)}(M)$ can be parametrized by the p entries in

$$P = \left\{a_{2i,4i+1} \mid m-p \leqslant i \leqslant m-1\right\}.$$

Moreover, if the Steinhaus graph associated with M is regular modulo 4, then Proposition 4.2 implies that $a_{1,\frac{n}{2}+1} = 0$ and $a_{2,2i+1} = a_{i,2i+1}$ for all $2 \leqslant i \leqslant \frac{n}{2} - 1$. It follows that the entries $a_{2i,n-2(m-p)+1}$ also depend on the parameters in P for all $1 \leqslant i \leqslant \frac{n}{2} - 3(m-p) + 2$. Finally, we can see that, if p is a solution of the following inequality

$$\frac{n}{2} - 3(m-p) + 2 \geqslant m - p - 1,$$

then, as in the proof of Proposition 3.6, the extension M of $T^{2(m-p-1)}(M)$ depends on the entries $a_{2i,n-2(m-p)+1}$ for $1 \leqslant i \leqslant m - p - 1$ and thus all the entries of the matrix M can be expressed by means of the p parameters in P. We conclude the proof by observing that the inequality is obtained when

$$p = \left\lceil \frac{n}{24} \right\rceil \geqslant \left\lceil \frac{n}{6} \right\rceil - \frac{n+6}{8}. \quad \square$$

Using these explicit parametrizations of the multi-symmetric Steinhaus matrices whose Steinhaus graphs are regular modulo 4, we obtain the following result by computer search:

Computational result. *For all positive integers $n \leqslant 1500$, the zero-edge graph on n vertices is the only Steinhaus graph on n vertices with a multi-symmetric Steinhaus matrix and which is regular modulo 4.*

This result can be easily proved for all odd numbers in the special case of regular Steinhaus graphs on n vertices whose Steinhaus matrices are multi-symmetric.

Theorem 5.3. *For all odd numbers n, there is no regular Steinhaus graph on n vertices whose Steinhaus matrix is multi-symmetric, except the zero-edge graph on n vertices.*

Proof. Let n be an odd number. Let G be a regular Steinhaus graph on n vertices and $M = (a_{i,j})$ its Steinhaus matrix. Then Proposition 4.3 implies that

$$\deg(V_i) \equiv 0 \pmod 4,$$

for all $1 \leqslant i \leqslant n$ and

$$a_{2,2i+2} = 0,$$

for all $1 \leqslant i \leqslant \frac{n-3}{2}$. If we denote by $\oplus$ the addition in $\mathbb{F}_2$ and by $+$ the addition in the integers, then we obtain

$$\begin{aligned}\deg(V_3) &= a_{1,3} + a_{2,3} + \sum_{j=4}^{n} a_{3,j} = (a_{1,2} \oplus a_{2,3} + a_{2,3}) + \sum_{j=2}^{\frac{n-3}{2}} (a_{3,2j+1} + a_{3,2j+2}) + 2a_{3,n} \\ &= 2a_{2,3} + \sum_{j=2}^{\frac{n-3}{2}} (a_{2,2j} \oplus a_{2,2j+1} + a_{2,2j+1} \oplus a_{2,2j+2}) + 2(a_{2,n-1} \oplus a_{2,n}) \\ &= 2\sum_{j=1}^{\frac{n-1}{2}} a_{2,2j+1} = 2(a_{1,2} + \sum_{j=3}^{n} a_{2,j}) = 2 \times \deg(V_2).\end{aligned}$$

This leads to $\deg(V_i) = 0$ for all $1 \leqslant i \leqslant n$ and thus G is the zero-edge graph on n vertices. □

Finally, the above computational result permits us to extend the verification of Conjecture 2 up to $n \leqslant 1500$ vertices. Indeed, as proved in the remark following Theorem 3.4, for a Steinhaus graph G on $2n$ vertices, if $G \setminus \{V_1, V_{2n}\}$ is the zero-edge graph on $2n-2$ vertices, then G is the star graph on $2n$ vertices which is not a regular graph. Therefore, by Theorem 3.4, we obtain the following theorem.

Theorem 5.4. *There is no regular Steinhaus graph of odd degree on* $2 < n \leqslant 1500$ *vertices.*

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