Construction of a general class of Dirichlet forms in terms of white noise analysis

Edouard A. Razafimanantena

Department of Mathematics, Kumamoto University, Kurokami 2-39-1, Kumamoto 860, Japan

Received 30 April 1990
Revised 18 October 1990

In the framework of white noise analysis a Gel'fand triple

\( (\mathcal{F}) \subset (L^2) \subset (\mathcal{F})^* \)

has been defined (e.g. Kubo and Yokoi, 1989), the space of smooth test functionals \( (\mathcal{F}) \) and the space of Hida distributions \( (\mathcal{F})^* \) play some important roles. It has been shown (e.g. Yokoi, 1990) that a positive Hida distribution \( \phi \) is given by a positive measure \( \nu_\phi \) on the space of real tempered distributions \( \mathcal{F}^* \). Thus the space \( (L^2)_\nu = L^2(\mathcal{F}; \mathcal{B}, \nu_\phi) \) can be defined, where \( \mathcal{B} \) is the Borel \( \sigma \)-algebra on \( \mathcal{F}^* \) generated by the weak topology.

The present article is concerned with a special choice of pre-Dirichlet forms with domain \( (\mathcal{F}) \) on \( (L^2)_\nu \) which is a generalization of the energy form (Hida, Potthoff and Streit, 1988) and of the type

\[
\varepsilon(F) = \left( \Phi; \sum_{i,k} H_{i,k} \cdot \delta_i \cdot \delta_k F \right),
\]

for each \( F \in \mathcal{F} \) and where \( (H_{i,k}; j, k \in \mathbb{N}_0) \) is a double sequence of test functionals satisfying some natural conditions. Some closability results are given in the last section under mild conditions.

0. Introduction

The general theory of Dirichlet forms with locally compact state spaces has its origin in classical work by Beurling and Deny (1958, 1959) and was considerably extended by Fukushima (1980) and Silverstein (1974). The infinite dimensional case (non-locally compact state space) is of particular interest for the development of infinite dimensional analysis and of quantum and stochastic models with infinitely many degrees of freedom.

Many authors have investigated the theory of Dirichlet forms over infinite dimensional spaces. In their works Albeverio and Röckner (1989, 1990) studied what, in their terminology, are called classical Dirichlet forms over infinite dimensional space \( E \). These are closed symmetric bilinear forms defined on a suitable domain of functions in \( L^2(E; \mu) \) of the type

\[
\varepsilon(u, v) = \int_E \langle \nabla u, \nabla v \rangle_H \, d\mu \tag{0.1}
\]
where \( \mu \) is a probability on \((E, \mathcal{B}(E))\), \( H \) is a Hilbert subspace of \( E \) and \( \nabla \) is a \( \mu \)-stochastic Gâteaux type derivative. In Albeverio and Röckner (1989) it is shown that if \( \varepsilon \) is closable over a certain core and if \( E \) is a Banach space, then there exists an \( E \)-valued diffusion process associated with \( \varepsilon \). Furthermore the analytical properties of the form \( \varepsilon \) provide a useful tool in understanding the probabilistic behavior of the diffusion.

Hida, Potthoff and Streit (1988) have defined energy forms on \( L^2(\mathcal{F}, \nu_\phi) \), for a given positive Hida distribution \( \Phi \), by

\[
\varepsilon(F, G) = \left\langle \Phi; \sum_j \partial_j F \cdot \overline{\partial_j G} \right\rangle, \quad F, G \in \mathcal{F}.
\]

(0.2)

Where \( \partial_j \) is the Gâteaux type derivative to the direction \( e_j \) and \( (e_j, j \in \mathbb{N}_0) \) is the complete orthonormal system on \( L^2(\mathbb{R}) \) (see (1.9)). Later some new approaches have been done (e.g. Albeverio, Hida, Potthoff, Röckner and Streit, 1990a, b).

In this paper we present a generalization of (0.2). Since the Gâteaux derivative of any test functional \( F \) to any direction \( x \) \((x \in \mathcal{F}^\ast)\) is a test functional and since the space of test functionals \( \mathcal{T} \) is an algebra it is quite natural that we define a pre-Dirichlet form \((\varepsilon_\phi, \mathcal{D}(\varepsilon_\phi))\) of the type

\[
\varepsilon_\phi(F, G) = \left\langle \Phi; \sum_{j,k} H_{j,k} \partial_j F \cdot \overline{\partial_k G} + H \cdot F \cdot \overline{G} \right\rangle
\]

(0.4)

where \( H \) and \( H_{j,k}, j \) and \( k \in \mathbb{N}_0 \), are test functionals, and for a given test functional \( F \) we shall denote its complex conjugate by \( \overline{F} \).

The organization of this paper is as follows. In Section 1 we do a quick review of the basic properties of our space of white noise. Section 2 defines the directional derivative. The definition of our class of Dirichlet forms and the main results are given in Section 3 and some closability conditions are given in Section 4.

**Remark.** (i) Since for the forms of the type

\[
\left\langle \Phi; H \cdot F \cdot \overline{G} \right\rangle
\]

all the Dirichlet form’s properties are easily verified we only consider the form

\[
\varepsilon_\phi(F, G) = \left\langle \Phi; \sum_{j,k} H_{j,k} \partial_j F \cdot \overline{\partial_k G} \right\rangle.
\]

(0.4')

(ii) The Markovian property of a pre-Dirichlet form of the type (0.4') is shown in (Razafimanantena, 1991).

1. White noise analysis

Let \( \mathcal{F} \) be the Schwartz space on \( \mathbb{R} \), and \( \mathcal{F}^\ast \) be the space of tempered distributions,
then we get the basic Gel'fand triplet
\[ \mathcal{F} \subset L^2(\mathbb{R}) \subset \mathcal{F}^*. \] (1.1)

Let \( \mu \) be the Gaussian measure of white noise on \( \mathcal{F}^* \) given by the characteristic functional
\[ \mathcal{C}(\xi) = \int_{\mathcal{F}^*} \exp(i(x; \xi)) \mu(dx) = \exp(-\frac{1}{2} \| \xi \|^2), \] (1.2)
\( \xi \in \mathcal{F}, \) where \( \| \cdot \| \) is the \( L^2(\mathbb{R}) \)-norm. Define the complex Hilbert space
\[ (L^2) = L^2(\mathcal{F}^*; d\mu). \] (1.3)

It is well known that \( (L^2) \) is isomorphic to the symmetric Fock space
\[ \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^n; n! d^n t) \] according to a transformation \( S, \) defined by
\[ SF(\xi) = \int_{\mathcal{F}^*} F(x + \xi) \mu(dx). \] (1.4)

Furthermore, an \( F \in (L^2) \) corresponds with an \( (F_n; n = 0, 1, \ldots) \in \bigoplus_n L^2(\mathbb{R}^n; n! d^n t) \) such that the transformation \( S \) is expressed in the form
\[ SF(\xi) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} F_n(t_1, \ldots, t_n) \xi(t_1) \cdots \xi(t_n) d^n t. \] (1.5)

Let us define the space of \textit{exponential functions} by
\[ \mathcal{A} = \text{span}\{ \exp(i(\cdot; \xi)): \xi \in \mathcal{F} \} \] (1.6)
and the space of \textit{polynomial functionals} by
\[ \mathcal{P} = \text{span}\{ P(\cdot; \xi_1, \ldots, \xi_n): n \in \mathbb{N}, \xi_1, \ldots, \xi_n \in \mathcal{F} \text{ and } P(t_1, \ldots, t_n) \text{ is a polynomial in real variables } t_1, \ldots, t_n \text{ with complex coefficients} \}. \] (1.7)

It is known that \( \mathcal{A} \) and \( \mathcal{P} \) are dense in \( (L^2) \) (Hida, 1980).

Let us introduce the Hamiltonian operator defined by
\[ A = -\frac{d^2}{dt^2} + t^2 + 1. \] (1.8)

Let \( (e_i)_{i=0,1,\ldots} \) be the complete orthonormal system (c.o.n.s.) in \( L^2(\mathbb{R}) \) given by
\[ e_j(u) = (2^j j! \sqrt{\pi})^{-1/2} H_j(u) \exp(-\frac{1}{2} u^2) \] (1.9)
where \( H_j(u) \) is the Hermite polynomial of degree \( j. \) Thus we get
\[ Ae_j = (2j + 2)e_j. \] (1.10)

By \( I'(A) \) we mean the \textit{second quantization} of \( A \) (Simon, 1974) and \( \| \cdot \|_2 \) is the \( (L^2) \)-norm, thus we can define a norm \( \| \cdot \|_{2,p} \) on \( \mathcal{A} \) by
\[ \| F \|_{2,p} = \| I(A^p) F \|_2. \] (1.11)
By \((\mathcal{S}_p)\) we mean the completion of \(\mathcal{A}\) according to the norm \(\|\cdot\|_{2,p}\) and \((\mathcal{S}_p)^*\) is the dual space of \((\mathcal{S}_p)\). Thus we get the chain
\[
\cdots \subset (\mathcal{S}_p)^* \subset (\mathcal{S}_{p+1})^* \subset \cdots \subset (L^2)^* \subset \cdots \subset (\mathcal{S}_{p+1}) \subset (\mathcal{S}_p) \subset \cdots
\] (1.12)

Define the space \((\mathcal{S})\) to be the projective limit of the chain (1.12) and \((\mathcal{S})^*\) to be its dual. It has been shown that \((\mathcal{S})\) is a nuclear Fréchet algebra (see Kubo and Yokoi, 1989). Thus we get a Gel'fand triplet
\[
(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*.
\] (1.13)

An element of \((\mathcal{S})\) is called a smooth white noise functional or a test functional, and an element of \((\mathcal{S})^*\) is a generalized white noise functional or Hida distribution.

**Lemma 1.1** (Kubo and Yokoi, 1989). \((\mathcal{S})\) is an algebra invariant under the complex conjugation such that
\[
\| \bar{F} \|_{2,p} = \| F \|_{2,p},
\] (1.14)

furthermore, there exists a constant \(K_1\) and a positive integer \(q\) so that
\[
\| F \cdot G \|_{2,p} \leq K_1 \| F \|_{2,p+q} \| G \|_{2,p+q}.
\] (1.15)

For each \(F, G \in (\mathcal{S})\) and \(p \geq 0\). □

By the notation \(:x^{\otimes i}:\), we mean the Wick product for elements of the space of tempered distributions \(\mathcal{S}^\#\) (Simon, 1974).

**Lemma 1.2** (Kubo and Yokoi, 1989). Any test functional \(F\) has a strongly continuous version \(\tilde{F}\) in \((L^2)\) given by
\[
\tilde{F}(x) = \sum_{j \in \mathbb{N}_0} \langle :x^{\otimes j}:; F_j \rangle
\] (1.16)

for some \(F_j \in \mathcal{S}(\mathbb{R}^j)\), the symmetric Schwartz space on \(\mathbb{R}^j\). Conversely, any \(F\) of the form (1.16) belongs to \((\mathcal{S})\) if and only if
\[
\sum_{j \in \mathbb{N}_0} j! \| (A^p)^{\otimes j} F_j \|^2 < \infty
\] (1.17)

for each \(p \in \mathbb{N}_0\). □

**Definition 1.1.** The set of positive test functionals is the cone
\[
(\mathcal{S})_+ = \{ F \in (\mathcal{S}); F(x) \geq 0 \text{ for any } x \in \mathcal{S}^* \}
\] (1.18)

and let us define the set of positive Hida distributions by
\[
(\mathcal{S})_+^* = \{ \Phi \in (\mathcal{S})^*; \langle \Phi; F \rangle \geq 0 \text{ for } F \in (\mathcal{S})_+ \}.
\] (1.19)
Theorem 1.1 (Yokoi, 1987). For any positive Hida distribution $\Phi$ there is a unique positive measure $\nu_{\Phi}$ on $\mathcal{F}^*$ representing $\Phi$ in such a way that

$$\langle \Phi; F \rangle = \int_{\mathcal{F}^*} F(x) \nu_{\Phi}(dx), \quad F \in (\mathcal{F}).$$

Thus we can define the complex Hilbert space $(L^2)_{\Phi}$ as $L^2(\mathcal{F}^*; d\nu_{\Phi})$. \hfill $\square$

Corollary 1.1. For any positive Hida distribution $\Phi$, let $\nu_{\Phi}$ be the unique positive measure on $\mathcal{F}^*$ representing $\Phi$. If $\text{supp}(\nu_{\Phi}) = \mathcal{F}^*$ then there is a one to one correspondence between $(\mathcal{F})$ and its classes in $(L^2)_{\Phi}$ and $(\mathcal{F})$ is dense in $(L^2)_{\Phi}$. \hfill $\square$

2. Directional derivatives

For any $F$ belongs to $(\mathcal{F})$ and $\xi$ belongs to $\mathcal{F}$ let us define by

$$\partial_\xi F = S^{-1}D_\xi SF$$

(2.1)

where $S$ is the transformation given by (1.4) and $D_\xi$ the Gâteaux type derivative of functional on $\mathcal{F}$. In addition another partial derivative is defined by

$$\frac{\partial F}{\partial \xi} = \lim_{s \to 0} s^{-1}(F(x+s\xi)-F(x))$$

(2.2)

if the limit of the right-hand side exists.

Lemma 2.1 (Albeverio et al., 1990b). The derivatives $\partial_\xi$ and $\partial/\partial \xi$ coincide on $\mathcal{F}$ and on $\mathcal{P}$ for any $\xi$ belonging to $\mathcal{F}$. \hfill $\square$

Lemma 2.2 (Hida, Potthoff and Streit, 1988). The restriction of the derivatives $\partial_\xi$ defines a continuous linear map on $(\mathcal{F})$ and there is an integer $q$ and there is a constant $K_2$ so that

$$\|\partial_\xi F\|_{2,p} \leq K_2\|\mathcal{A}^{-1}\xi\|_2\|F\|_{2,p+1}$$

(2.3)

for each $F \in (\mathcal{F})$, integer $p$ and $\xi \in \mathcal{F}$.

Furthermore, $\partial_\xi$ is well defined and obeys the chain rule for any differentiable $g \circ F$ and it obeys the product rule. \hfill $\square$

Let us denote by $\partial_j = \partial_{e_j}$ for $j \in \mathbb{N}_0$, where $e_j$ is given in (1.9).

Lemma 2.3 (Hida, Potthoff and Streit, 1988). An operator $\nabla$ on $(L^2)_{\Phi}$ into $(L^2)_{\Phi} \otimes \ell^2$ can be introduced by

$$\nabla F = (\partial_j F; \ j \in \mathbb{N}_0)$$

(2.4)

for each $F \in (\mathcal{F})$. \hfill $\square$
Remark. By some results in Potthoff and Jia-an (1989) the above operator $\mathcal{V}$ is the same as the operator $\mathcal{V}$ given in Albeverio and Röckner (1988a), written in its system of coordinates given by the c.o.n.s. $\{e_j; j \in \mathbb{N}_0\}$ in $L^2(\mathbb{R})$. Furthermore one can see (e.g. Potthoff and Röckner, 1989) that the form $(\mathcal{E}, (\mathcal{F}))$ given by

$$\mathcal{E}(F; G) = \int \nabla F \cdot \nabla \overline{G} \, d\mu,$$

is the minimal classical pre-Dirichlet form defined in Albeverio and Röckner (1988).

3. The forms

In this section we will define a form generated by a positive Hida distribution $\Phi$ and double sequence $(H_{j,k}; j, k \in \mathbb{N}_0)$ of test functionals satisfying the following three conditions:

(C1) $H_{j,k} = \overline{H}_{k,j}$, \quad $k, j \in \mathbb{N}_0$. \quad (3.1)

(C2) For each $p$ belongs to $\mathbb{N}_0$, there exists a positive constant $M_p$ such that

$$\tau_{k,p} \equiv \sum_j \| H_{j,k} \|_{2,p} \leq M_p, \quad k \in \mathbb{N}_0. \quad (3.2)$$

(C3) For each $x \in \mathcal{F}^*$ and a sequence of complex numbers $(\alpha_j; j = 0, 1, \ldots)$,

$$\sum_{j,k=0}^n H_{j,k}(x) \alpha_j \overline{\alpha}_k \geq 0. \quad (3.3)$$

Thus we can get

$$\left\| \sum_j \frac{H_{j,k} \cdot F_j}{\tau_{k,p+q}} \right\|_{2,p} \leq \sum_j \frac{\| H_{j,k} \|_{2,p+q}}{\tau_{k,p+q}} \| F_j \|_{2,p+q},$$

by (1.15). Since

$$\sum_j \frac{\| H_{j,k} \|_{2,p+q}}{\tau_{k,p+q}} = 1,$$

the right-hand side of the last inequality is a weighted average of the $\| F_j \|_{2,p+q}$, and since $t^2$ is a convex function of $t$, the second power of the right member does not exceed the weighted average of the $\| F_j \|_{2,p+q}^2$ with the same weights (Hardy, Littlewood and Polya, 1952), thus we get the following evaluations:

$$\left\| \sum_j \frac{H_{j,k} \cdot F_j}{\tau_{k,p+q}} \right\|_{2,p}^2 \leq K_1^2 \sum_j (\tau_{k,p+q})^{-1} \| H_{j,k} \|_{2,p+q} \| F_j \|_{2,p+q}^2,$$

$$\left\| \sum_j H_{j,k} \cdot F_j \right\|_{2,p}^2 \leq K_1^2 \sum_j \| H_{j,k} \|_{2,p+q} \| F_j \|_{2,p+q}^2 \leq K_1^2 M_{p+q} \sum_j \| H_{j,k} \|_{2,p+q} \| F_j \|_{2,p+q}^2$$
and
\[ \sum_k \left( \sum_j H_{j,k} \cdot F_j \right)_{2,p}^2 \leq K^2 \sum_j \left\| H_{j,k} \right\|_{2,p+q} \left\| F_j \right\|_{2,p+q}^2, \]
thus by \((C_3)\) we get the following:
\[ \sum_k \left( \sum_j H_{j,k} \cdot F_j \right)_{2,p}^2 \leq K^2 \sum_j F_j_{2,p+q}^2. \] (3.4)

Let us define the set \((\mathcal{F}) \otimes \ell^2\) by
\[ \mathcal{F} \otimes \ell^2 = \left\{ (F_j; j \in \mathbb{N}_0); F_j \in (\mathcal{F}) \text{ for each } j \in \mathbb{N}_0, \text{ and } \sum_j \| F_j \|_{2,p}^2 < \infty \text{ for any } p \geq 0 \right\}. \] (3.5)

It is easy to see that \((\mathcal{F}) \otimes \ell^2\) is densely imbedded in \((L^2)_{\phi} \otimes \ell^2\).

**Lemma 3.1.** For given \((F_j, j \in \mathbb{N}_0)\) and \((G_j, j \in \mathbb{N}_0)\) of elements of \((\mathcal{F}) \otimes \ell^2\) the \((L^2)\)-limit of \(\sum_j F_j \cdot G_j\) is a test functional.

**Proof.** This is easy to check by the definition of \((\mathcal{F})\) and the Schwarz inequality. \(\square\)

**Lemma 3.2.** Let us define a transformation \(T\) on \((L^2)_{\phi} \otimes \ell^2\) by
\[ T(F_j; j \in \mathbb{N}_0) = \left( \sum_j H_{j,k} \cdot F_j; k \in \mathbb{N}_0 \right). \] (3.6)

Then \(T\) is well defined and symmetric on \((\mathcal{F}) \otimes \ell^2\). Furthermore \(T\) leaves \((\mathcal{F}) \otimes \ell^2\) invariant and for any \(\mathcal{F} = (F_j; j \in \mathbb{N}_0) \in (\mathcal{F}) \otimes \ell^2\) we get
\[ \| T\mathcal{F} \|_{(L^2)_{\phi} \otimes \ell^2}^2 \leq K_1 \| \Phi \|_{2,-p_0} M^2_{2,p_0+2q} \sum_j \| F_j \|_{2,p_0+2q}^2 \] (3.7)
where \(p_0\) is the degree of \(\Phi\) and, \(K_1\) and \(q\) are as in Lemma 1.1.

**Remark.** We mean by a degree of a Hida distribution \(\Phi\) the least positive integer \(p\) such that \(\Phi \in (\mathcal{F})_{-p}\).

**Proof of Lemma 3.2.** From Lemma 1.1 we get
\[ \left\| \sum_j H_{j,k} \cdot F_j \right\|_{2,p} \leq \sum_j \left\| H_{j,k} \cdot F_j \right\|_{2,p} \leq K_1 \sum_j \left\| H_{j,k} \right\|_{2,p+q} \left\| F_j \right\|_{2,p+q}. \]

But the last estimation is finite by the Schwarz inequality and \((C_3)\), thus \(\sum_j H_{j,k} \cdot F_j\) belongs to \((\mathcal{F})\) for each \(k \in \mathbb{N}_0\). The inequality (3.4) gives that \(\sum_k \left\| \sum_j H_{j,k} \cdot F_j \right\|_{2,p}^2\) is finite, and obviously \(T\) is symmetric.
Now let us prove that $T$ is well defined, in fact

$$\|T\Phi\|_{(L^2)_{\Phi} \otimes \ell^2}^2 = \sum_k \|\sum_j H_{j,k} \cdot F_j\|_{(L^2)_{\Phi}}^2,$$

moreover

$$\left\| \sum_j H_{j,k} \cdot F_j \right\|_{(L^2)_{\Phi}}^2 = \left\langle \Phi; \left\| \sum_j H_{j,k} \cdot F_j \right\|_{(L^2)_{\Phi}}^2 \right\rangle \leq \|\Phi\|_{2,-p_0} \|\sum_j H_{j,k} \cdot F_j\|_{2,p_0}^2,$$

for some $p_0 \in \mathbb{N}_0$. Thus by Lemma 1.1,

$$\left\| \sum_j H_{j,k} \cdot F_j \right\|_{(L^2)_{\Phi}}^2 \leq K_1 \left\|\Phi\right\|_{2,-p_0} \left\| \sum_j H_{j,k} \cdot F_j \right\|_{2,p_0+q},$$

and using (3.4) we get

$$\|T\Phi\|_{(L^2)_{\Phi} \otimes \ell^2} \leq K_1 M^{2p_0+2q} \|\Phi\|_{2,-p_0} \sum_j \|F_j\|_{2,p_0+2q}^2 < \infty. \quad \square$$

From now on we assume that for a given positive Hida distribution $\Phi$ (as usual $\nu_{\Phi}$ denotes the positive measure representing $\Phi$) $\text{supp}(\nu_{\Phi}) = \mathcal{F}^*$, and it is easy to see that $(\partial_j F; j \in \mathbb{N}_0)$ belongs to $(\mathcal{F}) \otimes \ell^2$ for any $F \in (\mathcal{F})$. Thus we get the following proposition.

**Proposition 3.1.** For a given double sequence $(H_{j,k}; j, k \in \mathbb{N}_0)$ of test functionals satisfying $(C_1)$, $(C_2)$ and $(C_3)$, and for a given positive Hida distribution $\Phi$ we can define a densely defined positive definite Hermitian form $\varepsilon_{\Phi}$ with domain $\mathcal{D}(\varepsilon_{\Phi})$ in $(L^2)_{\Phi}$ as follows:

$$\mathcal{D}(\varepsilon_{\Phi}) = (\mathcal{F}),$$

$$\varepsilon_{\Phi}(F; G) = \int_{\mathcal{F}^*} \sum_{j,k} H_{j,k}(x) \cdot \partial_j F(x) \cdot \partial_k \tilde{G}(x) \nu_{\Phi}(dx)$$

$$= (TVF; \nabla \tilde{G})(L^2)_{\otimes \ell^2}$$

$$= \left\langle \Phi; \sum_{j,k} H_{j,k} \cdot \partial_j F \cdot \partial_k \tilde{G} \right\rangle.$$

**Proof.** The following estimate gives us the convergence of the double series:

$$\left\| \sum_{j,k} H_{j,k} \cdot \partial_j F \cdot \partial_k \tilde{G} \right\|_{2,p} \leq \sum_{j,k} \|H_{j,k} \cdot \partial_j F \cdot \partial_k \tilde{G}\|_{2,p}$$

$$\leq \sum_{j,k} K \|H_{j,k} \cdot \partial_j F\|_{2,p+q} \|\partial_k \tilde{G}\|_{2,p+q}.$$
In fact the last inequality follows from Lemma 2.2, Lemma 3.1 and the Schwarz inequality. Thus $\sum_{j,k} H_{j,k} \cdot \partial_j \bar{F} \cdot \partial_k G$ belongs to $(\mathcal{F})$ and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is well defined. The form is symmetric from $(C_1)$, and it is positive definite by $(C_2)$ since $\Phi$ is a positive Hida distribution. \( \square \)

**Example 3.1.** Let us take the c.o.n.s. given by (1.9) and define a double sequence $(H_{j,k}; j$ and $k \in \mathbb{N}_0)$ of test functionals by

$$H_{j,k} = \frac{1}{j!k!} \langle \cdot; e_j \rangle \cdot \langle \cdot; e_k \rangle.$$  \hspace{1cm} (3.11)

It is obvious that assumption $(C_1)$ and $(C_3)$ are satisfied. The following argument derives condition $(C_2)$:

For a test functional $F$ given by

$$F(x) = \langle x; \xi \rangle \quad \text{for a } \xi \in \mathcal{F},$$

we get

$$\Gamma(A^p) F = \langle \cdot; A^p \xi \rangle.$$  

Thus

$$\| F \|_{2,p} = \| A^p \xi \|_2, \quad p \in \mathbb{N}_0.$$  

Using this result, we get

$$\| \langle \cdot; e_j \rangle \|_{2,p} = (2j+2)^p,$$

now condition $(C_2)$ is proved by Lemma 4.3 in Section 4.

From now on let us fix the measure $\nu_{\Phi}$ which comes from a positive Hida distribution and the double sequence $(H_{j,k}; j, k \in \mathbb{N}_0)$ of test functionals satisfying assumptions $(C_1)$, $(C_2)$ and $(C_3)$. Suppose further more that $\text{supp}(\nu_{\Phi}) = \mathcal{F}^*$ and let us consider the associate form defined by (3.8) and (3.9). The following theorem is well known (see e.g. Weidmann, 1980, Theorem 5.32).

**Theorem 3.1.** Let $S$ be a symmetric operator on the (real or complex) Hilbert space $\mathcal{H}$, and assume that $(f; Sf)_{\mathcal{H}} \geq \gamma \|f\|_{\mathcal{H}}^2$ with some $\gamma \in \mathbb{R}$ for all $f \in \mathcal{D}(S)$, then for each $k \in (-\infty, \gamma)$ there exists a self-adjoint extension $T_k$ of $S$ such that $(f; T_k f)_{\mathcal{H}} \geq k \|f\|_{\mathcal{H}}^2$ for each $f \in \mathcal{D}(T_k)$. \( \square \)

**Proposition 3.2.** Our form is closable if for any $F \in (\mathcal{F})$, $T \nabla F \in \mathcal{D}(\nabla^*)$.

**Proof.** The proof is just a standard application of Theorem 3.1. \( \square \)
4. Closability results

Let us assume from now on that our double sequence of test functionals satisfies the following condition (C') stronger than (C2):

\[(C') \quad \text{For each } p \in \mathbb{N}_0, \text{ there exists a positive constant } M'_p \text{ so that} \]
\[\tau_{k,p} = \sum_j j! \|H_{j,k}\|_{2,p} \leq M'_p \quad \text{for } k \in \mathbb{N}_0. \tag{4.1}\]

**Lemma 4.1.** For any \((F_k; k \in \mathbb{N}_0) \in (\mathcal{S}) \otimes \ell^2\),
\[\sum_{k \in \mathbb{N}_0} \partial_k F_k \in (\mathcal{S}).\]

**Proof.** We get the result by the estimate
\[\left\| \sum_{k} \partial_k F_k \right\|_{2,p} \leq \sum_{k} \| \partial_k F_k \|_{2,p} \]
\[\leq \sum_k K_2 \|A^{-1} e_k\| \|F_k\|_{2,p+1} \]
\[= K_2 \sum_k (2k+1)^{\frac{1}{2}} \|F_k\|_{2,p+1} \]
\[< \infty. \] 

**Lemma 4.2.** For any \((F_k; k \in \mathbb{N}_0) \in (\mathcal{S}) \otimes \ell^2\), the test functional \(\sum_{k \in \mathbb{N}_0} (k!)^{-1} \langle \cdot ; e_k \rangle \cdot F_k\) is well defined.

**Proof.**
\[\left\| \sum_{k} (k!)^{-1} \langle \cdot ; e_k \rangle \cdot F_k \right\|_{2,p} \]
\[\leq \sum_{k} (k!)^{-1} \| \langle \cdot ; e_k \rangle \cdot F_k \|_{2,p} \]
\[\leq K_1 \sum_{k} (k!)^{-1} \| \langle \cdot ; e_k \rangle \|_{2,p+q} \|F_k\|_{2,p+q} \]
\[\leq K_1 \sum_{k} (k!)^{-1}(2k+2)^{p+q} \|F_k\|_{2,p+q} \]
\[< \infty. \]

**Proposition 4.1.** If the positive Hida distribution \(\Phi\) satisfies
\[\partial_k \Phi = B_k \Phi, \tag{4.2}\]
for each \(k \in \mathbb{N}_0\), for some \(B_k \in (S)\) and
\[\sum_{k} (k!)^{-1} \|B_k\|_{2,p} < \infty, \quad p \in \mathbb{N}_0, \tag{4.3}\]
then the corresponding form \(\varepsilon_{\Phi}\) is closable. Here we mean by \(\partial_k \Phi\) the distributional derivative defined by \(\langle \partial_k \Phi ; F \rangle = \langle \Phi ; \partial_k^* F \rangle\) for each \(F \in (\mathcal{S})\) and \(\partial_k^*\) is the \((L^2)\)-adjoint of \(\partial_k\).
Proof. Let $\mathcal{G} = (G_k, k \in \mathbb{N}_0) \in (\mathcal{F}) \otimes \ell^2$, then $\nabla^* \mathcal{G}$ acts as
\begin{equation}
\nabla^* \mathcal{G} = \sum_k (-\partial_k + x_k - B_k)G_k,
\end{equation}
where $x_k$ is the multiplication operator
\begin{equation}
x_kF(\cdot) \equiv \langle \cdot; e_k \rangle \cdot F(\cdot).
\end{equation}

By Lemma 4.1, $\sum_k -\partial_k G_k \in (\mathcal{F})$. Furthermore condition $(C')$ gives us that $(\sum_j j! k! H_{j,k} \cdot \partial_j F; k \in \mathbb{N}_0) \in (\mathcal{F}) \otimes \ell^2$ for each $F \in (\mathcal{F})$, thus
\begin{align*}
\left\| \sum_k (x_k - B_k) \left( \sum_j H_{j,k} \cdot \partial_j F \right) \right\|_{2,p} \\
\leq \left( K_1 \sum_k \| \langle \cdot; e_k \rangle \|_{2,p+q} \left\| \sum_j H_{j,k} \cdot \partial_j F \right\|_{2,p+q} \right) \\
+ \left( K_1 \sum_k \| B_k \|_{2,p+q} \left\| \sum_j H_{j,k} \cdot \partial_j F \right\|_{2,p+q} \right)
\end{align*}
and this estimation is finite by Lemma 4.2, assumption $(C')$ and (4.3). Now the closability is just a result of Proposition 3.2.

Example 4.1. Consider any strictly positive self-adjoint operator $\kappa$ on $L^2(\mathbb{R})$ such that $\mathcal{D}(\kappa) \supset \mathcal{F}$ and $\kappa$ leaves $\mathcal{F}$ invariant. Furthermore, suppose that the operator $\hat{\kappa} = \kappa / (\kappa + 1)$ is strictly positive and obeys
\begin{equation}
(\hat{\kappa} \Phi \langle \cdot; \xi \rangle) = \exp(-\frac{1}{2}(\xi; \hat{\kappa}/\xi))
\end{equation}
defines a positive Hida distribution $\Phi$ of order $p_0$ and
\begin{equation}
\partial_k \Phi = \langle \cdot; \kappa e_k \rangle \Phi,
\end{equation}
thus in this case (4.3) is equivalent to
\begin{equation}
\sum_k (k!)^{-1} \| A^p \kappa e_k \| < \infty, \quad p \in \mathbb{N}_0.
\end{equation}

Now let us do some investigation for a special form of double sequence $(H_{j,k}; j, k \in \mathbb{N}_0)$ satisfying the three conditions $(C_1)$, $(C')$ and $(C_2)$. The following lemma states our idea.

Lemma 4.3. For a given sequence $(H_j; j \in \mathbb{N}_0)$ of test functionals which satisfies the following property:
\begin{equation}
\sum_j \| H_j \|_{2,p} < +\infty \quad \text{for any } p \in \mathbb{N}_0
\end{equation}
we can define a double sequence \((H_{j,k}; j, k \in \mathbb{N}_0)\) of test functionals which satisfies the conditions \((C_1), (C_2)\) and \((C_3)\) by

\[
H_{j,k} = \frac{1}{j!k!} H_j \cdot \tilde{H}_k. \quad (4.11)
\]

Now let us define an operator \(T'\) onto \((L^2) \otimes \ell^2\) into \((L^2)\) by

\[
T'(\mathcal{F}) = \left( \sum_j H_j \cdot F_j \right), \quad \mathcal{F} = (F_j; j \in \mathbb{N}_0) \in \mathcal{S} \otimes \ell^2. \quad (4.12)
\]

It is easy to see that \(T'\) transforms \((\mathcal{S}) \otimes \ell^2\) into \(\mathcal{S}\) and in this case our energy form can be written in the form

\[
\varepsilon_{\Phi}(F, G) = \left\langle \Phi; \sum_{j,k} H_{j,k} \cdot \partial_j F \cdot \partial_k \tilde{G} \right\rangle = (T'\nabla F, T'\nabla G)_{(L^2)_{\Phi}}. \quad (4.13)
\]

From the general theory we have the following theorem.

**Theorem 4.1.** The energy form \(\varepsilon_{\Phi}\) is closable if and only if the adjoint operator \((T'\nabla)^*\) of \(T'\nabla\) is densely defined as an operator on \((L^2)_{\Phi}\). \(\square\)

In order to describe a convenient closability result in the case that \(\nu_{\Phi}\) is absolutely continuous with respect to \(\mu\) with density \(\Phi\), we introduce the spaces \((L^{p,q})\), \(p = 2, 3, \ldots\), and \(q \in \mathbb{N}_0\), which are the completions of the algebra \(\mathcal{A}\), defined in (1.6), under the norms

\[
\|F\|_{p,q} = \|(1 + H)^q F\|_p \quad (4.14)
\]

where \(-H\) is the Laplace–Beltrami operator given by

\[
H = \sum_k \partial_k^x \partial_k^y. \quad (4.15)
\]

Note that \((\mathcal{S}) \subset (L^{p,q}) \subset (L^2)\) for all \(p = 2, 3, \ldots, q \in \mathbb{N}_0\) and the embeddings are dense and continuous.

**Lemma 4.4** (Hida, Potthoff and Streit, 1988). Assume that \(\Phi > 0\) \((\mu\text{-a.e.) and } \Phi^{1/2} \in (L^{4,1}),\) then

\[
\partial_k \Phi^{1/2} = \frac{1}{2} \Phi^{-1/2} \partial_k \Phi \quad (4.16)
\]

for all \(k \in \mathbb{N}_0\) in \((L^2)\)-sense. \(\square\)

**Proposition 4.2.** Suppose that \(\nu_{\Phi}\) is absolutely continuous with respect to \(\mu\) with density \(\Phi\) and that \(\Phi > 0\) \(\mu\text{-a.e. and } \Phi^{1/2} \in (L^{4,1}),\) Suppose furthermore that a given sequence \((H_j; j \in \mathbb{N}_0)\) of test functionals satisfies (4.10) and there is a constant \(C\) so that \(H_j \leq C\) \(\mu\text{-a.e. for all } j \in \mathbb{N}_0\). Then the form given by (4.13) is closable.
Proof. For any $F \in \mathcal{A}$,
\[
(T' \nabla)^* F = \sum_k (-\partial_k + x_k - (\Phi^{-1} \partial_k \Phi)) H_k \cdot F.
\]
By previous computations we easily get that $\sum_k (-\partial_k + x_k) H_k \cdot F \in (\mathcal{F})$. Thus it remains to show that
\[
\sum_k \int \mu(dx) \Phi(x)(\Phi^{-1} \partial_k \Phi H_k \cdot F)^2 < \infty,
\]
the left hand side is evaluated as follows:
\[
\sum_k \int \mu(dx) \Phi(x)(\Phi^{-1} \partial_k \Phi)^2 (\Phi_k \cdot F)^2
\leq \sum_k \int \mu(dx) \Phi(x)(\Phi^{-1} \partial_k \Phi)^2 C \|F\|_\infty
\]
\[
= C \|F\|_\infty 4 \sum_k \mu(dx) |\partial_k \Phi^{1/2}|
\]
\[
= 4C \|F\|_\infty \|H^{1/2} \Phi^{1/2}\| < \infty \quad \text{(by Lemma 4.4)}.
\]
Thus by Theorem 4.1 the closability is obtained. \qed

Acknowledgement

I have profited very much by helpful suggestions and discussions with Professor T. Hida and Professor L. Streit at Nagoya University in October 1989 where the starting point of this investigation took place I enjoyed fruitful discussions with Professor S. Albeverio, Professor J. Potthoff and Professor M. Röckner in March 1990. I wish to thank all members of the probability seminar at Kumamoto University for stimulating works. Hearty thanks to Professor M. Hitsuda who introduced me to Hida calculus and to Professor Y. Oshima who gave me much information on Dirichlet spaces.

References

M. Fukushima, Dirichlet Forms and Markov Processes (Kondasha and North Holland, Amsterdam, 1980).
T. Hida, Brownian Motion (Springer, Berlin, 1980).
J. Potthoff and M. Röckner, On the contraction property of energy forms on infinite dimensional space, Preprint Nr 369/89, BiBoS Univ. Bielefeld (Bielefeld, 1989).
E.A. Razafimananterena, Contraction property for a class of Dirichlet forms via white noise analysis, Preprint, Kumamoto Univ. (Kumamoto, 1991).