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Bousfield localization on formal schemes

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Abstract

Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be a noetherian formal scheme and consider $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ its derived category of sheaves with quasi-coherent torsion homology. We show that there is a bijection between the set of rigid (i.e., $\otimes$-ideals) localizing subcategories of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ and subsets in $\mathfrak{X}$, generalizing previous work by Neeman. If, moreover, $\mathfrak{X}$ is separated, the associated localization and acyclization functors are described in certain cases. When $Z \subset \mathfrak{X}$ is a stable for specialization subset, its associated acyclization is $\mathbf{R}\Gamma_Z$. When X is a scheme, the corresponding localizing subcategories are generated by perfect complexes and we recover Thomason's classification of thick subcategories. On the other hand, if $Y \subset \mathfrak{X}$ is generically stable, we show that the associated localization functor is $\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathbf{R}\Gamma_{\mathfrak{X}\setminus Y}\mathcal{O}'_{\mathfrak{X}}, \mathcal{G})$.

Introduction

The techniques of localization have a long tradition in several areas of mathematics. They have the virtue of concentrating our attention on some part of the structure in sight

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allowing us to handle more manageable pieces of information. One of the clear examples of this technique is the localization in algebra where one studies a module centering the attention around a point of the spectrum of the base ring, i.e., a prime ideal. The idea was transported to topology by Adams and later Bousfield proved that there are plenty of localizations in stable homotopy. In the past decade it became clear that one could successfully transpose homotopy techniques to the study of derived categories (over rings and schemes). In particular, in our previous work, we have shown that for the derived category of a Grothendieck category we also have plenty of localizations. In that paper, [4], we applied the result to the existence of unbounded resolutions and we hinted that, in the case of the derived category of quasi-coherent sheaves over a nice scheme, there should be a connection between localizations in the derived category and the geometric structure of the underlying space.

The present paper realizes that goal extending the work of Neeman [14, Theorem 3.3], who classified all Bousfield localizations in the derived category of modules over a noetherian ring $\mathbf{D}(R)$, to the classification of the Bousfield localizations of the derived category of sheaves with quasi-coherent *torsion* homology over a noetherian formal scheme (Theorem 4.12). This category is a basic ingredient in Grothendieck duality [2]. Also, if the formal scheme is just an *usual* noetherian scheme, it gives the derived category of sheaves with quasi-coherent homology. Thus we obtain an analog of the *chromatic tower* in stable homotopy for these kind of schemes and formal schemes. It is clear that the monoidal structure of the derived category is an essential part of the cohomological formalism. In fact, to get the classification, we were forced to consider only *rigid* localizing subcategories. This means, roughly speaking, that the localizing subcategory is an ideal in the monoidal sense (see Section 3). This condition is needed in order to have compatibility with open sets. It holds for all localizing subcategories in the affine case, that is why it was not considered by Neeman.

The classification theorem is more useful if the localization functor associated to a subset of the formal scheme can be expressed in geometrically meaningful terms with respect to this subset. This can be done for noetherian separated formal schemes under certain conditions over the subset. The most rich case is the case of stable for specialization subsets (that recover the classical system of supports). They provide localizations that have the property of being compatible with the tensor product. They are also characterized by being associated to a right-derived functor and they correspond to the smashing localizations of topologists. All of this is contained in Theorem 5.3. These kind of localizations correspond to Lipman's notion of idempotent pairs [13]. The associated localizing subcategory is characterized in terms of homological support (Theorem 5.6). With this tool at hand we see that our classification of tensor triangulated categories (or smashing localizations) agrees with Thomason classification of thick $\otimes$-subcategories of the derived category quasi-coherent sheaves [18], when both make sense, i.e., for a noetherian separated scheme.

The dual notion of *tensor compatible* is that of *Hom compatible* localization. They correspond to stable for generalization subsets, which are complementary of stable for specialization subsets. The Hom compatible localizations can be described via a certain formal duality relation with the tensor compatible localization associated to its complementary subsets (Theorem 5.14). If the stable for generalization subset is an open

set, the localization functor agrees with the left-derived of a completion. This relates the results of [1] to this circle of ideas.

While our work does not exhaust all the possible questions about these topics, we believe that it can be useful for the current program of extracting information on a space looking at its derived category.

Now, let us describe briefly the contents of the paper. The first section recalls the concepts and notations used throughout and we give a detailed overview of the symmetric closed structure in the derived categories we are going to consider. In the next section, we specify the relationship between cohomology with supports and the algebraic version defined in terms of ext sheaves. We make a detailed study of the cohomology with respect to a system of supports in the case of a formal scheme and interpret the classical results in terms of Bousfield localization. In the third section we discuss the basic properties of *rigid localizing subcategories* and give a counterexample of a non-rigid localizing subcategory generated by a set. In Section 4 we state and prove the classification theorem, the rigid localizing subcategories in the derived category of quasi-coherent torsion sheaves on a noetherian formal scheme $\mathfrak{X}$ are in one-to-one correspondence with the subsets in the underlying space of $\mathfrak{X}$. The arguments are close in spirit to [14], with the modifications needed to make them work in the present context. In the last section we give a description of the acyclization functor associated to a stable for specialization subset as the derived functor of the sections with support and connect it to smashing localizations and to Lipman's idempotent pair. We characterize the localizing subcategory associated to such a subset by means of homological support. This result gives us a comparison of Thomason classification and ours for a noetherian separated scheme. Finally, by adjointness, we obtain also a description of the localization functor associated to generically stable subsets.

The question of describing localizations for subsets that are neither stable for specialization nor generically stable remains open for the moment.

1. Basic facts and set-up

1.1. Preliminaries

For formal schemes, we will follow the terminology of [7, Section 10] and of [2]. In this paper, we will always consider noetherian schemes and noetherian formal schemes.

Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be a noetherian formal scheme and let $\mathcal{I}$ be an ideal of definition of $\mathfrak{X}$. In what follows, we will identify an usual (noetherian) scheme with a formal scheme whose ideal of definition is 0. Denote by $\mathcal{A}(\mathfrak{X})$ the category of all $\mathcal{O}_{\mathfrak{X}}$-modules. The powers of $\mathcal{I}$ define a torsion class (see [17, pp. 139–141]) whose associated torsion functor is

$$\Gamma_{\mathcal{I}}\mathcal{F} := \varinjlim_{n>0} \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}\left(\mathcal{O}_{\mathfrak{X}}/\mathcal{I}^n, \mathcal{F}\right)$$

for $\mathcal{F} \in \mathcal{A}(\mathfrak{X})$. This functor does not depend on $\mathcal{I}$ but on the topology it determines in the rings of sections $\mathcal{O}_{\mathfrak{X}}$, therefore we will denote it by $\Gamma'_{\mathfrak{X}}$. Let $\mathcal{A}_{\mathrm{t}}(\mathfrak{X})$ be the full subcategory of $\mathcal{A}(\mathfrak{X})$ consisting of sheaves $\mathcal{F}$ such that $\Gamma'_{\mathfrak{X}}\mathcal{F} = \mathcal{F}$; it is a *plump* subcategory of $\mathcal{A}(\mathfrak{X})$.

This means it is closed for kernels, cokernels, and extensions (cf. [2, beginning of §1]). Most important for us is the subcategory $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}) := \mathcal{A}_{\mathrm{t}}(\mathfrak{X}) \cap \mathcal{A}_{\mathrm{qc}}(\mathfrak{X})$. It is again a plump subcategory of $\mathcal{A}(\mathfrak{X})$ by [2, Corollary 5.1.3] and it defines a triangulated subcategory of $\mathbf{D}(\mathfrak{X}) := \mathbf{D}(\mathcal{A}(\mathfrak{X}))$, the derived category of $\mathcal{A}(\mathfrak{X})$, it is $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, the full subcategory of $\mathbf{D}(\mathfrak{X})$ formed by complexes whose homology lies in $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$. If $\mathfrak{X} = X$ is an usual scheme, then $\mathcal{A}_{\mathrm{t}}(X) = \mathcal{A}(X)$ and $\mathcal{A}_{\mathrm{qct}}(X) = \mathcal{A}_{\mathrm{qc}}(X)$.

The inclusion functor $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}) \to \mathcal{A}(\mathfrak{X})$ has a right adjoint denoted $Q^{\mathrm{t}}_{\mathfrak{X}}$ (see [2, Corollary 5.1.5]). By the existence of K-injective resolutions ([16, Theorem 4.5] or [4, Theorem 5.4]) it is possible to get right-derived functors from functors with source a category of sheaves, as a consequence we have a functor $\mathbf{R}Q^{\mathrm{t}}_{\mathfrak{X}} : \mathbf{D}(\mathfrak{X}) \to \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$. If $\mathfrak{X}$ is either separated or of finite Krull dimension, this functor induces an equivalence between $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ and $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ by [2, Proposition 5.3.1]. In these cases, we will identify $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ and $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$. To avoid potential confusions, let us point out that all left- and right-derived functors defined over $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, or over $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ when this category is equivalent to the former, are defined using K-flat and K-injective resolutions in $\mathbf{K}(\mathfrak{X})$.

The categories $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$ and $\mathcal{A}(\mathfrak{X})$ are Grothendieck categories so we can apply the machinery developed in [4]. In particular, if $\mathcal{L}$ is the smallest localizing subcategory of $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ or of $\mathbf{D}(\mathfrak{X})$ that contains a given set, then there is a localization functor ℓ such that $\mathcal{L}$ is the full subcategory of $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ or of $\mathbf{D}(\mathfrak{X})$, respectively, whose objects are sent to 0 by ℓ (see [4, Theorem 5.7]). The category $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ is a localizing subcategory of $\mathbf{D}(\mathfrak{X})$, therefore if $\mathcal{L}$ is the smallest localizing subcategory of $\mathbf{D}(\mathfrak{X})$ that contains a given set of objects in $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, the localization functor ℓ defined over $\mathbf{D}(\mathfrak{X})$ lands inside $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, therefore $\mathcal{L}$ is characterized again as the full subcategory of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ whose objects are sent to 0 by ℓ. If $\mathfrak{X}$ is either separated or of finite Krull dimension, the localizations of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ are identified with those of $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$. For the general formalism of Bousfield localization in triangulated categories the reader may consult [4, §1].

1.2. Monoidal structures

The categories $\mathcal{A}(\mathfrak{X})$ and $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$ are *symmetric closed*, in the sense of Eilenberg and Kelly, see [9]. For every $\mathcal{F} \in \mathbf{K}(\mathcal{A}(\mathfrak{X}))$ there is a K-flat resolution $\mathcal{P}_{\mathcal{F}} \to \mathcal{F}$, this follows from [16, Proposition 5.6]. As a consequence, there exists a derived functor

$$\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} - : \mathbf{D}(\mathfrak{X}) \to \mathbf{D}(\mathfrak{X})$$

defined by $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} = \mathcal{P}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G}$. Also the functor $\mathcal{H}om^{\cdot}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, -)$ has a right-derived functor defined by $\mathbf{R}\mathcal{H}om^{\cdot}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G}) = \mathcal{H}om^{\cdot}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{J}_{\mathcal{G}})$ where $\mathcal{G} \to \mathcal{J}_{\mathcal{G}}$ denotes a K-injective resolution of $\mathcal{G}$. The usual relations hold providing $\mathbf{D}(\mathfrak{X})$ with the structure of symmetric closed category. Observe that the unit object is $\mathcal{O}_{\mathfrak{X}}$.

Given $\mathcal{F}, \mathcal{G} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, the complex $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G}$ has quasi-coherent torsion homology. Indeed, it is a local question, and for affine noetherian formal schemes, a complex in $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ is quasi-isomorphic to a complex made by locally free sheaves so the homology of $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G}$ is quasi-coherent. Furthermore, for any $\mathcal{F} \in \mathbf{D}_{\mathrm{t}}(\mathfrak{X})$ and $\mathcal{E} \in \mathbf{D}(\mathfrak{X})$, the complex $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E} \in \mathbf{D}_{\mathrm{t}}(\mathfrak{X})$. Again, this is a local question so it can be checked using [2, Proposition 5.2.1(a)] and the complex $\mathcal{K}^{\cdot}_{\infty}$ in its proof. Therefore, for each $\mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$,

the functor $\mathcal{F}\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} - : \mathbf{D}_{\mathsf{qct}}(\mathfrak{X}) \to \mathbf{D}(\mathfrak{X})$ takes values in $\mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$. So it provides an internal tensor product. One can see that the category $\mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$ has a symmetric monoidal structure. The unit object is $\mathbf{R}\Gamma'_{\mathfrak{X}}\mathcal{O}_{\mathfrak{X}}$ where by $\mathbf{R}\Gamma'_{\mathfrak{X}}$ we denote the right-derived functor of $\Gamma'_{\mathfrak{X}}$. We will denote this object by $\mathcal{O}'_{\mathfrak{X}}$ for convenience.

If furthermore $\mathfrak{X}$ is either separated or of finite Krull dimension, the category $\mathbf{D}_{\mathsf{qct}}(\mathfrak{X}) = \mathbf{D}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}))$ possesses the richer structure of symmetric closed category. The internal hom is defined as

$$\mathcal{H}\mathsf{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\mathcal{G}) := \mathbf{R}\mathcal{Q}^{\mathbf{t}}_{\mathfrak{X}}\mathbf{R}\mathcal{H}om^{\cdot}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F},\mathcal{G})$$

for $\mathcal{F},\mathcal{G}\in\mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$. It is also important to note that the $\otimes$-hom adjunction is internal, i.e., it holds replacing the usual hom-group with the internal hom we have just defined, namely, we have a canonical isomorphism

$$\mathcal{H}\mathsf{om}^{\cdot}_{\mathfrak{X}}\big(\mathcal{F}\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{G},\mathcal{M}\big) = \mathcal{H}\mathsf{om}^{\cdot}_{\mathfrak{X}}\big(\mathcal{F},\mathcal{H}\mathsf{om}^{\cdot}_{\mathfrak{X}}(\mathcal{G},\mathcal{M})\big)$$

where $\mathcal{F},\mathcal{G}$, and $\mathcal{M}\in\mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$.

If the reader is only interested in usual schemes, then it is enough to consider the quasi-coherence of the derived tensor product. In this case the topology in the sections of the structural sheaf is discrete, Γ'_X is the identity functor and so the unit object is $\mathcal{O}_X$. For the internal hom-sheaf, in the separated or finite Krull dimension case, one uses the derived "coherator" functor $\mathbf{R}\mathcal{Q}$ defined in [10, §3] taking

$$\mathcal{H}\mathsf{om}^{\cdot}_{X}(\mathcal{F},\mathcal{G}) := \mathbf{R}\mathcal{Q}\mathbf{R}\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$$

for $\mathcal{F}$ and $\mathcal{G}\in\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(X))$.

2. Cohomology with supports on formal schemes

2.1. Algebraic supports

Given $\mathcal{F}\in\mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$ and $Z\subset\mathfrak{X}$ a closed subset, for the right-derived functor of sheaf of sections with support along Z we have that $\mathbf{R}\Gamma_Z\mathcal{F}\in\mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$ because in the distinguished triangle

$$\mathbf{R}\Gamma_Z\mathcal{F}\to\mathcal{F}\to\mathbf{R}j_*j^*\mathcal{F}\xrightarrow{+}, \tag{1}$$

where $j:\mathfrak{X}\setminus Z\hookrightarrow\mathfrak{X}$ denotes the canonical open embedding, $\mathbf{R}j_*j^*\mathcal{F}\in\mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$ [2, Proposition 5.2.6 and Corollary 5.2.11]. On the other hand, the closed subset Z is the support of a coherent sheaf $\mathcal{O}_{\mathfrak{X}}/\mathcal{Q}$ where $\mathcal{Q}$ is an open coherent ideal in $\mathcal{O}_{\mathfrak{X}}$. The functor

$$\Gamma'_Z := \Gamma_{\mathcal{Q}} = \varinjlim_{n>0}\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}\big(\mathcal{O}_{\mathfrak{X}}/\mathcal{Q}^n,-\big)$$

of "sections with algebraic support along Z" does not depend on $\mathcal{Q}$ but only on Z. The natural map $\Gamma'_Z \to \Gamma_Z$ is an isomorphism when applied to sheaves in $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$. Furthermore, the natural morphism in $\mathbf{D}(\mathfrak{X})$ obtained by deriving $\theta_{Z,\mathcal{F}}: \mathbf{R}\Gamma'_Z\mathcal{F} \to \mathbf{R}\Gamma_Z\mathcal{F}$ is an isomorphism for all $\mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$. Indeed, this is a local question, so we can assume that $\mathfrak{X}$ is affine with $\mathfrak{X} = \mathrm{Spf}(A)$ where A is a noetherian adic ring. Let $\kappa: \mathrm{Spf}(A) \to \mathrm{Spec}(A)$ be the canonical map. Let $X := \mathrm{Spec}(A)$. The set Z can be considered as a closed subset of either X or $\mathfrak{X}$. We will use Γ'_Z and Γ_Z for the corresponding pair of endofunctors in $\mathcal{A}(\mathfrak{X})$ and $\mathcal{A}(X)$. This will not cause any confusion, because the context will make it clear in which category we are working. By [2, Proposition 5.2.4] it is enough to show that $\kappa_*\theta_{Z,\mathcal{F}}$ is an isomorphism. But this is true because the diagram

$$\begin{array}{ccc} \kappa_*\mathbf{R}\Gamma'_Z\mathcal{F} & \xrightarrow{\kappa_*\theta_{Z,\mathcal{F}}} & \kappa_*\mathbf{R}\Gamma_Z\mathcal{F} \\ \downarrow & & \downarrow \\ \mathbf{R}\Gamma'_Z\kappa_*\mathcal{F} & \longrightarrow & \mathbf{R}\Gamma_Z\kappa_*\mathcal{F} \end{array}$$

commutes and all the unlabeled maps are isomorphisms (for the map in the bottom use loc. cit. and [1, Corollary 3.2.4]).

Given $\mathcal{E}, \mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ there is a bifunctorial map

$$\psi_Z(\mathcal{E},\mathcal{F}): \mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{R}\Gamma_Z\mathcal{F} \to \mathbf{R}\Gamma_Z\big(\mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}\big)$$

defined as follows. Assume $\mathcal{E}$ is K-flat and $\mathcal{F}$ is K-injective and choose a quasi-isomorphism $\mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} \to \mathcal{J}$ with $\mathcal{J}$ K-injective. The composed map (of complexes) $\mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Gamma_Z\mathcal{F} \to \mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} \to \mathcal{J}$ has image into $\Gamma_Z\mathcal{J}$ and we define $\psi_Z(\mathcal{E},\mathcal{F})$ to be the resulting factorization

$$\mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{R}\Gamma_Z\mathcal{F} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Gamma_Z\mathcal{F} \xrightarrow{\psi_Z(\mathcal{E},\mathcal{F})} \Gamma_Z\mathcal{J} \xrightarrow{\sim} \mathbf{R}\Gamma_Z\big(\mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}\big).$$

This map is a quasi-isomorphism if Z is closed. The question is local so using again [2, Propositions 5.2.4 and 5.2.8], we restrict to the analogous question for an ordinary scheme X and a closed subset $Z \subset X$. We conclude by [1, Corollary 3.2.5].

2.2. Systems of supports on formal schemes

In general, a subset $Z \subset \mathfrak{X}$ *stable for specialization* is a union $Z = \bigcup_{\alpha\in I} Z_\alpha$ of a directed system of closed subsets $\{Z_\alpha \mid \alpha \in I\}$ of $\mathfrak{X}$ and $\Gamma_Z = \varinjlim_{\alpha\in I} \Gamma_{Z_\alpha}$, this corresponds to the classical case of a "system of supports." Writing $\Gamma'_Z = \varinjlim_{\alpha\in I} \Gamma'_{Z_\alpha}$ the canonical map $\Gamma'_Z \to \Gamma_Z$ induces natural maps $\theta_{Z,\mathcal{F}}: \mathbf{R}\Gamma'_Z\mathcal{F} \to \mathbf{R}\Gamma_Z\mathcal{F}$ for all $\mathcal{F} \in \mathbf{D}(\mathfrak{X})$. If $\mathcal{F} \to \mathcal{J}$ is a K-injective resolution, we have that

$$\theta_{Z,\mathcal{F}}: \mathbf{R}\Gamma'_Z\mathcal{F} = \Gamma'_Z\mathcal{J} = \varinjlim_{\alpha\in I} \Gamma'_{Z_\alpha}\mathcal{J} \xrightarrow{\varinjlim_{\alpha\in I}\theta_{Z_\alpha,\mathcal{F}}} \varinjlim_{\alpha\in I} \Gamma_{Z_\alpha}\mathcal{J} = \mathbf{R}\Gamma_Z\mathcal{F},$$

therefore, for all $\mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, $\theta_{Z,\mathcal{F}}$ is a quasi-isomorphism.

Mimicking the case of a closed subset, for $\mathcal{E}, \mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ there is a bifunctorial map

$$\psi_Z(\mathcal{E},\mathcal{F}) : \mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{R}\Gamma_Z\mathcal{F} \to \mathbf{R}\Gamma_Z\big(\mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}\big)$$

that is a quasi-isomorphism. To check this fact we may assume $\mathcal{E}$ is K-flat and $\mathcal{F}$ is K-injective and choose a quasi-isomorphism $\mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} \to \mathcal{J}$ with $\mathcal{J}$ a K-injective resolution and consider the commutativity of the diagram of complexes

$$\begin{array}{ccc} \mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \Gamma_Z\mathcal{F} & \xrightarrow{\psi_Z(\mathcal{E},\mathcal{F})} & \Gamma_Z\mathcal{J} \\ \uparrow & & \uparrow \\ \varinjlim_{\alpha\in I}\big(\mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \Gamma_{Z_\alpha}\mathcal{F}\big) & \xrightarrow{\varinjlim_{\alpha\in I}\psi_{Z_\alpha}(\mathcal{E},\mathcal{F})} & \varinjlim_{\alpha\in I}\Gamma_{Z_\alpha}\mathcal{J} \end{array}$$

2.3. Bousfield triangles for systems of supports

Let $Z \subset \mathfrak{X}$ be a subset stable for specialization as in the previous paragraph. The endofunctor $\mathbf{R}\Gamma_Z : \mathbf{D}_{\mathrm{qct}}(\mathfrak{X}) \to \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ together with the natural transformation $\rho : \mathbf{R}\Gamma_Z \to \mathrm{id}$ is a Bousfield acyclization functor. Let us see why. We need to check that ρ induces a canonical isomorphism $\rho(\mathbf{R}\Gamma_Z\mathcal{M}) = (\mathbf{R}\Gamma_Z\rho)(\mathcal{M})$, for all $\mathcal{M} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$. Indeed, it follows from the previous paragraph that it is enough to check this for $\mathcal{M} \in \mathbf{D}^{+}_{\mathrm{qct}}(\mathfrak{X})$, specifically for $\mathcal{M} = \mathcal{O}'_{\mathfrak{X}}$. The question is local, so arguing as at the end of Section 2.1, we can suppose that $\mathfrak{X} = X$ is a noetherian affine scheme and $\mathcal{M}$ a bounded-below complex formed by quasi-coherent injective sheaves. In this case $\Gamma_Z\mathcal{M}$ is a bounded-below complex formed by quasi-coherent injective sheaves, too (cf. [17, Propositions VI.7.1 and VII.4.5]). But the functor Γ_Z is idempotent from which it follows that

$$\mathbf{R}\Gamma_Z\mathbf{R}\Gamma_Z\mathcal{M} = \Gamma_Z\Gamma_Z\mathcal{M} = \Gamma_Z\mathcal{M} = \mathbf{R}\Gamma_Z\mathcal{M}.$$

Using the notation of Section 2.1 for a closed subset $Z \subset \mathfrak{X}$, the triangle (1) is a Bousfield localization triangle for each $\mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$.

In general, let $Z \subset \mathfrak{X}$ be a subset stable for specialization, therefore it can be considered as the union of a directed system $\{Z_\alpha \mid \alpha \in I\}$ of closed subsets of $\mathfrak{X}$. For every $\alpha \in I$, let $U_\alpha := \mathfrak{X} \setminus Z_\alpha$ be the complementary open subset and $j_\alpha : U_\alpha \to \mathfrak{X}$ be the canonical open embedding. Let $L_Z : \mathcal{A}(\mathfrak{X}) \to \mathcal{A}(\mathfrak{X})$ be the endofunctor defined as $L_Z := \varinjlim_{\alpha\in I} j_{\alpha *}j^*_\alpha$. For every $\mathcal{M} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ the triangle

$$\mathbf{R}\Gamma_Z\mathcal{M} \xrightarrow{\rho(\mathcal{M})} \mathcal{M} \to \mathbf{R}L_Z\mathcal{M} \xrightarrow{+}$$

is the Bousfield localization triangle whose associated acyclization functor is $\mathbf{R}\Gamma_Z$.

For every $\mathcal{E} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, the commutative diagram

$$\begin{array}{ccc} \mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{R}\Gamma_Z \mathcal{O}'_{\mathfrak{X}} & \xrightarrow{\mathcal{E}\otimes\rho(\mathcal{O}'_{\mathfrak{X}})} & \mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}'_{\mathfrak{X}} \\ {\scriptstyle \psi_Z(\mathcal{E},\mathcal{O}'_{\mathfrak{X}})}\downarrow \wr & & \downarrow \wr \\ \mathbf{R}\Gamma_Z \mathcal{E} & \xrightarrow{\rho(\mathcal{E})} & \mathcal{E} \end{array}$$

can be completed to an isomorphism of distinguished triangles

$$\begin{array}{ccccccc} \mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{R}\Gamma_Z \mathcal{O}'_{\mathfrak{X}} & \longrightarrow & \mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}'_{\mathfrak{X}} & \longrightarrow & \mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{R}L_Z \mathcal{O}'_{\mathfrak{X}} & \xrightarrow{+} & \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ \mathbf{R}\Gamma_Z \mathcal{E} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathbf{R}L_Z \mathcal{E} & \xrightarrow{+} & . \end{array}$$

Note that, in particular, $\mathbf{R}\Gamma_Z$ and $\mathbf{R}L_Z$ are endofunctors of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ that commute with coproducts, and two Bousfield acyclization or localization functors of this type commute. If $Z, W \subset \mathfrak{X}$ are stable for specialization subsets, then $\Gamma_{Z\cap W} = \Gamma_Z \Gamma_W$. One can check, following the same kind of arguments at the beginning of this subsection, that the canonical map $\mathbf{R}\Gamma_{Z\cap W}\mathcal{F} \to \mathbf{R}\Gamma_Z \mathbf{R}\Gamma_W \mathcal{F}$ is an isomorphism for every $\mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$.

2.4. *Computing the functor* $\mathbf{R}L_{\mathfrak{X}\setminus\mathfrak{X}_x}$

Let $x \in \mathfrak{X}$. Consider the affine formal scheme $\mathfrak{X}_x := \mathrm{Spf}(\widehat{\mathcal{O}_{\mathfrak{X},x}})$ where the adic topology in the ring $\mathcal{O}_{\mathfrak{X},x}$ is given by $\mathcal{I}_x$. If $\mathfrak{X} = \mathrm{Spf}\, B$ and $\mathfrak{p}$ is the prime ideal corresponding to the point x, then $\mathcal{O}_{\mathfrak{X},x} = B_{\{\mathfrak{p}\}}$. Denote by $i_x : \mathfrak{X}_x \hookrightarrow \mathfrak{X}$ the canonical inclusion map. Consider the functors

$$\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}_x) \underset{i_x^*}{\overset{\mathbf{R}i_{x*}}{\rightleftarrows}} \mathbf{D}_{\mathrm{qct}}(\mathfrak{X}),$$

which are defined by virtue of [2, Proposition 5.2.6 and Corollary 5.2.11] using the fact that i_x is an adic map.

Given $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, we have that

$$\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathbf{R}\Gamma_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{F}_1, \mathbf{R}i_{x*}i_x^*\mathcal{F}_2\big) \cong \mathrm{Hom}_{\mathbf{D}(\mathfrak{X}_x)}\big(i_x^*\mathbf{R}\Gamma_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{F}_1, i_x^*\mathcal{F}_2\big) = 0$$

because $i_x^*\mathbf{R}\Gamma_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{F}_1 = 0$. Indeed, write $\mathfrak{X} \setminus \mathfrak{X}_x = \bigcup_{\alpha\in I} Z_\alpha$ a filtered union of closed subsets $\{Z_\alpha \mid \alpha \in I\}$, and let $\mathcal{F}_1 \to \mathcal{J}$ be a K-injective resolution; then

$$i_x^*\mathbf{R}\Gamma_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{F}_1 = i_x^*\Gamma_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{J} = i_x^* \varinjlim_{\alpha\in I} \Gamma_{Z_\alpha}\mathcal{J} = \varinjlim_{\alpha\in I} i_x^*\Gamma_{Z_\alpha}\mathcal{J} = 0.$$

It follows that for each $\mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ there is a unique map $\mathbf{R}L_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{F} \to \mathbf{R}i_{x*}i_x^*\mathcal{F}$ making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathbf{R}L_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{F} \\ \| & & \downarrow{\scriptstyle h_{\mathcal{F}}} \\ \mathcal{F} & \longrightarrow & \mathbf{R}i_{x*}i_x^*\mathcal{F}. \end{array}$$

Furthermore, h is a natural transformation of Δ-functors and it is an isomorphism, i.e., $h_{\mathcal{F}}$ is a quasi-isomorphism for every $\mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$. Let us show this. First of all, we can assume that $\mathfrak{X}$ is affine. Indeed, choose an affine open subset $\mathfrak{U} \subset \mathfrak{X}$ such that $x \in \mathfrak{U}$, then one can describe $\mathfrak{X} \setminus \mathfrak{X}_x$ as a filtered union of closed subsets $\{Z_\alpha \mid \alpha \in I\}$ such that each $\mathfrak{U}_\alpha := \mathfrak{X} \setminus Z_\alpha$ is an affine open subset of $\mathfrak{U}$. Let us denote by $j : \mathfrak{U} \hookrightarrow \mathfrak{X}$, $j_\alpha : \mathfrak{U}_\alpha \hookrightarrow \mathfrak{X}$, and $i'_x : \mathfrak{X}_x \hookrightarrow \mathfrak{U}$ the canonical morphisms. Note that $j \circ i'_x = i_x$. For every $\mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ we have an isomorphism $\mathbf{R}L_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{F} \xrightarrow{\sim} \mathbf{R}j_*j^*\mathbf{R}L_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{F}$ because $\mathbf{R}\Gamma_{\mathfrak{X}\setminus\mathfrak{U}}\mathbf{R}L_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{F} = \mathbf{R}\Gamma_{\mathfrak{X}\setminus\mathfrak{U}}\mathbf{R}\Gamma_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathbf{R}L_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{F} = 0$ (see Section 2.3). Using flat base change [2, Proposition 7.2], we see that the canonical map $\mathbf{R}i_{x*}i_x^*\mathcal{F} \xrightarrow{\sim} \mathbf{R}j_*j^*\mathbf{R}i_{x*}i_x^*\mathcal{F}$ is also an isomorphism. So, we are left to prove that $j^*h_{\mathcal{F}}$ is an isomorphism, or, equivalently, that $h_{j^*\mathcal{F}} : \mathbf{R}L_{\mathfrak{U}\setminus\mathfrak{X}_x}(j^*\mathcal{F}) \to \mathbf{R}i'_{x*}i'^*_x(j^*\mathcal{F})$ is an isomorphism. Then, let us treat the case $\mathfrak{X} = \operatorname{Spf} A$ with A a complete noetherian ring. Both endofunctors $\mathbf{R}L_{\mathfrak{X}\setminus\mathfrak{X}_x}$ and $\mathbf{R}i_{x*}i_x^*$ commute with coproducts by 2.3 and [2, Proposition 3.5.2], respectively. To prove that $h_{\mathcal{F}}$ is a quasi-isomorphism for every $\mathcal{F} \in \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})) = \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ it is enough to check it for $\mathcal{F} \in \mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$, because the smallest localizing subcategory containing $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$ is all of $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$. In this case the morphisms $j_\alpha : U_\alpha \hookrightarrow \mathfrak{X}$ and $i_x : \mathfrak{X}_x \hookrightarrow \mathfrak{X}$ are affine. Therefore, by [2, Lemma 3.4.2], for $\mathcal{F} \in \mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$ and $i > 0$,

$$\mathcal{H}^i(\mathbf{R}L_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{F}) = \varinjlim_{\alpha\in I} \mathcal{H}^i\left(\mathbf{R}j_{\alpha*}j_\alpha^*\mathcal{F}\right) = 0, \qquad \mathcal{H}^i\left(\mathbf{R}i_{x*}i_x^*\mathcal{F}\right) = 0,$$

and for $i = 0$,

$$\mathcal{H}^0(\mathbf{R}L_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{F}) = \varinjlim_{\alpha} j_{\alpha*}j_\alpha^*\mathcal{F} \xrightarrow{\mathcal{H}^0(h_{\mathcal{F}})} i_{x*}i_x^*\mathcal{F} = \mathcal{H}^0\left(\mathbf{R}i_{x*}i_x^*\mathcal{F}\right)$$

is the natural map. Let us show that $\mathcal{H}^0(h_{\mathcal{F}})$ is an isomorphism. Using [2, Proposition 5.2.4], we are reduced to the particular case $\mathfrak{X} = X = \operatorname{Spec} A$ is an usual affine scheme, x corresponds to a prime ideal $\mathfrak{p} \subset A$, M is an A-module and $\mathcal{F} = \widetilde{M}$. Then $\mathcal{H}^0(h_{\mathcal{F}})$ corresponds to the canonical isomorphism of A-modules

$$\varinjlim_{f\in A\setminus\mathfrak{p}} M_f \xrightarrow{\sim} M_{\mathfrak{p}}.$$

Therefore, for $\mathfrak{X}$ a noetherian formal scheme and every $\mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ one has a natural Bousfield triangle

$$\mathbf{R}\Gamma_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{F} \to \mathcal{F} \to \mathbf{R}i_{x*}i_x^*\mathcal{F} \xrightarrow{+}. \tag{2}$$

Recall that the canonical triangle

$$\mathbf{R}\Gamma_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{O}'_{\mathfrak{X}} \to \mathcal{O}'_{\mathfrak{X}} \to \mathbf{R}i_{x*}i_x^*\mathcal{O}'_{\mathfrak{X}} \xrightarrow{+}$$

tensored by $\mathcal{F}$ provides a triangle

$$\mathbf{R}\Gamma_{\mathfrak{X}\setminus\mathfrak{X}_x}\mathcal{O}'_{\mathfrak{X}} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} \to \mathcal{F} \to \mathbf{R}i_{x*}i_x^*\mathcal{O}'_{\mathfrak{X}} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} \xrightarrow{+}$$

that is naturally isomorphic to (2) by Section 2.3.

3. Rigid localizing subcategories

Let $\mathbf{T}$ be a triangulated category with all coproducts. This is the case for $\mathbf{D}(\mathfrak{X})$ and $\mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$ for a noetherian formal scheme $\mathfrak{X}$, and also for $\mathbf{D}(X)$ and $\mathbf{D}_{\mathsf{qc}}(X)$ for an usual scheme X. A triangulated subcategory $\mathcal{L}$ of $\mathbf{T}$ is called *localizing* if it is stable for coproducts in $\mathbf{T}$. If $\mathbf{T}$ is one of the aforementioned derived categories, it is not ensured that $\mathcal{L} \subset \mathbf{T}$ is well-behaved with respect to the tensorial structure. It turns out that we need such compatibility in order to localize on open subsets. So let us establish the following definition. A localizing subcategory $\mathcal{L} \subset \mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$ is called *rigid* if for every $\mathcal{F} \in \mathcal{L}$ and $\mathcal{G} \in \mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$, we have that $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} \in \mathcal{L}$. This condition has been independently considered by Thomason for thick subcategories by the same reason (see [18, Definition 3.9], where they are called $\otimes$-*subcategories*). Our route to find this condition came from a paper by one of the authors where localizations are considered in the abelian context, see [11, 2.3].

Proposition 3.1. *Suppose that $\mathfrak{X}$ is furthermore either separated or of finite Krull dimension. Let $\mathcal{L}$ be a localizing subcategory of $\mathbf{D}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}))$. If $\mathcal{L}$ is rigid, then, for every $\mathcal{F}, \mathcal{G} \in \mathbf{D}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}))$ such that $\mathcal{G}$ is $\mathcal{L}$-local (i.e., $\mathcal{G} \in \mathcal{L}^{\perp}$), then $\boldsymbol{\mathcal{H}}\mathsf{om}_{\mathfrak{X}}^{\cdot}(\mathcal{F}, \mathcal{G})$ is $\mathcal{L}$-local. If moreover ${}^{\perp}(\mathcal{L}^{\perp}) = \mathcal{L}$, the converse is true.*

Proof. Let $\mathcal{H} \in \mathcal{L}$, then

$$\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{H}, \boldsymbol{\mathcal{H}}\mathsf{om}_{\mathfrak{X}}^{\cdot}(\mathcal{F}, \mathcal{G})\big) = \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{H} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}, \mathcal{G}\big) = 0, \tag{3}$$

because $\mathcal{G} \in \mathcal{L}^{\perp}$ and $\mathcal{H} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} \in \mathcal{L}$. Conversely, if (3) holds for every $\mathcal{G} \in \mathcal{L}^{\perp}$, then $\mathcal{H} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} \in {}^{\perp}(\mathcal{L}^{\perp}) = \mathcal{L}$. □

Remark. The condition ${}^{\perp}(\mathcal{L}^{\perp}) = \mathcal{L}$ holds if $\mathcal{L}$ is the localizing subcategory of objects whose image is 0 by a Bousfield localization (see [4, Proposition 1.6]). We will see later (Corollary 4.14) that every rigid localizing subcategory of $\mathbf{D}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}))$ arises in this way.

Proposition 3.2. *If $\mathfrak{X}$ is affine, every localizing subcategory of $\mathbf{D}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}))$ is rigid.*

Proof. Take $\mathfrak{X} = \mathrm{Spf}\, A$ where A is a noetherian adic ring. Every quasi-coherent torsion sheaf comes from an A-module and therefore it has a free resolution. Let $\kappa : \mathrm{Spf}\, A \to$

Spec A the canonical morphism and $X := \operatorname{Spec} A$. Let $\mathcal{L}$ be a localizing subcategory of $\mathbf{D}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}))$. The full subcategory $\mathbf{T}$ of $\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(X))$ defined by

$$\mathbf{T} = \left\{\mathcal{N} \in \mathbf{D}\big(\mathcal{A}_{\mathsf{qc}}(X)\big) \;\middle|\; \kappa^*\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{M} \in \mathcal{L},\ \forall \mathcal{M} \in \mathcal{L}\right\}$$

is triangulated and stable for coproducts. It is clear that $\mathcal{O}_X \in \mathbf{T}$, therefore $\mathbf{T} = \mathbf{D}(\mathcal{A}_{\mathsf{qc}}(X))$. Now, given $\mathcal{G} \in \mathbf{D}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}))$, $\mathcal{G} = \kappa^*\kappa_*\mathcal{G}$, and $\kappa_*\mathcal{G} \in \mathbf{D}(\mathcal{A}_{\mathsf{qc}}(X)) = \mathbf{T}$ [2, Proposition 5.1.2], therefore $\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{M} \in \mathcal{L}$, for every $\mathcal{M} \in \mathcal{L}$. □

Example. Not all localizing subcategories are rigid. Let us show an example of a non-rigid localizing subcategory. Our example is based in Thomason's example [18, Example 3.13] of a thick subcategory that it is not a $\otimes$-subcategory. Consider the projective line over a field together with its canonical map $\pi\colon \mathbf{P}^1_k \to \operatorname{Spec} k$. Denote by $\mathbf{D}(\mathbf{P}^1_k)_{\mathsf{cp}}$ the full subcategory[2] of $\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(\mathbf{P}^1_k))$ formed by perfect complexes (i.e., quasi-isomorphic to a bounded complex of locally free finite-type sheaves). Let $\mathcal{L}$ the smallest localizing subcategory of $\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(\mathbf{P}^1_k))$ generated by $\mathcal{E} := \mathbf{L}\pi^*\tilde{k}$. Note that $\mathcal{E} \in \mathbf{D}(\mathbf{P}^1_k)_{\mathsf{cp}}$ and that $\mathcal{L}$ is the smallest localizing subcategory that contains the thick subcategory $\mathcal{A} = \{\mathcal{F} \in \mathbf{D}(\mathbf{P}^1_k)_{\mathsf{cp}} \mid \mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{F} = \mathcal{F}\}$, which is a thick subcategory of $\mathbf{D}(\mathbf{P}^1_k)_{\mathsf{cp}}$, constructed by Thomason in loc. cit. Every object $\mathcal{M} \in \mathcal{L}$ is such that $\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{M} = \mathcal{M}$ because both $\mathbf{L}\pi^*$ and $\mathbf{R}\pi_*$ commute with coproducts and the equality holds for $\mathcal{E}$. Observe that $\mathcal{L}$ is the essential image of $\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(\operatorname{Spec} k))$ by the functor $\mathbf{L}\pi^*$. The localizing category $\mathcal{L}$ is not rigid. Indeed, take $\mathcal{M} \in \mathcal{L}$, $\mathcal{M} \neq 0$, we will show that $\mathcal{M} \otimes \mathcal{O}(-1) \notin \mathcal{L}$. Let $\mathcal{F} := \mathbf{R}\pi_*\mathcal{M}$, then

$$\begin{aligned}\mathbf{R}\pi_*\big(\mathcal{M} \otimes \mathcal{O}(-1)\big) &= \mathbf{R}\pi_*\big(\mathbf{L}\pi^*(\mathcal{F}) \otimes \mathcal{O}(-1)\big) \quad [12, (3.9.4)]\\ &\simeq \mathcal{F} \otimes \mathbf{R}\pi_*\mathcal{O}(-1) \quad [8, 2.12.16]\\ &\simeq 0.\end{aligned}$$

We conclude that $\mathcal{M} \otimes \mathcal{O}(-1)$ is not an object in $\mathcal{L}$ because $\mathcal{M} \otimes \mathcal{O}(-1) \neq 0 = \mathbf{L}\pi^*\mathbf{R}\pi_*(\mathcal{M} \otimes \mathcal{O}(-1))$.

Remark. The rigidity condition may seem strange but, in fact, these are the localizations that behave well when restricted to open subsets and "are detected" by ample sheaves when they exist. We suggest the interested reader to adapt [18, Proposition 3.11] and its corollary to our situation. We will not get into these details because we do not need them.

4. Localizing subcategories and subsets

We keep denoting by $\mathfrak{X}$ a noetherian formal scheme and $\mathcal{I}$ its ideal of definition. Let $x \in \mathfrak{X}$, we denote by $i_x\colon \mathfrak{X}_x \hookrightarrow \mathfrak{X}$ the canonical inclusion map where $\mathfrak{X}_x = \operatorname{Spf}(\widehat{\mathcal{O}_{\mathfrak{X},x}})$ (completion with respect to $\mathcal{I}_x$).

[2] Denoted as $\mathbf{D}(\mathbf{P}^1_k)_{\mathrm{parf}}$ in [18].

We will denote by $\kappa(x)$ the residue field of the local ring $\widehat{\mathcal{O}_{\mathfrak{X},x}}$, or, equivalently, of $\mathcal{O}_{\mathfrak{X},x}$, by $\mathcal{K}_x$ the quasi-coherent torsion sheaf over $\mathrm{Spf}(\widehat{\mathcal{O}_{\mathfrak{X},x}})$ associated to the $\widehat{\mathcal{O}_{\mathfrak{X},x}}$-module $\kappa(x)$ and $\mathcal{K}(x) := \mathbf{R}i_{x*}(\mathcal{K}_x)$. Observe that $\mathcal{K}(x) = \mathbf{R}\Gamma_{\overline{\{x\}}}\mathcal{K}(x) = \mathbf{R}i_{x*}i_x^*\mathcal{K}(x)$. If $\mathfrak{X} = X$ is an usual scheme and x is a closed point, $\mathcal{K}(x)$ has been denoted $\mathcal{O}_x$ in recent literature, but we will not use this notation to avoid potential confusions.

Let Z be any subset of the underlying space of $\mathfrak{X}$. We define the subcategory $\mathcal{L}_Z$ as the smallest localizing subcategory of $\mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$ that contains the set of quasi-coherent torsion sheaves $\{\mathcal{K}(x) \mid x \in Z\}$. If $Z = \{x\}$, we will denote $\mathcal{L}_Z$ simply by $\mathcal{L}_x$. Note that if $x \in Z$, then $\mathcal{L}_x \subset \mathcal{L}_Z$.

Lemma 4.1. *If $\mathcal{F} \in \mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$ and $x \in \mathfrak{X}$, then $\mathbf{R}\Gamma_{\overline{\{x\}}}(\mathbf{R}i_{x*}i_x^*\mathcal{F})$ belongs to the localizing subcategory $\mathcal{L}_x$.*

Proof. Let $\mathcal{Q}_0$ be a sheaf of coherent ideals in $\mathcal{O}_{\mathfrak{X}}$ such that $\mathrm{Supp}(\mathcal{O}_{\mathfrak{X}}/\mathcal{Q}_0) = \overline{\{x\}}$ and denote $\mathcal{Q} := i_x^*\mathcal{Q}_0$. Recall, by [2, §5.4],

$$\begin{aligned}\mathbf{R}\Gamma_{\overline{\{x\}}}\big(\mathbf{R}i_{x*}i_x^*\mathcal{F}\big) &= \operatorname{holim}_{\substack{\longrightarrow\\ n>0}} \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}\big(\mathcal{O}_{\mathfrak{X}}/\mathcal{Q}_0^n, i_{x*}\mathcal{J}\big)\\ &\cong \operatorname{holim}_{\substack{\longrightarrow\\ n>0}} \mathbf{R}i_{x*}\,\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_x}}\big(\mathcal{O}_{\mathfrak{X}_x}/\mathcal{Q}^n, \mathcal{J}\big),\end{aligned}$$

where $i_x^*\mathcal{F} \to \mathcal{J}$ is a K-injective resolution.

Let $\mathcal{G} := \varinjlim_{n>0} \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_x}}(\mathcal{O}_{\mathfrak{X}_x}/\mathcal{Q}^n, \mathcal{J})$ and let us consider the filtration

$$0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset \mathcal{G},$$

where $\mathcal{G}_n := \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_x}}(\mathcal{O}_{\mathfrak{X}_x}/\mathcal{Q}^n, \mathcal{J})$, i.e., the subcomplex of $\mathcal{J}$ annihilated by $\mathcal{Q}^n$. The successive quotients $\mathcal{G}_n/\mathcal{G}_{n-1}$ are complexes of quasi-coherent $\mathcal{K}_x$-modules and, therefore, isomorphic in $\mathbf{D}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}_x))$ to a direct sum of shifts of $\mathcal{K}_x$. The functor $\mathbf{R}i_{x*}$ preserves coproducts, therefore every $\mathbf{R}i_{x*}(\mathcal{G}_n/\mathcal{G}_{n-1})$ is an object of $\mathcal{L}_x$. We deduce by induction, using the distinguished triangles

$$\mathbf{R}i_{x*}\mathcal{G}_{n-1} \to \mathbf{R}i_{x*}\mathcal{G}_n \to \mathbf{R}i_{x*}(\mathcal{G}_n/\mathcal{G}_{n-1}) \xrightarrow{+}$$

that every $\mathbf{R}i_{x*}\mathcal{G}_n$ is in $\mathcal{L}_x$ for every $n \in \mathbb{N}$. But we have

$$\mathbf{R}\Gamma_{\overline{\{x\}}}\big(\mathbf{R}i_{x*}i_x^*\mathcal{F}\big) \cong \operatorname{holim}_{\substack{\longrightarrow\\ n>0}} \mathbf{R}i_{x*}\mathcal{G}_n,$$

and the result follows from the fact that a localizing subcategory is stable for homotopy direct limits [4, Lemma 3.5 and its proof]. □

Let E_x be an injective hull of the $\mathcal{O}_{\mathfrak{X},x}$-module $\kappa(x)$, then E_x is a $\mathcal{I}_x$-torsion $\widehat{\mathcal{O}_{\mathfrak{X},x}}$-module. Let then $\mathcal{E}_x$ be the sheaf in $\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}_x)$ determined by $\Gamma(\mathfrak{X}_x, \mathcal{E}_x) = E_x$.

Corollary 4.2. *The object $\mathcal{E}(x) := \mathbf{R}i_{x*}\mathcal{E}_x$ belongs to $\mathcal{L}_x$.*

Proof. Use the previous lemma and the fact that $\mathcal{E}(x) = \mathbf{R}\Gamma_{\overline{\{x\}}}(\mathbf{R}i_{x*}i_x^*\mathcal{E}(x))$. □

Lemma 4.3. *Let $\mathcal{M} \in \mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$ and $\mathcal{L}$ the smallest localizing subcategory of $\mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$ that contains $\mathcal{M}$. If $\mathcal{G} \in \mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$ is such that $\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} = 0$, then $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} = 0$, for every $\mathcal{F} \in \mathcal{L}$.*

Proof. The Δ-functor $- \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G}$ preserves coproducts and therefore the full subcategory whose objects are those $\mathcal{F} \in \mathcal{L}$ such that $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} = 0$ is localizing, but it contains $\mathcal{M}$, therefore it is $\mathcal{L}$. □

Proposition 4.4. *The smallest localizing subcategory $\mathcal{L}$ of $\mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$ that contains $\mathcal{K}(x)$ for every $x \in \mathfrak{X}$ is the whole $\mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$.*

Proof. Let $\mathcal{F} \in \mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$ and $\mathcal{C}$ denote the family of subsets $Y \subset \mathfrak{X}$ stable for specialization such that $\mathbf{R}\Gamma_Y\mathcal{F} \in \mathcal{L}$. If $\{W_\alpha\}_{\alpha\in I}$ is a chain in $\mathcal{C}$, then

$$\mathbf{R}\Gamma_{\bigcup W_\alpha}\mathcal{F} = \varinjlim_{\alpha\in I} \Gamma_{W_\alpha}\mathcal{J},$$

for a K-injective resolution $\mathcal{F} \to \mathcal{J}$. By [4, Theorems 2.2, 3.1], $\mathbf{R}\Gamma_{\bigcup W_\alpha}\mathcal{F} = \Gamma_{\bigcup W_\alpha}\mathcal{J} \in \mathcal{L}$, because each $\mathbf{R}\Gamma_{W_\alpha}\mathcal{F} = \Gamma_{W_\alpha}\mathcal{J} \in \mathcal{L}$, so $\bigcup W_\alpha \in \mathcal{C}$.

The set $\mathcal{C}$ is stable for filtered unions, therefore, there is a maximal element in $\mathcal{C}$ which we will denote by W. We will see that $W = \mathfrak{X}$ from which it follows that $\mathcal{F} \cong \mathbf{R}\Gamma_{\mathfrak{X}}\mathcal{F} \in \mathcal{L}$.

Indeed, otherwise suppose $\mathfrak{X} \setminus W \neq \emptyset$. As $\mathfrak{X}$ is noetherian, the family of closed subsets

$$\mathcal{C}' = \left\{ \overline{\{z\}} \;\middle|\; z \in \mathfrak{X} \text{ and } \overline{\{z\}} \cap (\mathfrak{X} \setminus W) \neq \emptyset \right\}$$

has a minimal subset $\overline{\{y\}}$. If $x \in \overline{\{y\}} \cap (\mathfrak{X} \setminus W)$, then $\overline{\{x\}} \in \mathcal{C}'$, but $\overline{\{y\}}$ is minimal, so $x = y$ and $W \cup \overline{\{y\}} = W \cup \{y\}$. Consider now the inclusion $i_y : \mathfrak{X}_y \to \mathfrak{X}$ and the distinguished triangle in $\mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$

$$\mathbf{R}\Gamma_W\mathcal{F} \to \mathbf{R}\Gamma_{W\cup\{y\}}\mathcal{F} \to \mathbf{R}\Gamma_{\overline{\{y\}}}(\mathbf{R}i_{y*}i_y^*\mathcal{F}) \xrightarrow{+}$$

obtained applying $\mathbf{R}\Gamma_{W\cup\{y\}}$ to the canonical triangle

$$\mathbf{R}\Gamma_{\mathfrak{X}\setminus\mathfrak{X}_y}\mathcal{F} \to \mathcal{F} \to \mathbf{R}i_{y*}i_y^*\mathcal{F} \xrightarrow{+}.$$

We deduce that $\mathbf{R}\Gamma_{W\cup\{y\}}\mathcal{F} \in \mathcal{L}$, because $W \in \mathcal{C}$ and $\mathbf{R}\Gamma_{\overline{\{y\}}}(\mathbf{R}i_{y*}i_y^*\mathcal{F}) \in \mathcal{L}_y \subset \mathcal{L}$ by Lemma 4.1, contradicting the maximality of W. □

Corollary 4.5. *Let $\mathcal{G} \in \mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$. We have that $\mathcal{G} = 0$ if and only if $\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\mathcal{K}(x)[n], \mathcal{G}) = 0$ for all $x \in \mathfrak{X}$ and $n \in \mathbb{Z}$.*

Proof. Immediate from Proposition 4.4. □

Corollary 4.6. *Let $\mathcal{G} \in \mathbf{D}_{\mathsf{qct}}(\mathfrak{X})$ be such that $\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} = 0$ for every $x \in \mathfrak{X}$, then $\mathcal{G} = 0$.*

Proof. It is a consequence of Proposition 4.4 and Lemma 4.3. □

Lemma 4.7. *If $x \neq y$, then $\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{K}(y) = 0$.*

Proof. There exist an affine open subset $\mathfrak{U} \subset \mathfrak{X}$ such that it only contains one of the points, for instance assume that $x \in \mathfrak{U}$ and $y \notin \mathfrak{U}$. Denote by $j : \mathfrak{U} \hookrightarrow \mathfrak{X}$ the canonical inclusion map. Now, using Section 2.4,

$$\begin{aligned}\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{K}(y) &\cong \mathbf{R}j_*j^*\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{K}(y) \cong \mathbf{R}j_*j^*\mathcal{O}'_{\mathfrak{X}} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{K}(y)\\ &\cong \mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{R}j_*j^*\mathcal{K}(y) = 0,\end{aligned}$$

because $j^*\mathcal{K}(y) = 0$. □

Corollary 4.8. *For every subset $Z \subset \mathfrak{X}$, the localizing subcategory $\mathcal{L}_Z$ is rigid.*

Proof. The full subcategory $\mathcal{S} \subset \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ defined by

$$\mathcal{S} = \left\{\mathcal{N} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X}) \;\middle|\; \mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{M} \in \mathcal{L}_Z,\ \forall \mathcal{M} \in \mathcal{L}_Z\right\}$$

is a localizing subcategory of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$. For $x \in \mathfrak{X}$, $\mathcal{K}(x) \cong \mathbf{R}\Gamma_{\overline{\{x\}}}\mathbf{R}i_{x*}i_x^*\mathcal{K}(x)$, so using Sections 2.3 and 2.4, we have that

$$\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{M} \cong \mathbf{R}\Gamma_{\overline{\{x\}}}\mathbf{R}i_{x*}i_x^*\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{M} \cong \mathbf{R}\Gamma_{\overline{\{x\}}}\mathbf{R}i_{x*}i_x^*\big(\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{M}\big).$$

Therefore if $x \in Z$, then $\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{M} \in \mathcal{L}_x \subset \mathcal{L}_Z$ by Lemma 4.1, and for $x \notin Z$, by Lemmas 4.7 and 4.3, $\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{M} = 0$ it is also in $\mathcal{L}_Z$. Necessarily $\mathcal{S} = \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ by Proposition 4.4. □

Corollary 4.9. *If Z and Y are subsets of $\mathfrak{X}$ such that $Z \cap Y = \emptyset$, then $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} = 0$ for every $\mathcal{F} \in \mathcal{L}_Z$ and $\mathcal{G} \in \mathcal{L}_Y$.*

Proof. This follows from the previous lemma and Lemma 4.3. □

Corollary 4.10. *Given $x \in \mathfrak{X}$ and $\mathcal{F} \in \mathcal{L}_x$ we have that*

$$\mathcal{F} = 0 \quad \Leftrightarrow \quad \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{K}(x) = 0.$$

Proof. By Lemmas 4.7 and 4.3, given $\mathcal{F} \in \mathcal{L}_x$, for all $y \in \mathfrak{X}$, with $y \neq x$ we have that $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{K}(y) = 0$, therefore if also $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{K}(x) = 0$, it follows that $\mathcal{F} = 0$ by Corollary 4.6. □

Corollary 4.11. *Let $\mathcal{L}$ be a localizing subcategory of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ and $\mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$. If $\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} \in \mathcal{L}$ for every $x \in \mathfrak{X}$, then $\mathcal{F} \in \mathcal{L}$.*

Proof. Let $\mathcal{L}' = \{\mathcal{G} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X}) \mid \mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} \in \mathcal{L}\}$. The subcategory $\mathcal{L}'$ is a localizing subcategory of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ such that $\mathcal{K}(x) \in \mathcal{L}'$ for all $x \in \mathfrak{X}$. By Proposition 4.4, we deduce that $\mathcal{L}' = \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, in particular $\mathcal{O}'_{\mathfrak{X}} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} = \mathcal{F} \in \mathcal{L}$. □

Remark. If the localizing subcategory $\mathcal{L}$ is rigid, then $\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} \in \mathcal{L}$ for all $x \in \mathfrak{X}$ if, and only if, $\mathcal{F} \in \mathcal{L}$.

Theorem 4.12. *For a noetherian formal scheme $\mathfrak{X}$ there is a bijection between the class of rigid localizing subcategories of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ and the set of all subsets of $\mathfrak{X}$.*

Proof. Denote by $\mathsf{Loc}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ the class of rigid localizing subcategories of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ and by $\mathsf{P}(\mathfrak{X})$ the set of all subsets of $\mathfrak{X}$. Let us define a couple of maps:

$$\mathsf{Loc}\big(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})\big) \underset{\phi}{\overset{\psi}{\rightleftarrows}} \mathsf{P}(\mathfrak{X})$$

and check that they are mutual inverses. Define for $Z \subset \mathfrak{X}$, $\phi(Z) := \mathcal{L}_Z$ which is rigid by Corollary 4.8, and for a rigid localizing subcategory $\mathcal{L}$ of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, $\psi(\mathcal{L}) := \{x \in \mathfrak{X} \mid \exists \mathcal{G} \in \mathcal{L} \text{ with } \mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} \neq 0\}$.

Let us check first that $\psi \circ \phi = \mathrm{id}$. Let $Z \subset \mathfrak{X}$ and $x \in Z$, by definition $\mathcal{K}(x) \in \mathcal{L}_Z$ and clearly $\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{K}(x) \neq 0$ by Corollary 4.6 and Lemma 4.7, therefore $x \in \psi(\phi(Z))$, so $Z \subset \psi(\phi(Z))$. Conversely let $x \in \psi(\phi(Z))$, by definition there is $\mathcal{G} \in \mathcal{L}_Z$ such that $\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} \neq 0$, by Corollary 4.9, $x \in Z$.

Now we have to prove that $\phi \circ \psi = \mathrm{id}$. Let $\mathcal{L}$ be a rigid localizing subcategory of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$. We will see first that $\mathcal{L}_{\psi(\mathcal{L})} \subset \mathcal{L}$ and for this it will be enough to check that $\mathcal{K}(x) \in \mathcal{L}$ for every $x \in \psi(\mathcal{L})$. So let $x \in \psi(\mathcal{L})$, there is a $\mathcal{G} \in \mathcal{L}$ such that $\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} \neq 0$. On the other hand, $\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G}$ belongs to $\mathcal{L}$ because $\mathcal{L}$ is rigid. We have that

$$\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} \cong \bigoplus_{\alpha \in J} \mathcal{F}_\alpha,$$

where J is a set of indices and $\mathcal{F}_\alpha = \mathcal{K}(x)[s_\alpha]$ with $s_\alpha \in \mathbb{Z}$. Indeed, it is enough to take a free resolution $\mathcal{M} \to i_x^*\mathcal{G}$ of the complex of quasi-coherent torsion $\mathcal{O}_{\mathfrak{X}_x}$-modules $i_x^*\mathcal{G}$ and to consider the chain of natural isomorphisms

$$\begin{aligned}
\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} &\cong \mathbf{R}i_{x*}i_x^*\big(\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G}\big) \\
&\cong \mathbf{R}i_{x*}\big(\mathcal{K}_x \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}_x}} i_x^*\mathcal{G}\big) \quad [12, (3.2.4)] \\
&\cong \mathbf{R}i_{x*}\big(\mathcal{K}_x \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}_x}} \mathcal{M}\big),
\end{aligned}$$

and use the fact that both functors $\mathcal{K}_x \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}_x}} -$ and $\mathbf{R}i_{x*}$ commute with coproducts. But $\mathcal{L}$ is localizing, so stable for coproducts and, as a consequence, for direct summands (see [5] or [4, footnote, p. 227]). From this, $\bigoplus_{\alpha \in J} \mathcal{F}_\alpha \in \mathcal{L}$ implies $\mathcal{K}(x) \in \mathcal{L}$, as required.

Finally, let us see that $\mathcal{L} \subset \mathcal{L}_{\psi(\mathcal{L})}$. Let $\mathcal{F} \in \mathcal{L}$, by Corollary 4.11 to see that $\mathcal{F} \in \mathcal{L}_{\psi(\mathcal{L})}$ it is enough to prove that $\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} \in \mathcal{L}_{\psi(\mathcal{L})}$ for every $x \in \mathfrak{X}$. Suppose that the non-trivial situation $\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} \neq 0$ holds. In this case, $x \in \psi(\mathcal{L})$, therefore we conclude that $\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} \in \mathcal{L}_x \subset \mathcal{L}_{\psi(\mathcal{L})}$ using Corollary 4.8 that tells us that $\mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}$ belongs to the localizing subcategory generated by $\mathcal{K}(x)$. □

Remark. In view of Proposition 3.2, the previous result is a generalization of [14, Theorem 2.8] from noetherian *affine* schemes to the bigger category of noetherian formal schemes.

Corollary 4.13. *For a noetherian scheme X there is a bijection between the class of rigid localizing subcategories of $\mathbf{D}_{\mathrm{qc}}(X)$ and the set of all subsets of X.*

Corollary 4.14. *Every rigid localizing subcategory of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ has associated a localization functor.*

Proof. Theorem 4.12 says that a rigid localizing subcategory $\mathcal{L} \subset \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ is the smallest localizing subcategory that contains the set $\{\mathcal{K}(x) \mid x \in \psi(\mathcal{L})\}$. It follows from [4, Theorem 5.7] that there is an associated localization functor for $\mathcal{L}$. □

The following consequences of the previous discussion will be used in the next section.

Lemma 4.15. *Let $\mathcal{L}$ be a rigid localizing subcategory of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ and $z \in \mathfrak{X}$. If $z \notin \psi(\mathcal{L})$, then $\mathcal{K}(z)$ is a $\mathcal{L}$-local object.*

Proof. Let $\mathcal{N} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ consider the natural map

$$\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{N}, \mathcal{K}(z)\big) \xrightarrow{\alpha} \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{K}(z), \mathcal{K}(z) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{K}(z)\big),$$

and the map

$$\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{K}(z), \mathcal{K}(z) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{K}(z)\big) \xrightarrow{\beta} \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{N}, \mathcal{K}(z)\big)$$

induced by the canonical maps

$$\mathcal{O}_{\mathfrak{X}} \to \mathcal{K}(z) \quad \text{and} \quad \mathcal{K}(z) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{K}(z) \to \mathcal{K}(z).$$

It is clear that $\beta \circ \alpha = \mathrm{id}$. By Corollary 4.9, we have that $\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} = 0$ for all $\mathcal{N} \in \mathcal{L}$ and $\mathcal{G} \in \mathcal{L}_z$, and necessarily,

$$\mathrm{Hom}_{\mathfrak{X}}\big(\mathcal{N}, \mathcal{K}(z)\big) = 0,$$

therefore, $\mathcal{K}(z)$ is $\mathcal{L}$-local. □

Lemma 4.16. *Suppose that $\mathfrak{X}$ is either separated or of finite Krull dimension and let $\mathcal{L}$ be a rigid localizing subcategory of $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ and $z \in \mathfrak{X}$. If $z \notin \psi(\mathcal{L})$, then $\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{G}, \mathcal{F})$ is a $\mathcal{L}$-local objects for every $\mathcal{F} \in \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ and $\mathcal{G} \in \mathcal{L}_z$.*

Proof. By Corollary 4.9, we have that

$$\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{N}, \boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{G}, \mathcal{F})\big) \cong \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G}, \mathcal{F}\big) = 0$$

for every $\mathcal{N} \in \mathcal{L}$, from which it follows that $\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{G}, \mathcal{F})$ is $\mathcal{L}$-local. □

5. Compatibility of localization with the monoidal structure

In this section $\mathfrak{X}$ will denote a noetherian scheme that is either separated or of finite Krull dimension. Let $\mathcal{L}$ be a localizing subcategory of $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ with associated Bousfield localization functor ℓ. For every $\mathcal{F} \in \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ there is a canonical distinguished triangle

$$\gamma\mathcal{F} \to \mathcal{F} \to \ell\mathcal{F} \xrightarrow{+} \tag{4}$$

such that $\gamma\mathcal{F} \in \mathcal{L}$ and $\ell\mathcal{F} \in \mathcal{L}^{\perp}$ (in other words, $\ell\mathcal{F}$ is $\mathcal{L}$-local). The functor γ is called the acyclization or colocalization associated to $\mathcal{L}$ and was denoted ℓ^a in [4]. Here we have changed the notation for clarity. The endofunctors γ and ℓ are idempotent in a functorial sense as explained in Section 1 of loc. cit. For all $\mathcal{F}, \mathcal{G} \in \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ we have the following canonical isomorphisms:

$$\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\gamma\mathcal{F}, \gamma\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\gamma\mathcal{F}, \mathcal{G}),$$
$$\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\ell\mathcal{F}, \ell\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\mathcal{F}, \ell\mathcal{G})$$

induced by $\gamma\mathcal{G} \to \mathcal{G}$ and $\mathcal{F} \to \ell\mathcal{F}$, respectively.

Lemma 5.1. *With the previous notation, the following are equivalent*:

(i) *The localizing subcategory $\mathcal{L}$ is rigid.*
(ii) *The natural transformation $\gamma\mathcal{G} \to \mathcal{G}$ induces isomorphisms*

$$\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\gamma\mathcal{F}, \gamma\mathcal{G}) \cong \boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\gamma\mathcal{F}, \mathcal{G})$$

for every $\mathcal{F}, \mathcal{G} \in \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$.
(iii) *The natural transformation $\mathcal{F} \to \ell\mathcal{F}$ induces isomorphisms*

$$\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\ell\mathcal{F}, \ell\mathcal{G}) \cong \boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F}, \ell\mathcal{G})$$

for every $\mathcal{F}, \mathcal{G} \in \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$.

Proof. Let us show (i) ⇒ (ii). Let $\mathcal{N} \in \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$, we have the following chain of isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{N}, \boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\gamma\mathcal{F}, \gamma\mathcal{G})\big) &\cong \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \gamma\mathcal{F}, \gamma\mathcal{G}\big)\\ &\overset{a}{\cong} \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \gamma\mathcal{F}, \mathcal{G}\big)\\ &\cong \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{N}, \boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\gamma\mathcal{F}, \mathcal{G})\big),\end{aligned}$$

where a is an isomorphism because $\mathcal{L}$ is rigid and therefore $\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \gamma\mathcal{F} = \gamma(\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \gamma\mathcal{F})$. Having an isomorphism for every $\mathcal{N} \in \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ forces the target complexes to be isomorphic.

We will see now (ii) ⇒ (iii). From (4), we have a distinguished triangle

$$\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\ell\mathcal{F}, \ell\mathcal{G}) \to \boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F}, \ell\mathcal{G}) \to \boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\gamma\mathcal{F}, \ell\mathcal{G}) \xrightarrow{+}$$

but its third point is null, considering

$$\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\gamma\mathcal{F}, \ell\mathcal{G}) \overset{\text{(ii)}}{\cong} \boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\gamma\mathcal{F}, \gamma\ell\mathcal{G}) = 0,$$

because $\gamma\ell\mathcal{G} = 0$.

Finally, let us see that (iii) ⇒ (i). Take $\mathcal{F} \in \mathcal{L}$ and $\mathcal{N} \in \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$. To see that $\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} \in \mathcal{L}$ it is enough to check that $\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}, \mathcal{G}) = 0$ for every $\mathcal{G} \in \mathcal{L}^{\perp}$ because ${}^{\perp}(\mathcal{L}^{\perp}) = \mathcal{L}$. But this is true:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}, \mathcal{G}\big) &\cong \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{N}, \boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G})\big)\\ &\overset{b}{\cong} \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{N}, \boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\ell\mathcal{F}, \ell\mathcal{G})\big)\\ &= 0,\end{aligned}$$

where b is an isomorphism, as follows from (iii) and the fact that $\mathcal{G} = \ell\mathcal{G}$, and the last equality holds because $\mathcal{F} \in \mathcal{L}$ and so $\ell\mathcal{F} = 0$. □

Example. Let Z be a closed subset of $\mathfrak{X}$, or more generally, a set stable for specialization.[3] Recall the functor sections with support $\Gamma_Z : \mathcal{A}_{\mathrm{qct}}(\mathfrak{X}) \to \mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$. From Section 2.3, we see that $\mathbf{R}\Gamma_Z : \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})) \to \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$, its derived functor, together with the natural transformation $\mathbf{R}\Gamma_Z \to \mathrm{id}$ posses the formal properties of an acyclization such that the associated localizing subcategory

$$\mathcal{L} = \big\{\mathcal{M} \in \mathbf{D}\big(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})\big) \;\big|\; \mathbf{R}\Gamma_Z(\mathcal{M}) = \mathcal{M}\big\}$$

is rigid.

[3] See Section 2.2.

The functor $\mathbf{R}\Gamma_Z$ has the following property:

$$\mathbf{R}\Gamma_Z\big(\mathcal{K}(x)\big) = \begin{cases} 0 & \text{if } x \notin Z, \\ \mathcal{K}(x) & \text{if } x \in Z. \end{cases}$$

Indeed, if $x \notin Z$ by Lemma 4.15, $\mathbf{R}\Gamma_Z(\mathcal{K}(x)) = 0$. On the contrary, if $x \in Z$, then $\mathcal{K}(x) \in \mathcal{L}_x \subset \mathcal{L}_Z$, so $\mathbf{R}\Gamma_Z(\mathcal{K}(x)) = \mathcal{K}(x)$. It follows that $\mathcal{L}$ has to agree with $\mathcal{L}_Z$ by Theorem 4.12 and, consequently, $\mathbf{R}\Gamma_Z$ is γ_Z, the acyclization functor associated to the localizing subcategory $\mathcal{L}_Z$. This acyclization functor satisfies a special property, namely, $\gamma_Z(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \mathcal{G})$ and $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \gamma_Z\mathcal{G}$ are canonically isomorphic, see Section 2.2.

5.2. Let $\mathcal{L}$ be a rigid localizing subcategory of $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ and $\mathcal{F}, \mathcal{G} \in \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$. The morphism $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \gamma\mathcal{G} \to \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \mathcal{G}$ induced by $\gamma\mathcal{G} \to \mathcal{G}$ factors naturally through $\gamma(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \mathcal{G})$ giving a natural morphism

$$t : \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \gamma\mathcal{G} \to \gamma\big(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \mathcal{G}\big).$$

Let us denote by

$$p : \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \ell\mathcal{G} \to \ell\big(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \mathcal{G}\big)$$

a morphism such that the diagram

$$\begin{array}{ccccccc}
\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \gamma\mathcal{G} & \longrightarrow & \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \mathcal{G} & \longrightarrow & \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \ell\mathcal{G} & \xrightarrow{+} & \\
\downarrow{\scriptstyle t} & & \| & & \downarrow{\scriptstyle p} & & \\
\gamma\big(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \mathcal{G}\big) & \longrightarrow & \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \mathcal{G} & \longrightarrow & \ell\big(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \mathcal{G}\big) & \xrightarrow{+} &
\end{array}$$

is a morphism of distinguished triangles. In fact, the triangle is functorial in the sense that the map p is uniquely determined by t due to the fact that $[\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \gamma\mathcal{G}, \ell(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \mathcal{G})[-1]) = 0]$.

We say that the localization ℓ is $\otimes$-*compatible* (or that $\mathcal{L}$ is $\otimes$-compatible or that γ is $\otimes$-compatible) if the canonical morphism t, or equivalently p, is an isomorphism.

We remind the reader our convention that $\mathcal{O}'_\mathfrak{X}$ denotes $\mathbf{R}\Gamma'_\mathfrak{X}\mathcal{O}_\mathfrak{X}$.

Theorem 5.3. *In the previous hypothesis we have the following equivalent statements*:

(i) *The localization associated to $\mathcal{L}$ is $\otimes$-compatible.*
(ii) *For every $\mathcal{E} \in \mathcal{L}^\perp$ and $\mathcal{F} \in \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ we have that $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} \mathcal{E} \in \mathcal{L}^\perp$.*
(iii) *The functor ℓ preserves coproducts.*
(iv) *A coproduct of $\mathcal{L}$-local objects is $\mathcal{L}$-local.*
(v) *The set $Z := \psi(\mathcal{L})$ is stable for specialization and its associated acyclization functor is $\gamma = \mathbf{R}\Gamma_Z$.*

Proof. Let us begin proving the non-trivial part of (i) $\Leftrightarrow$ (ii). Indeed, suppose that (ii) holds and for $\mathcal{F}, \mathcal{G} \in \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ consider the triangle

$$\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \gamma\mathcal{G} \to \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} \to \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \ell\mathcal{G} \xrightarrow{+};$$

we have that $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \gamma\mathcal{G} \in \mathcal{L}$ because $\mathcal{L}$ is rigid, on the other hand, $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \ell\mathcal{G} \in \mathcal{L}^{\perp}$ because $\ell\mathcal{G} \in \mathcal{L}^{\perp}$. The fact that the natural maps

$$\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \gamma\mathcal{G} \xrightarrow{t} \gamma\big(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G}\big) \quad \text{and} \quad \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \ell\mathcal{G} \xrightarrow{p} \ell\big(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G}\big)$$

are isomorphisms follows from [4, Proposition 1.6, (vi) $\Rightarrow$ (i)].

Let us see now that (i) $\Rightarrow$ (iii). If the localization associated to $\mathcal{L}$ is $\otimes$-compatible we have that, for $\mathcal{F} \in \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$,

$$\ell\mathcal{F} \cong \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \ell\mathcal{O}'_{\mathfrak{X}}$$

from which is clear that ℓ preserves coproducts.

The implication (iii) $\Rightarrow$ (iv) is obvious because $\ell\mathcal{F} \cong \mathcal{F}$ if, and only if, $\mathcal{F} \in \mathcal{L}^{\perp}$. To see that (iv) $\Rightarrow$ (v), we will use an argument similar to the one in the affine case [14, Lemma 3.7]. Assume that ℓ preserves coproducts. Let $x \in Z$ and $z \in \overline{\{x\}}$. If $z \notin Z$, then $\mathcal{K}(z) \in \mathcal{L}^{\perp}$ (Lemma 4.15) and $\mathcal{L}_z \subset \mathcal{L}^{\perp}$ because by (iv) $\mathcal{L}^{\perp}$ is localizing, and it follows by Corollary 4.2 that also $\mathcal{E}(z) \in \mathcal{L}^{\perp}$. But $\mathcal{E}(x) \in \mathcal{L}$ which contradicts the existence on a non-zero map $\mathcal{E}(x) \to \mathcal{E}(z)$ because $z \in \overline{\{x\}}$. Therefore Z, is stable for specialization and $\gamma \cong \mathbf{R}\Gamma_Z$ by the example following Lemma 5.1. The same example shows that (v) $\Rightarrow$ (i). $\square$

Remark. In the category of stable homotopy, Ho**Sp**, the localizations for which condition (iii) holds are called *smashing*. This can be characterized by a condition analogous to (i) in terms of its monoidal structure via the smash product, $\wedge$. So, the previous result classifies smashing localizations in $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$.

Corollary 5.4. *There is a bijection between the class of $\otimes$-compatible localizations of $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ and the set of subsets stable for specialization of $\mathfrak{X}$.*

In [13, §1.4], Lipman defines an *idempotent pair* for a closed category. In the case in which the closed category is $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$, it is a pair $(\mathcal{E}, \alpha)$ where $\mathcal{E} \in \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ and $\alpha : \mathcal{E} \to \mathcal{O}'_{\mathfrak{X}}$ is such that $\mathrm{id}_{\mathcal{E}} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \alpha$ and $\alpha \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathrm{id}_{\mathcal{E}}$ are equal isomorphisms from $\mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}$ to $\mathcal{E}$.

Corollary 5.5. *There is a bijective correspondence between $\otimes$-compatible localizations and idempotent pairs in $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$.*

Proof. A $\otimes$-compatible localization associated to the stable for specialization subset Z gives an idempotent pair $(\mathbf{R}\Gamma_Z(\mathcal{O}'_{\mathfrak{X}}), t)$ with $t : \mathbf{R}\Gamma_Z(\mathcal{O}'_{\mathfrak{X}}) \to \mathcal{O}'_{\mathfrak{X}}$ the canonical map. The

condition that $\mathrm{id}\otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}} t$ and $t\otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}}\mathrm{id}$ are equal isomorphisms is simply the fact that $\mathbf{R}\Gamma_Z$ is an acyclization functor associated to a $\otimes$-compatible localization.

Given an idempotent pair $(\mathcal{E},\alpha)$, define the endofunctor γ by $\gamma(\mathcal{F}) := \mathcal{F}\otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}}\mathcal{E}$ and analogously for morphisms. The idempotence of γ follows from the condition of idempotent pair, which also ensures that it is $\otimes$-compatible. These constructions are mutually inverse because if $Z\subset\mathfrak{X}$ is the stable for specialization subset associated to γ, then $\mathbf{R}\Gamma_Z(\mathcal{O}'_\mathfrak{X}) = \gamma(\mathcal{O}'_\mathfrak{X}) = \mathcal{O}'_\mathfrak{X}\otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}}\mathcal{E} = \mathcal{E}$. □

For a complex $\mathcal{F}\in\mathbf{D}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}))$ we define its *homological support* as the union of the supports of its homologies, i.e., $\operatorname{Supph}\mathcal{F} := \bigcup_{i\in\mathbb{Z}}\operatorname{Supp}\mathcal{H}^i(\mathcal{F})$. Note that $\operatorname{Supph}\mathcal{F}$ is always a subset of $\mathfrak{X}$ stable for specialization. In fact, it can be characterized in terms of cohomology with supports, as the following result shows.

Theorem 5.6. *Let $Z\subset\mathfrak{X}$ be a stable for specialization subset, for $\mathcal{F}\in\mathbf{D}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}))$, we have the following equivalent conditions*:

(i) $\mathbf{R}\Gamma_Z\mathcal{F}\simeq\mathcal{F}$.
(ii) $\mathcal{F}\in\mathcal{L}_Z$.
(iii) $\operatorname{Supph}\mathcal{F}\subset Z$.

Proof. The equivalence (i) ⇔ (ii) follows from the fact that $\mathcal{L}_Z$ is a localizing subcategory with associated Bousfield acyclization $\mathbf{R}\Gamma_Z$ as is explained in the example following Lemma 5.1. The implication (i) ⇒ (iii) is clear because $\operatorname{Supph}\mathbf{R}\Gamma_Z\mathcal{F}\subset Z$, as $\mathbf{R}\Gamma_Z\mathcal{F}$ is computed by a complex formed by sheaves already supported in Z.

Let us show then that (iii) ⇒ (ii). By Corollary 4.11, it is enough to check that $\mathcal{K}(x)\otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}}\mathcal{F}\in\mathcal{L}_Z$, for all $x\in\mathfrak{X}$. If $x\in Z$, then $\mathcal{K}(x)\otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}}\mathcal{F}\in\mathcal{L}_x\subset\mathcal{L}_Z$. For $x\notin Z$, $\mathfrak{X}_x\cap Z=\emptyset$ because Z is stable for specialization. Let us consider the chain of isomorphisms:

$$\mathcal{K}(x)\otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}}\mathcal{F}\simeq\mathbf{R}i_{x*}i_x^*\mathcal{K}(x)\otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}}\mathcal{F}\overset{(2.4)}{\simeq}\mathcal{K}(x)\otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}}\mathbf{R}i_{x*}i_x^*\mathcal{F}.$$

Note that $\mathbf{R}i_{x*}i_x^*\mathcal{F}=0$ because $\operatorname{Supph}\mathcal{F}\subset Z\subset\mathfrak{X}\setminus\mathfrak{X}_x$, therefore we conclude that $\mathcal{K}(x)\otimes^{\mathbf{L}}_{\mathcal{O}_\mathfrak{X}}\mathcal{F}=0$. □

This last result allows us to compare our classification of $\otimes$-compatible localizations with Thomason's localization. It says [18, Theorem 3.15] that there is a bijection between the set of subsets stable for specialization of a quasi-compact quasi-separated scheme X and the set of thick triangulated $\otimes$-subcategories of $\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(X))_{\mathsf{cp}}$. We recall that a triangulated subcategory $\mathcal{B}\subset\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(X))_{\mathsf{cp}}$ is called *thick* if it is stable for direct summands and is called by Thomason a $\otimes$-subcategories if it is a $\otimes$-ideal, i.e., the same condition that we use to define rigid localizing subcategories. If X is noetherian and separated we are able to compare this classification with ours, which is expressed in Corollary 5.4. We have the following proposition.

Proposition 5.7. *Let X be a noetherian separated scheme. There is a bijection between the set of $\otimes$-compatible localizing subcategories of $\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))$ and the set of thick triangulated $\otimes$-subcategories of $\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))_{\mathrm{cp}}$. This bijection is compatible with the classification of both sets in terms of stable for specialization subsets of X.*

Proof. Denote by $\mathsf{Loc}_{\otimes}(\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X)))$ the set of $\otimes$-compatible localizing subcategories of $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ and by $\mathsf{Th}_{\otimes}(\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))_{\mathrm{cp}})$ the set of thick triangulated $\otimes$-subcategories of $\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))_{\mathrm{cp}}$. Let us define a couple of maps:

$$\mathsf{Loc}_{\otimes}\big(\mathbf{D}\big(\mathcal{A}_{\mathrm{qc}}(X)\big)\big) \underset{g}{\overset{f}{\rightleftarrows}} \mathsf{Th}_{\otimes}\big(\mathbf{D}\big(\mathcal{A}_{\mathrm{qc}}(X)\big)_{\mathrm{cp}}\big),$$

and check that they are mutual inverses. For a $\otimes$-compatible localizing subcategory $\mathcal{L}$ we define $f(\mathcal{L}) := \mathcal{L} \cap \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))_{\mathrm{cp}}$ which is clearly a thick triangulated $\otimes$-subcategory. For such a subcategory $\mathcal{B}$ we define $g(\mathcal{B})$ as the smallest localizing subcategory $\mathcal{L}(\mathcal{B})$ of $\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))$ that contains $\mathcal{B}$. Let us show that $\mathcal{L}(\mathcal{B})$ is $\otimes$-compatible. For $\mathcal{N} \in \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))_{\mathrm{cp}}$, define $\mathcal{L}_0 = \{\mathcal{M} \in \mathcal{L}(\mathcal{B}) \mid \mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{N} \in \mathcal{L}(\mathcal{B})\}$. Note that $\mathcal{L}_0$ is a localizing subcategory of $\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))$ and that $\mathcal{B} \subset \mathcal{L}_0 \subset \mathcal{L}(\mathcal{B})$, so $\mathcal{L}_0 = \mathcal{L}(\mathcal{B})$. Therefore, $\mathcal{L}' := \{\mathcal{N} \in \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X)) \mid \mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{N} \in \mathcal{L}(\mathcal{B}), \forall \mathcal{M} \in \mathcal{L}(\mathcal{B})\}$ is a localizing subcategory of $\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))$ that contains $\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))_{\mathrm{cp}}$. Applying [15, Proposition 2.5], we conclude that $\mathcal{L}' = \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))$, therefore $\mathcal{L}(\mathcal{B})$ is rigid. The coproduct of $\mathcal{L}(\mathcal{B})$-local objects is again $\mathcal{L}(\mathcal{B})$-local because $\mathcal{L}(\mathcal{B})$ is generated by perfect complexes. Then, $\mathcal{L}(\mathcal{B})$ is $\otimes$-compatible by Theorem 5.3.

First, let us see that $f(g(\mathcal{B})) = \mathcal{B}$. By the cited Thomason's result there is a stable for specialization subset Z of X such that $\mathcal{B}$ is the class of all perfect complex with homological support contained in Z. It follows that the smallest localizing subcategory that contains $\mathcal{B}$, $\mathcal{L}(\mathcal{B})$, is contained in $\mathcal{L}_Z$ because all of is complexes are supported in Z by Theorem 5.6. Now $\mathcal{L}(\mathcal{B})$ is $\otimes$-compatible, so there is a stable for specialization subset $Z' \subset Z$ of X such that $\mathcal{L}(\mathcal{B}) = \mathcal{L}_{Z'}$. But Z' has to agree with Z, otherwise by [18, Lemma 3.4] we could find a perfect complex in $\mathcal{B}$ with homological support outside Z', a contradiction. So, necessarily $\mathcal{L}(\mathcal{B}) = \mathcal{L}_Z$ and $\mathcal{L}(\mathcal{B}) \cap \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))_{\mathrm{cp}} = \mathcal{B}$.

Take now a $\otimes$-compatible localizing subcategory $\mathcal{L} \subset \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))$. By Corollary 5.4, there is a subset $Z \subset \mathfrak{X}$ stable for specialization such that $\mathcal{L} = \mathcal{L}_Z$ which means that the objects in $\mathcal{B} := \mathcal{L} \cap \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))_{\mathrm{cp}}$ are perfect complexes whose homological support is contained in Z. The localizing subcategory $\mathcal{L}' := g(f(\mathcal{L}))$ is the smallest one that contains the objects of $\mathcal{B}$, so $\mathcal{L}' \subset \mathcal{L}$. The localizing subcategory $\mathcal{L}'$ is $\otimes$-compatible, then there is a stable for specialization subset $Z' \subset Z$ of X such that $\mathcal{L}' = \mathcal{L}_{Z'}$. But observe that Z' has to agree with Z arguing as before with the perfect complexes in the thick $\otimes$-subcategory $f(\mathcal{L})$. □

Corollary 5.8. *In the previous situation, a $\otimes$-compatible localizing subcategory of $\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))$ is generated by perfect complexes.*

5.9. Let $\mathcal{L}$ be a rigid localizing subcategory of $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ and $\mathcal{F},\mathcal{G}\in\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$. The morphism $\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\mathcal{G})\to\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\ell\mathcal{G})$ induced by $\mathcal{G}\to\ell\mathcal{G}$ factors through $\ell\,\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\mathcal{G})$ by Proposition 3.1. So, it gives a natural morphism

$$q:\ell\,\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\mathcal{G})\to\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\ell\mathcal{G}).$$

Let us denote by

$$h:\gamma\,\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\mathcal{G})\to\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\gamma\mathcal{G})$$

the morphism such that the diagram

$$\begin{array}{ccccccc}
\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\gamma\mathcal{G}) & \longrightarrow & \boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\mathcal{G}) & \longrightarrow & \boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\ell\mathcal{G}) & \xrightarrow{+} & \\
\uparrow{\scriptstyle h} & & \| & & \uparrow{\scriptstyle q} & & \\
\gamma\,\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\mathcal{G}) & \longrightarrow & \boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\mathcal{G}) & \longrightarrow & \ell\,\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\mathcal{G}) & \xrightarrow{+} &
\end{array}$$

is a morphism of distinguished triangles. Again, h and q determine each other.

With the notation of the previous remark, we say that the localization ℓ is *$\mathcal{H}om$-compatible* (or that $\mathcal{L}$ is $\mathcal{H}om$-compatible or that γ is $\mathcal{H}om$-compatible) if the canonical morphism q, or equivalently h, is an isomorphism.

5.10. Let $\mathcal{L}_Z$ be a $\otimes$-compatible localizing subcategory of $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ whose associated (stable for specialization) subset is $Z\subset\mathfrak{X}$. Let us apply the functor $\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}(-,\mathcal{F})$, where $\mathcal{F}\in\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$, to the canonical triangle

$$\gamma_Z\mathcal{O}'_{\mathfrak{X}}\to\mathcal{O}'_{\mathfrak{X}}\to\ell_Z\mathcal{O}'_{\mathfrak{X}}\xrightarrow{+}$$

associated to $\mathcal{L}_Z$. We have added the associated subsets as subindices for clarity. We obtain:

$$\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}\big(\ell_Z\mathcal{O}'_{\mathfrak{X}},\mathcal{F}\big)\to\mathcal{F}\to\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}\big(\gamma_Z\mathcal{O}'_{\mathfrak{X}},\mathcal{F}\big)\xrightarrow{+}. \tag{5}$$

Proposition 5.11. *The canonical natural transformations*

$$\mathrm{id}\to\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}\big(\gamma_Z\mathcal{O}'_{\mathfrak{X}},-\big)\quad\textit{and}\quad\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}\big(\ell_Z\mathcal{O}'_{\mathfrak{X}},-\big)\to\mathrm{id},$$

correspond to a $\mathcal{H}om$-compatible localization and its corresponding acyclization in $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$, respectively. Its associated subset of $\mathfrak{X}$ is $\mathfrak{X}\setminus Z$.

Proof. Note that (5) is a Bousfield localization triangle because $\mathcal{L}_Z$ is $\otimes$-compatible. The associated localizing subcategory

$$\mathcal{L}=\big\{\mathcal{M}\in\mathbf{D}\big(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})\big)\;\big|\;\boldsymbol{\mathcal{H}}\mathrm{om}^{\cdot}_{\mathfrak{X}}\big(\gamma_Z\mathcal{O}'_{\mathfrak{X}},\mathcal{M}\big)=0\big\}$$

satisfies that ${}^{\perp}(\mathcal{L}^{\perp})=\mathcal{L}$ [4, Proposition 1.6]. Furthermore, the canonical isomorphisms

$$\begin{aligned}\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}\big(\mathcal{F},\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}\big(\gamma_Z\mathcal{O}'_{\mathfrak{X}},\mathcal{G}\big)\big)&\cong\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}\big(\mathcal{F}\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}}\gamma_Z\mathcal{O}'_{\mathfrak{X}},\mathcal{G}\big)\\&\cong\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}\big(\gamma_Z\mathcal{O}'_{\mathfrak{X}},\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\mathcal{G})\big)\end{aligned}$$

show that $\mathcal{L}$ is rigid (Proposition 3.1) and $\mathcal{H}om$-compatible.

Let us check that $\mathcal{L}=\mathcal{L}_{\mathfrak{X}\setminus Z}$. Let $z\in\mathfrak{X}$, we will consider two possibilities depending on the point being or not in Z. First, if $z\in\mathfrak{X}\setminus Z$, it follows that $\mathcal{K}(z)\in\mathcal{L}_Z^{\perp}$ by Lemma 4.15 and therefore we have that

$$\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}\big(\gamma_Z\mathcal{O}'_{\mathfrak{X}},\mathcal{K}(z)\big)\xleftarrow{\sim}\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}\big(\gamma_Z\mathcal{O}'_{\mathfrak{X}},\gamma_Z\mathcal{K}(z)\big)=0,$$

then

$$\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}\big(\ell_Z\mathcal{O}'_{\mathfrak{X}},\mathcal{K}(z)\big)\xrightarrow{\sim}\mathcal{K}(z).$$

For $z\in Z$ we will show that $\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\ell_Z\mathcal{O}'_{\mathfrak{X}},\mathcal{K}(z))=0$. By Proposition 4.4, it is enough to prove that

$$\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{K}(y),\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}\big(\ell_Z\mathcal{O}'_{\mathfrak{X}},\mathcal{K}(z)\big)\big)=0,\quad\forall y\in\mathfrak{X},$$

equivalently that

$$\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{K}(y)\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}}\ell_Z\mathcal{O}'_{\mathfrak{X}},\mathcal{K}(z)\big)=0,\quad\forall y\in\mathfrak{X}.$$

The localization functor ℓ_Z is $\otimes$-compatible so $\mathcal{K}(y)\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}}\ell_Z\mathcal{O}'_{\mathfrak{X}}\cong\ell_Z\mathcal{K}(y)$ will be zero if $y\in Z$. On the other hand, if $y\in\mathfrak{X}\setminus Z$ we conclude because $\mathcal{K}(y)\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}}\ell_Z\mathcal{O}'_{\mathfrak{X}}\in\mathcal{L}_y$ and $\mathcal{K}(z)\in\mathcal{L}_y^{\perp}$ (Lemma 4.15). □

5.12. Note that the following adjunction is completely formal:

$$\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\gamma_Z\mathcal{F},\mathcal{G})\xrightarrow{\sim}\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\ell_{\mathfrak{X}\setminus Z}\mathcal{G}).$$

Indeed, it is the composition of the following natural isomorphisms:

$$\begin{aligned}\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\gamma_Z\mathcal{F},\mathcal{G})&\cong\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}\big(\gamma_Z\mathcal{O}'_{\mathfrak{X}}\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{F},\mathcal{G}\big)\cong\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}\big(\mathcal{F},\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\gamma_Z\mathcal{O}'_{\mathfrak{X}},\mathcal{G})\big)\\&\cong\mathcal{H}\mathrm{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F},\ell_{\mathfrak{X}\setminus Z}\mathcal{G}).\end{aligned}$$

Example. Let now Z be a closed subset of X which we assume it is an ordinary (noetherian separated) scheme. and $\mathbf{L}\Lambda_Z:\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(X))\to\mathbf{D}(\mathcal{A}(X))$ the left-derived functor of the completion along the closed subset Z (which exist because it can be computed using quasi-coherent flat resolutions, as proved in [1]). In loc. cit. it is also shown there is a natural isomorphism

$$\mathcal{H}\mathrm{om}^{\cdot}_{X}(\mathbf{R}\Gamma_Z\mathcal{O}_X,\mathcal{G})\xrightarrow{\sim}\mathbf{R}Q\mathbf{L}\Lambda_Z(\mathcal{G}).$$

This result together with the previous adjunction is often referred to as *Greenlees–May duality* because it generalizes a result from [6] in the affine case.

5.13. In general, if $Z \in \mathfrak{X}$ is a stable for specialization subset of $\mathfrak{X}$, we will define for every $\mathcal{G} \in \mathbf{D}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}))$:

$$\mathbf{\Lambda}_Z(\mathcal{G}) := \boldsymbol{\mathcal{H}}\mathsf{om}^{\cdot}_{\mathfrak{X}}\big(\mathbf{R}\Gamma_Z\mathcal{O}'_{\mathfrak{X}}, \mathcal{G}\big).$$

Theorem 5.14. *For a rigid localizing subcategory $\mathcal{L} \subset \mathbf{D}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}))$, the following are equivalent*:

(i) *The localization associated to $\mathcal{L}$ is $\mathcal{H}$om-compatible.*
(ii) *For every $\mathcal{N} \in \mathcal{L}$ and $\mathcal{F} \in \mathbf{D}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}))$ we have that $\boldsymbol{\mathcal{H}}\mathsf{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F}, \mathcal{N}) \in \mathcal{L}$.*
(iii) *The set $Y := \psi(\mathcal{L})$ is generically stable*[4] *and its associated localization functor is $\mathbf{\Lambda}_Z$ with $Z = \mathfrak{X} \setminus Y$.*

Proof. Let us see first that (ii) $\Rightarrow$ (iii). Let $z \in Y$ and $x \in \mathfrak{X}$ such that $z \in \overline{\{x\}}$. With the notation of Corollary 4.2, if $x \notin Y$, then by Lemma 4.16, $\boldsymbol{\mathcal{H}}\mathsf{om}^{\cdot}_{\mathfrak{X}}(\mathcal{E}(x), \mathcal{E}(z)) \in \mathcal{L}^{\perp}$. By (ii), $\boldsymbol{\mathcal{H}}\mathsf{om}^{\cdot}_{\mathfrak{X}}(\mathcal{E}(x), \mathcal{E}(z))$ belongs to $\mathcal{L}$ because $\mathcal{E}(z) \in \mathcal{L}$. Therefore $\boldsymbol{\mathcal{H}}\mathsf{om}^{\cdot}_{\mathfrak{X}}(\mathcal{E}(x), \mathcal{E}(z)) = 0$ and we have

$$\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{E}(x), \mathcal{E}(z)\big) \cong \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}\big(\mathcal{O}'_{\mathfrak{X}}, \boldsymbol{\mathcal{H}}\mathsf{om}^{\cdot}_{\mathfrak{X}}\big(\mathcal{E}(x), \mathcal{E}(z)\big)\big) = 0,$$

a contradiction. Necessarily, the set $Z = \mathfrak{X} \setminus Y$ is stable for specialization and $\ell_Y = \mathbf{\Lambda}_Z$.

The implication (iii) $\Rightarrow$ (i) follows from the previous remarks and the bijective correspondence established in Theorem 4.12.

To finish, (i) $\Rightarrow$ (ii) is straightforward because for every $\mathcal{N} \in \mathcal{L}$, we have that

$$\boldsymbol{\mathcal{H}}\mathsf{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F}, \mathcal{N}) = \boldsymbol{\mathcal{H}}\mathsf{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F}, \gamma\mathcal{N}) \overset{\text{(i)}}{\cong} \gamma\,\boldsymbol{\mathcal{H}}\mathsf{om}^{\cdot}_{\mathfrak{X}}(\mathcal{F}, \mathcal{N}) \in \mathcal{L}. \qquad \square$$

Corollary 5.15. *The functor γ associated to a $\mathcal{H}$om-compatible localization in $\mathbf{D}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}))$ commutes with* products*, in particular, the corresponding localizing class $\mathcal{L}$ is closed for products.*

Proof. It is an immediate consequence of Theorem 5.14(iii) and that every complex in $\mathbf{K}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}))$ admits a K-flat resolution by [3, Proposition 2.1.3] and a K-injective resolution. $\square$

Corollary 5.16. *For a noetherian separated formal scheme $\mathfrak{X}$ there is a bijection between the class of $\mathcal{H}$om-compatible localizations of $\mathbf{D}(\mathcal{A}_{\mathsf{qct}}(\mathfrak{X}))$ and the set of generically stable subsets of $\mathfrak{X}$.*

[4] I.e., an arbitrary intersection of open subsets.

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