Norm closed operator ideals in Lorentz sequence spaces

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ABSTRACT

In this paper, we study the structure of closed algebraic ideals in the algebra of operators acting on a Lorentz sequence space.

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1. Introduction

1.1. Ideals

This paper is concerned with the study of the structure of closed algebraic ideals in the algebra $L(X)$ of all bounded linear operators on a Banach space X.

Throughout the paper, by a ***subspace*** of a Banach space we mean a closed subspace; a vector subspace of X which is not necessarily closed will be referred to as ***linear subspace***. A (two-sided) ***ideal*** in $L(X)$ is a linear subspace J of $L(X)$ such that $ATB \in J$ whenever $T \in J$ and $A, B \in L(X)$. The ideal J is called ***proper*** if $J \neq L(X)$. The ideal J is ***non-trivial*** if J is proper and $J \neq \{0\}$.

The spaces for which the structure of closed ideals in $L(X)$ is well understood are very few. It was shown in [7] that the only non-trivial closed ideal in the algebra $L(\ell_2)$ is the ideal of compact operators. This result was generalized in [13] to the spaces ℓ_p $(1 \leqslant p < \infty)$ and c_0. A space constructed recently in [5] is another space with this property. In [15] and [16], it was shown that the algebras $L((\bigoplus_{k=1}^{\infty} \ell_2^k)_{c_0})$ and $L((\bigoplus_{k=1}^{\infty} \ell_2^k)_{\ell_1})$ have exactly two non-trivial closed ideals. There are no other separable spaces for which the structure of closed ideals in $L(X)$ is completely known.

Partial results about the structure of closed ideals in $L(X)$ were obtained in [20, 5.3.9] for $X = L_p[0,1]$ $(1 < p < \infty$, $p \neq 2)$ and in [22] and [23] for $L(\ell_p \oplus \ell_q)$ $(1 \leqslant p, q < \infty)$. The purpose of this paper is to investigate the structure of ideals in $L(d_{w,p})$ where $d_{w,p}$ is a Lorentz sequence space (see the definition in Section 1.3).

For two closed ideals J_1 and J_2 in $L(X)$, we will denote by $J_1 \wedge J_2$ the largest closed ideal J in $L(X)$ such that $J \subseteq J_1$ and $J \subseteq J_2$ (that is, $J_1 \wedge J_2 = J_1 \cap J_2$), and we will denote by $J_1 \vee J_2$ the smallest closed ideal J in $L(X)$ such that $J_1 \subseteq J$ and $J_2 \subseteq J$. We say that J_2 is a ***successor*** of J_1 if $J_1 \subsetneq J_2$. If, in addition, no closed ideal J in $L(X)$ satisfies $J_1 \subsetneq J \subsetneq J_2$, then we call J_2 an ***immediate successor*** of J_1.

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It is well known that if X is a Banach space then every nonzero ideal in the algebra $L(X)$ must contain the ideal $\mathcal{F}(X)$ of all finite-rank operators on X. It follows that, at least in the presence of the approximation property (in particular, if X has a Schauder basis), every nonzero closed ideal in $L(X)$ contains the closed ideal $\mathcal{K}(X)$ of all compact operators.

Two ideals closely related to $\mathcal{K}(X)$ are the closed ideal $\mathcal{SS}(X)$ of strictly singular operators and the closed ideal $\mathcal{FSS}(X)$ of finitely strictly singular operators on X. Recall that an operator $T \in L(X)$ is called **strictly singular** if no restriction $T|_Z$ of T to an infinite-dimensional subspace Z of X is an isomorphism. An operator T is **finitely strictly singular** if for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that any subspace Z of X with $\dim Z \geqslant N$ contains a vector $z \in Z$ satisfying $\|Tz\| < \varepsilon \|z\|$. It is not hard to show that $\mathcal{K}(X) \subseteq \mathcal{FSS}(X) \subseteq \mathcal{SS}(X)$ (see [17,19,22,4] for more information about these classes of operators).

If X is a Banach space and $T \in L(X)$ then the ideal in $L(X)$ generated by T is denoted by J_T. It is easy to see that $J_T = \{\sum_{i=1}^{n} A_i T B_i \colon A_i, B_i \in L(X)\}$. It follows that if $S \in L(X)$ factors through T, i.e., $S = ATB$ for some $A, B \in L(X)$ then $J_S \subseteq J_T$.

1.2. Basic sequences

The main tool in this paper is the notion of a basic sequence. In this subsection, we will fix some terminology and remind some classical facts about basic sequences. For a thorough introduction to this topic, we refer the reader to [9] or [12].

If (x_n) is a sequence in a Banach space X then its closed span will be denoted by $[x_n]$. We say that a basic sequence (x_n) **dominates** a basic sequence (y_n) and write $(x_n) \succcurlyeq (y_n)$ if the convergence of a series $\sum_{n=1}^{\infty} a_n x_n$ implies the convergence of the series $\sum_{n=1}^{\infty} a_n y_n$. We say that (x_n) is **equivalent** to (y_n) and write $(x_n) \sim (y_n)$ if $(x_n) \succcurlyeq (y_n)$ and $(y_n) \succcurlyeq (x_n)$.

Remark 1.1. It follows from the Closed Graph Theorem that $(x_n) \succcurlyeq (y_n)$ if and only if the linear map from $\mathrm{span}\{x_n\}$ to $\mathrm{span}\{y_n\}$ defined by the formula $T : x_n \mapsto y_n$ is bounded.

If (x_n) is a basis in a Banach space X, $z = \sum_{i=1}^{\infty} z_i x_i \in X$, and $A \subseteq \mathbb{N}$ then the vector $\sum_{i \in A} z_i x_i$ will be denoted by $z|_A$ (provided the series converges; this is always the case when the basis is unconditional). We will refer to $z|_A$ as the **restriction of z to A**. The restrictions $z|_{[n,\infty) \cap \mathbb{N}}$ and $z|_{(n,\infty) \cap \mathbb{N}}$, where $n \in \mathbb{N}$, will be abbreviated as $z|_{[n,\infty)}$ and $z|_{(n,\infty)}$, respectively. We say that a vector v is a **restriction** of z if there exists $A \subseteq \mathbb{N}$ such that $v = z|_A$. The vector $z = \sum_{i=1}^{\infty} z_i x_i$ will also be denoted by $z = (z_i)$. If $z = \sum_{i=1}^{\infty} z_i x_i$ then the **support** of z is the set $\mathrm{supp}\, z = \{i \in \mathbb{N} \colon z_i \neq 0\}$.

Every 1-unconditional basis (x_n) in a Banach space X defines a Banach lattice order on X by $\sum_{i=1}^{\infty} a_i x_i \geqslant 0$ if and only if $a_i \geqslant 0$ for all $i \in \mathbb{N}$ (see, e.g., [18, p. 2]). For $x \in X$, we have $|x| = x \vee (-x)$. A Banach lattice is said to have **order continuous norm** if the condition $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \to 0$. For an introduction to Banach lattices and standard terminology, we refer the reader to [1, §1.2].

If (x_n) is a basic sequence in a Banach space X, then a sequence (y_n) in $\mathrm{span}\{x_n\}$ is a **block sequence** of (x_n) if there are a strictly increasing sequence (p_n) in $\mathbb{N}$ and a sequence of scalars (a_i) such that $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i x_i$ for all $n \in \mathbb{N}$.

The following two facts are classical and will sometimes be used without any references. The first fact is known as the Principle of Small Perturbations (see, e.g., [12, Theorem 4.23]).

Theorem 1.2. *Let X be a Banach space, (x_n) a basic sequence in X, and (x_n^*) the correspondent biorthogonal functionals defined on $[x_n]$. If (y_n) is a sequence such that $\sum_{n=1}^{\infty} \|x_n^*\| \cdot \|x_n - y_n\| < 1$ then (y_n) is a basic sequence equivalent to (x_n). Moreover, if $[x_n]$ is complemented in X then so is $[y_n]$. If $[x_n] = X$ then $[y_n] = X$.*

The next fact, which is often called the Bessaga–Pełczyński selection principle, is a result of combining the "gliding hump" argument (see, e.g., [9, Lemma 5.1]) with the Principle of Small Perturbations.

Theorem 1.3. *Let X be a Banach space with a seminormalized basis (x_n) and let (x_n^*) be the correspondent biorthogonal functionals. Let (y_n) be a seminormalized sequence in X such that $x_n^*(y_k) \xrightarrow{k \to \infty} 0$ for all $n \in \mathbb{N}$. Then (y_n) has a subsequence (y_{n_k}) which is basic and equivalent to a block sequence (u_k) of (x_n). Moreover, $y_{n_k} - u_k \to 0$, and u_k is a restriction of y_{n_k}.*

1.3. Lorentz sequence spaces

Let $1 \leqslant p < \infty$ and $w = (w_n)$ be a sequence in $\mathbb{R}$ such that $w_1 = 1$, $w_n \downarrow 0$, and $\sum_{i=1}^{\infty} w_i = \infty$. The Lorentz sequence space $d_{w,p}$ is a Banach space of all vectors $x \in c_0$ such that $\|x\|_{d_{w,p}} < \infty$, where

$$\big\|(x_n)\big\|_{d_{w,p}} = \left(\sum_{n=1}^{\infty} w_n x_n^{*p} \right)^{1/p}$$

is the norm in $d_{w,p}$. Here (x_n^*) is the **non-increasing rearrangement** of the sequence $(|x_n|)$. An overview of properties of Lorentz sequence spaces can be found in [17, Section 4.e].

The vectors (e_n) in $d_{w,p}$ defined by $e_n(i) = \delta_{ni}$ $(n, i \in \mathbb{N})$ form a 1-symmetric basis in $d_{w,p}$. In particular, (e_n) is 1-unconditional, hence $d_{w,p}$ is a Banach lattice. We call (e_n) the unit vector basis of $d_{w,p}$. The unit vector basis of ℓ_p will be denoted by (f_n) throughout the paper.

Remark 1.4. It is proved in [3, Lemma 1] and [10, Lemma 15] that if (u_n) is a seminormalized block sequence of (e_n) in $d_{w,p}$, $u_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$, such that $a_i \to 0$, then there is a subsequence (u_{n_k}) such that $(u_{n_k}) \sim (f_n)$ and $[u_{n_k}]$ is complemented in $d_{w,p}$. Further, it was shown in [3, Corollary 3] that if (y_n) is a seminormalized block sequence of (e_n) then there is a seminormalized block sequence (u_n) of (y_n) such that $u_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$, with $a_i \to 0$. Therefore, every infinite-dimensional subspace of $d_{w,p}$ contains a further subspace which is complemented in $d_{w,p}$ and isomorphic to ℓ_p [10, Corollary 17].

Remark 1.5. Remark 1.4 yields, in particular, that $d_{w,p}$ does not contain copies of c_0. Since the basis (e_n) of $d_{w,p}$ is unconditional, the space $d_{w,p}$ is weakly sequentially complete by [2, Theorem 4.60] (see also [17, Theorem 1.c.10]). Also, [2, Theorem 4.56] guarantees that $d_{w,p}$ has order continuous norm. In particular, if $x \in d_{w,p}$ then $\|x|_{[n,\infty)}\| \to 0$ as $n \to \infty$.

Remark 1.6. It was shown in [14] that if $p > 1$ then $d_{w,p}$ is reflexive. This can also be easily obtained from Remark 1.4 (cf. [17, Theorem 1.c.12]).

Remark 1.7. The unit vector basis (e_n) of $d_{w,p}$ is weakly null. Indeed, by Rosenthal's ℓ_1-theorem (see [21]; also [17, Theorem 2.e.5]), (e_n) is weakly Cauchy. Since it is symmetric, $(e_n) \sim (e_{2n} - e_{2n-1})$.

The next proposition will be used often in this paper.

Proposition 1.8. *(See [3, Proposition 5 and Corollary 2].) If (u_n) is a seminormalized block sequence of (e_n) then $(f_n) \succcurlyeq (u_n)$. If (u_n) does not contain subsequences equivalent to (f_n) then also $(u_n) \succcurlyeq (e_n)$.*

The following lemma is standard.

Lemma 1.9. *Let (x_n) be a block sequence of (e_n), $x_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$. If (y_n) is a basic sequence such that $y_n = \sum_{i=p_n+1}^{p_{n+1}} b_i e_i$, where $|b_i| \leqslant |a_i|$ for all $i \in \mathbb{N}$, then (x_n) is basic and $(x_n) \succcurlyeq (y_n)$.*
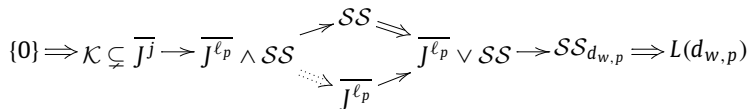
Proof. Let

$$\gamma_i = \begin{cases} \frac{b_i}{a_i}, & \text{if } a_i \neq 0, \\ 0, & \text{if } a_i = 0. \end{cases}$$

Define an operator $T \in L(d_{w,p})$ by $T(\sum_{i=1}^{\infty} c_i e_i) = \sum_{i=1}^{\infty} c_i \gamma_i e_i$. Then T is, clearly, linear and, since the basis (e_n) is 1-unconditional, T is bounded with $\|T\| \leqslant 1$. In particular, $T|_{[x_n]}$ is bounded. Also, $T(x_n) = y_n$ for all $n \in \mathbb{N}$, hence $(x_n) \succcurlyeq (y_n)$. $\square$

1.4. Outline of the results

The purpose of the paper is to uncover the structure of ideals in $L(d_{w,p})$. We show that (some of) these ideals can be arranged into the following diagram.

$$\{0\} \Longrightarrow \mathcal{K} \subsetneqq \overline{J^j} \longrightarrow \overline{J^{\ell_p}} \wedge \mathcal{SS} \begin{array}{c} \nearrow \mathcal{SS} \Longrightarrow \\ \\ \dashrightarrow \\ J^{\ell_p} \end{array} \overline{J^{\ell_p}} \vee \mathcal{SS} \longrightarrow \mathcal{SS}_{d_{w,p}} \Longrightarrow L(d_{w,p})$$

(the notations will be defined throughout the paper). On this diagram, a single arrow between ideals, $J_1 \longrightarrow J_2$, means that $J_1 \subseteq J_2$. A double arrow between ideals, $J_1 \Longrightarrow J_2$, means that J_2 is the only immediate successor of J_1 (in particular, $J_1 \neq J_2$), whereas a dotted double arrow between ideals, $J_1 \dashrightarrow J_2$, only shows that J_2 is an immediate successor for J_1 (in particular, J_1 may have other immediate successors).

While working with the diagram above, we obtain several important characterizations of some ideals in $L(d_{w,p})$. In particular, we show that $\mathcal{FSS}(d_{w,p}) = \mathcal{SS}(d_{w,p})$ (Theorem 3.5). We also characterize the ideal of weakly compact operators (Theorem 3.6) and Dunford–Pettis operators (Theorem 5.7) on $d_{w,p}$. We show in Theorem 4.7 that $\overline{J^j}$ is the only immediate successor of $\mathcal{K}$ under some assumption on the weights w. In the last section of the paper, we show that all strictly singular operators from ℓ_1 to $d_{w,1}$ can be approximated by operators factoring through the formal identity operator $j : \ell_1 \to d_{w,1}$ (see Section 4 for the definition). We also obtain a result on factoring positive operators from $\mathcal{SS}(d_{w,p})$ through the formal identity operator (Theorem 6.12).

2. Operators factorable through ℓ_p

Let X and Y be Banach spaces and $T \in L(X)$. We say that T **factors through** Y if there are two operators $A \in L(X, Y)$ and $B \in L(Y, X)$ such that $T = BA$.

The following two lemmas are standard. We present their proofs for the sake of completeness.

Lemma 2.1. *Let X and Y be Banach spaces and $T \in L(X, Y)$, $S \in L(Y, X)$ be such that $ST = \mathrm{id}_X$. Then T is an isomorphism and* Range T *is a complemented subspace of Y isomorphic to X.*

Proof. For all $x \in X$, we have $\|x\| = \|STx\| \leqslant \|S\| \|Tx\|$, so $\|Tx\| \geqslant \frac{1}{\|S\|} \|x\|$. This shows that T is an isomorphism. In particular, Range T is a closed subspace of Y isomorphic to X.

Put $P = TS \in L(Y)$. Then $P^2 = TSTS = T\,\mathrm{id}_X S = TS = P$, hence P is a projection. Clearly, Range $P \subseteq$ Range T. Also, $PT = TST = T$, so Range $T \subseteq$ Range P. Therefore Range $P =$ Range T, and Range T is complemented. $\quad\square$

Lemma 2.2. *Let X and Y be Banach spaces such that Y is isomorphic to $Y \oplus Y$. Then the set $J = \{T \in L(X): T \text{ factors through } Y\}$ is an ideal in $L(X)$.*

Proof. It is clear that J is closed under multiplication by operators in $L(X)$. In particular, J is closed under scalar multiplication. Let $A, B \in J$. Write $A = A_1 A_2$ and $B = B_1 B_2$, where $A_1, B_1 \in L(Y, X)$ and $A_2, B_2 \in L(X, Y)$. Then $A + B = UV$ where $V : x \in X \mapsto (A_2 x, B_2 x) \in Y \oplus Y$ and $U : (x, y) \in Y \oplus Y \mapsto A_1 x + B_1 y \in Y$. Clearly, UV factors through $Y \oplus Y \simeq Y$. Hence $A + B \in J$. $\quad\square$

We will denote the set of all operators in $L(d_{w,p})$ which factor through a Banach space Y by J^Y.

Theorem 2.3. *The sets J^{ℓ_p} and $\overline{J^{\ell_p}}$ are proper ideals in $L(d_{w,p})$.*

Proof. Since ℓ_p is isomorphic to $\ell_p \oplus \ell_p$, it follows from Lemma 2.2 that J^{ℓ_p} is an ideal in $L(d_{w,p})$. Let us show that $J^{\ell_p} \neq L(d_{w,p})$.

Assume that $J^{\ell_p} = L(d_{w,p})$, then the identity operator I on $d_{w,p}$ belongs to J. Write $I = ST$ where $T \in L(d_{w,p}, \ell_p)$ and $S \in L(\ell_p, d_{w,p})$. By Lemma 2.1, the range of T is complemented in ℓ_p and is isomorphic to $d_{w,p}$. This is a contradiction because all complemented infinite-dimensional subspaces of ℓ_p are isomorphic to ℓ_p (see, e.g., [17, Theorem 2.a.3]), while $d_{w,p}$ is not isomorphic to ℓ_p (see [6] for the case $p = 1$ and [14] for the case $1 < p < \infty$; see also [17, p. 176]).

Being the closure of a proper ideal, $\overline{J^{\ell_p}}$ is itself a proper ideal (see, e.g., [11, Corollary VII.2.4]). $\quad\square$

Proposition 2.4. *There exists a projection $P \in L(d_{w,p})$ such that* Range P *is isomorphic to ℓ_p. For every such P we have $J_P = J^{\ell_p}$.*

Proof. Such projections exist by Remark 1.4. Let $Y =$ Range P, $U : Y \to \ell_p$ be an isomorphism onto, and $i : Y \to d_{w,p}$ be the inclusion map. It is easy to see that $P = (iU^{-1})(UP)$, hence $P \in J^{\ell_p}$, so that $J_P \subseteq J^{\ell_p}$.
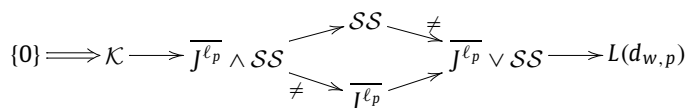
On the other hand, if $T \in J^{\ell_p}$ is arbitrary, $T = AB$ with $A \in L(\ell_p, d_{w,p})$, $B \in L(d_{w,p}, \ell_p)$, then one can write $T = (AUP)P(iU^{-1}B)$, so that $T \in J_P$. Thus $J^{\ell_p} \subseteq J_P$. $\quad\square$

Corollary 2.5. *The ideal $\overline{J^{\ell_p}}$ properly contains the ideal of compact operators $\mathcal{K}(d_{w,p})$.*

Proof. It was already mentioned in the introductory section that compact operators form the smallest closed ideal in $L(d_{w,p})$. Since a projection onto a subspace isomorphic to ℓ_p is not compact, it follows that $\mathcal{K}(d_{w,p}) \neq \overline{J^{\ell_p}}$. $\quad\square$

3. Strictly singular operators

In this section we will study properties of strictly singular operators in $L(d_{w,p})$. Since projections onto the subspaces of $d_{w,p}$ isomorphic to ℓ_p are clearly not strictly singular, it follows from Proposition 2.4 that $\mathcal{SS}(d_{w,p}) \neq J^{\ell_p}$. Moreover, $\mathcal{SS} \neq \overline{J^{\ell_p}} \vee \mathcal{SS}$ and $\overline{J^{\ell_p}} \wedge \mathcal{SS} \neq \overline{J^{\ell_p}}$. So, the ideals we discussed so far can be arranged as follows:

$$\{0\} \Longrightarrow \mathcal{K} \longrightarrow \overline{J^{\ell_p}} \wedge \mathcal{SS} \nearrow \overset{\mathcal{SS}}{\underset{\overline{J^{\ell_p}}}{\searrow}} \overset{\neq}{\searrow} \overline{J^{\ell_p}} \vee \mathcal{SS} \longrightarrow L(d_{w,p})$$

The following theorem shows that there can be no other closed ideals between $\mathcal{SS}$ and $\overline{J^{\ell_p}} \vee \mathcal{SS}$ on this diagram.

Theorem 3.1. *Let $T \in L(d_{w,p})$. If $T \notin \mathcal{SS}(d_{w,p})$ then $J^{\ell_p} \subseteq J_T$.*

Proof. Let $T \notin \mathcal{SS}(d_{w,p})$. Then there exists an infinite-dimensional subspace Y of $d_{w,p}$ such that $T|_Y$ is an isomorphism. By Remark 1.4, passing to a subspace, we may assume that Y is complemented in $d_{w,p}$ and isomorphic to ℓ_p. Let (x_n) be a basis of Y equivalent to the unit vector basis of ℓ_p. Define $z_n = Tx_n$, then (z_n) is also equivalent to the unit vector basis of ℓ_p. By Remark 1.4, (z_n) has a subsequence (z_{n_k}) such that $[z_{n_k}]$ is complemented in $d_{w,p}$ and isomorphic to ℓ_p.

Denote $W = [x_{n_k}]$. Then W and $T(W)$ are both complemented subspaces of $d_{w,p}$ isomorphic to ℓ_p. Let P and Q be projections onto W and $T(W)$, respectively. Put $S = (T|_W)^{-1}$, $S \in L(T(W), d_{w,p})$. Then it is easy to see that $P = (SQ)TP$. Since SQ and P are in $L(d_{w,p})$, we have $J_P \subseteq J_T$. By Proposition 2.4, $J^{\ell_p} \subseteq J_T$. $\quad\square$

Corollary 3.2. $\overline{J^{\ell_p}} \vee \mathcal{SS}(d_{w,p})$ *is the only immediate successor of* $\mathcal{SS}(d_{w,p})$ *and* $\overline{J^{\ell_p}}$ *is an immediate successor of* $\overline{J^{\ell_p}} \wedge \mathcal{SS}(d_{w,p})$.

Now we will investigate the ideal of finitely strictly singular operators on $d_{w,p}$. To prove the main statement (Theorem 3.5), we will need the following lemma due to Milman [19] (see also a thorough discussion in [22]). This lemma will be used more than once in the paper.

Lemma 3.3. *(See [19].) If F is a k-dimensional subspace of c_0 then there exists a vector $x \in F$ such that x attains its sup-norm on at least k coordinates* (*that is, x^* starts with a constant block of length k*).

We will also use the following simple lemma.

Lemma 3.4. *Let $s_n = \sum_{i=1}^n w_i$ ($n \in \mathbb{N}$) where $w = (w_i)$ is the sequence of weights for $d_{w,p}$. If $x \in d_{w,p}$, $y = x^*$, and $N \in \mathbb{N}$ then $0 \leqslant y_N \leqslant \frac{\|x\|}{s_N^{1/p}}$.*

Proof. $\|x\|^p = \|y\|^p = \sum_{i=1}^{\infty} y_i^p w_i \geqslant y_N^p \sum_{i=1}^N w_i = y_N^p s_N$. $\quad\square$

Theorem 3.5. *Let X and Y be subspaces of $d_{w,p}$. Then $\mathcal{FSS}(X,Y) = \mathcal{SS}(X,Y)$. In particular, $\mathcal{FSS}(\ell_p, d_{w,p}) = \mathcal{SS}(\ell_p, d_{w,p})$ and $\mathcal{FSS}(d_{w,p}) = \mathcal{SS}(d_{w,p})$.*

Proof. Let $T \in L(X,Y)$. Suppose that T is not finitely strictly singular. We will show that it is not strictly singular. Since T is not finitely strictly singular, there exist a constant $c > 0$ and a sequence F_n of subspaces of X with $\dim F_n \geqslant n$ such that for each n and for all $x \in F_n$ we have $\|Tx\| \geqslant c\|x\|$.

Fix a sequence (ε_k) in $\mathbb{R}$ such that $1 > \varepsilon_k \downarrow 0$. We will inductively construct a sequence (x_k) in X and two strictly increasing sequences (n_k), (m_k) in $\mathbb{N}$ such that:

(i) (x_k) and (Tx_k) are seminormalized; we will denote Tx_k by u_k;
(ii) for all $k \in \mathbb{N}$, $\operatorname{supp} x_k \subseteq [n_k, \infty)$ and $\operatorname{supp} u_k \subseteq [m_k, \infty)$;
(iii) if $k \geqslant 2$ then $\|x_{k-1}|_{[n_k, \infty)}\| < \varepsilon_k$, $\|u_{k-1}|_{[m_k, \infty)}\| < \varepsilon_k$, and all the coordinates of u_{k-1} where the sup-norm is attained are less than m_k;
(iv) for each $k \in \mathbb{N}$, the vector u_k^* begins with a constant block of length at least k.

That is, (x_n) and (u_n) are two almost disjoint sequences and u_n's have long "flat" sections.

Take x_1 to be any nonzero vector in F_1 and put $n_1 = m_1 = 1$. Suppose we have already constructed $x_1, \ldots, x_{k-1}$, $n_1, \ldots, n_{k-1}$, and $m_1, \ldots, m_{k-1}$ such that the conditions (i)–(iv) are satisfied. Choose $n_k \in \mathbb{N}$ and $m_k \in \mathbb{N}$ such that $n_k > n_{k-1}$, $m_k > m_{k-1}$ and the condition (iii) is satisfied.

Consider the space

$$V = \{y = (y_i) \in F_{n_k + m_k + k}: y_i = 0 \text{ for } i < n_k\} \subseteq F_{n_k + m_k + k}.$$

It follows from $\dim F_{n_k + m_k + k} \geqslant n_k + m_k + k$ that $\dim V \geqslant m_k + k$. Since $V \subseteq F_{n_k + m_k + k}$, $\|Ty\| \geqslant c\|y\|$ for all $y \in V$. In particular, $\dim(TV) \geqslant m_k + k$. Define

$$Z = \{z = (z_i) \in TV: z_i = 0 \text{ for } i < m_k\}.$$

It follows that $\dim Z \geqslant k$.

Clearly, $\operatorname{supp} y \subseteq [n_k, \infty)$ for all $y \in V$ and $\operatorname{supp} z \subseteq [m_k, \infty)$ for all $z \in Z$. By Lemma 3.3, we can choose $u_k \in Z$ such that u_k is normalized and u_k^* starts with a constant block of length k. Put $x_k = (T|_V)^{-1}(u_k) \in Y$. Since $x_k \in V$ and $\|u_k\| = 1$, it follows that $\frac{1}{\|T\|} \leqslant \|x_k\| \leqslant \frac{1}{c}$, so the conditions (i)–(iv) are satisfied for (x_k).

For each $k \in \mathbb{N}$, let $x_k' = x_k|_{[n_k, n_{k+1})}$ and $u_k' = u_k|_{[m_k, m_{k+1})}$. Passing to tails of sequences, if necessary, we may assume that both (x_k') and (u_k') are seminormalized block sequences of (e_n).

Since the non-increasing rearrangement of each u'_k starts with a constant block of length k by (iii), the coefficients in u'_k converge to zero by Lemma 3.4. Therefore, passing to a subsequence, we may assume by Remark 1.4 that (u'_k) is equivalent to the unit vector basis (f_n) of ℓ_p. Using Theorem 1.2 and passing to a further subsequence, we may also assume that $(x_k) \sim (x'_k)$ and $(u_k) \sim (u'_k)$.

By Proposition 1.8, the sequence (x'_k) is dominated by (f_n). Notice that the condition $u_k = Tx_k$ implies $(x_k) \succcurlyeq (u_k)$. Therefore, we get the following chain of dominations and equivalences of basic sequences:

$$(f_n) \succcurlyeq (x'_k) \sim (x_k) \succcurlyeq (u_k) \sim (u'_k) \sim (f_n).$$

It follows that all the dominations in this chain are, actually, equivalences. In particular, $(x_k) \sim (u_k)$. Thus, T is an isomorphism on the space $[x_k]$, hence T is not strictly singular. $\quad\square$

Recall that an operator T on a Banach space X is weakly compact if the image of the unit ball of X under T is relatively weakly compact. Alternatively, T is weakly compact if and only if for every bounded sequence (x_n) in X there exists a subsequence (x_{n_k}) of (x_n) such that (Tx_{n_k}) is weakly convergent.

If $1 < p < \infty$ then $d_{w,p}$ is reflexive, and, hence, every operator in $L(d_{w,p})$ is weakly compact. In case $p = 1$ we have the following.

Theorem 3.6. *Let $T \in L(d_{w,1})$. Then T is weakly compact if and only if T is strictly singular.*

Proof. Suppose that T is strictly singular. We will show that T is weakly compact.

Let (x_n) be a bounded sequence in X. By Rosenthal's ℓ_1-theorem, there is a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) is either equivalent to the unit vector basis (f_n) of ℓ_1 or is weakly Cauchy. In the latter case, (Tx_{n_k}) is also weakly Cauchy. If $(x_{n_k}) \sim (f_n)$ then, since T is strictly singular, (Tx_{n_k}) cannot have subsequences equivalent to (f_n). Hence, using Rosenthal's theorem one more time and passing to a further subsequence, we may assume that, again, (Tx_{n_k}) is weakly Cauchy. Since $d_{w,1}$ is weakly sequentially complete, the sequence (Tx_{n_k}) is weakly convergent. It follows that T is weakly compact.

Conversely, let J be the closed ideal of weakly compact operators in $L(d_{w,1})$. By the first part of the proof, J is a successor of $\mathcal{SS}(d_{w,1})$. Suppose that $J \neq \mathcal{SS}(d_{w,1})$. By Theorem 3.1, $J^{\ell_1} \subseteq J$. This, however, is a contradiction since a projection onto a copy of ℓ_1 (which belongs to J^{ℓ_1} by Proposition 2.4) is not weakly compact. $\quad\square$

4. Operators factorable through the formal identity

The operator $j : \ell_p \to d_{w,p}$ defined by $j(e_n) = f_n$ is called ***the formal identity operator from ℓ_p to $d_{w,p}$***. It follows immediately from the definition of the norm in $d_{w,p}$ that $\|j\| = 1$.

We will denote by the symbol J^j the set of all operators $T \in L(d_{w,p})$ which can be factored as $T = AjB$ where $A \in L(d_{w,p})$ and $B \in L(d_{w,p}, \ell_p)$.

Proposition 4.1. *J^j is an ideal in $L(d_{w,p})$.*

Proof. It is clear from the definition that the set J^j is closed under both right and left multiplication by operators from $L(d_{w,p})$. We have to show that if T_1 and T_2 are in J^j then $T_1 + T_2$ is in J^j, as well.

Write $T_1 = A_1 j B_1$, $T_2 = A_2 j B_2$ with $A_1, A_2 \in L(d_{w,p})$ and $B_1, B_2 \in L(d_{w,p}, \ell_p)$. Let $A \in L(d_{w,p} \oplus d_{w,p}, d_{w,p})$ and $B \in L(d_{w,p}, \ell_p \oplus \ell_p)$ be defined by

$$A(x_1, x_2) = A_1 x_1 + A_2 x_2 \quad \text{and} \quad Bx = (B_1 x, B_2 x).$$

Define also $U : \ell_p \to \ell_p \oplus \ell_p$ and $V : d_{w,p} \to d_{w,p} \oplus d_{w,p}$ by

$$U((x_n)) = \big((x_{2n-1}), (x_{2n})\big), \quad \text{and} \quad V((x_n)) = \big((x_{2n-1}), (x_{2n})\big).$$

Since the bases of ℓ_p and $d_{w,p}$ are both unconditional, U and V are bounded.

Now observe that for each $x = (x_n) \in d_{w,p}$ we can write

$$AVjU^{-1}Bx = AVjU^{-1}(B_1 x, B_2 x) = A(jB_1 x, jB_2 x) = A_1 jB_1 x + A_2 jB_2 x = T_1 x + T_2 x.$$

This shows that $T_1 + T_2 = AVjU^{-1}B$ with $AV \in L(d_{w,p})$ and $U^{-1}B \in L(d_{w,p}, \ell_p)$, hence $T_1 + T_2 \in J^j$. $\quad\square$

As we already mentioned before, the space $d_{w,p}$ contains many complemented copies of ℓ_p. Consider the operator $jUP \in L(d_{w,p})$ where P is a projection onto any subspace Y isomorphic to ℓ_p and $U : Y \to \ell_p$ is an isomorphism onto. It turns out that the ideal generated by any such operator does not depend on the choice of Y and, in fact, coincides with J^j.

Proposition 4.2. *Let Y be a complemented subspace of $d_{w,p}$ isomorphic to ℓ_p, $P \in L(d_{w,p})$ be a projection with range Y, and $U : Y \to \ell_p$ be an isomorphism onto. If $T = jUP$ then $J_T = J^j$.*

Proof. Clearly, $J_T \subseteq J^j$. Let $S \in J^j$. Then $S = AjB$ where $A \in L(d_{w,p})$ and $B \in L(d_{w,p}, \ell_p)$. It follows that

$$S = AjB = Aj(UPU^{-1})B = AT(U^{-1}B) \in J_T. \qquad \square$$

The next goal is to show that the ideal $\overline{J^j}$ "sits" between $\mathcal{K}(X)$ and $\mathcal{SS}(X) \wedge \overline{J^{\ell_p}}$.

Theorem 4.3. *The formal identity operator* $j : \ell_p \to d_{w,p}$ *is finitely strictly singular.*

Proof. Let $\varepsilon > 0$ be arbitrary. Take $n \in \mathbb{N}$ such that $\frac{1}{n} \sum_{i=1}^n w_i < \varepsilon$; such n exists by $w_n \to 0$. Since (w_n) is also a decreasing sequence, it follows that $w_i < \varepsilon$ for all $i \geqslant n$.

Let $Y \subseteq \ell_p$ be a subspace with $\dim Y \geqslant n$. By Lemma 3.3, there exists a vector $x \in Y$ such that $\|x\|_{\ell_p} = 1$ and x attains its sup-norm on at least n coordinates. Denote $\delta = \|x\|_{\sup} > 0$. Then $\|x\|_{\ell_p} \geqslant n^{1/p}\delta$, so $\delta \leqslant n^{-1/p}$.

Observe that the non-increasing rearrangement x^* of x satisfies the condition that $x_i^* = \delta$ for all $1 \leqslant 1 \leqslant n$. Therefore

$$\|jx\|_{d_{w,p}}^p = \sum_{i=1}^\infty x_i^{*p} w_i \leqslant \delta^p \sum_{i=1}^n w_i + \varepsilon \sum_{i=n+1}^\infty x_i^{*p} \leqslant \delta^p n \varepsilon + \varepsilon \|x\|_{\ell_p}^p \leqslant 2\varepsilon.$$

Hence $\|jx\|_{d_{w,p}} \leqslant (2\varepsilon)^{1/p}$. $\square$

Corollary 4.4. *The following inclusions hold:* $\mathcal{K}(d_{w,p}) \subsetneq \overline{J^j}$ *and* $J^j \subseteq \mathcal{SS}(d_{w,p}) \wedge J^{\ell_p}$.

Proof. Let Y, P, and U be as in Proposition 4.2. Then $jUP \in J^j$. If $x_n = U^{-1}f_n \in d_{w,p}$ then (x_n) is seminormalized and $jUPx_n = e_n$. Hence the sequence $(jUPx_n)$ has no convergent subsequences, so that jUP is not compact.

The inclusion $J^j \subseteq \mathcal{SS}(d_{w,p}) \wedge J^{\ell_p}$ is obvious since j is strictly singular. $\square$

Conjecture 4.5. *The ideal* $\overline{J^j}$ *is the only immediate successor of* $\mathcal{K}(d_{w,p})$.

In [3] and [10] (see also [17]), conditions on the weights $w = (w_n)$ are given under which $d_{w,p}$ has exactly two non-equivalent symmetric basic sequences. We will show that the conjecture holds true in this case.

Lemma 4.6. *If* $T \in \mathcal{SS}(d_{w,p}) \setminus \mathcal{K}(d_{w,p})$ *then there exists a seminormalized basic sequence* (x_n) *in* $d_{w,p}$ *such that* $(f_n) \succcurlyeq (x_n)$ *and* (Tx_n) *is weakly null and seminormalized.*

Proof. Let (z_n) be a bounded sequence in $d_{w,p}$ such that (Tz_n) has no convergent subsequences. Then (z_n) has no convergent subsequences either. Applying Rosenthal's ℓ_1-theorem and passing to a subsequence, we may assume that (z_n) is either equivalent to the unit vector basis of ℓ_1 or is weakly Cauchy.

Case: (z_n) *is equivalent to the unit vector basis of* ℓ_1. Since a reflexive space cannot contain a copy of ℓ_1, we conclude that $p = 1$, so $(z_n) \sim (f_n)$. Again, by Rosenthal's theorem, (Tz_n) has a subsequence which is either equivalent to (f_n) or is weakly Cauchy. If $(Tz_{n_k}) \sim (f_n)$ then T is an isomorphism on the space $[z_{n_k}]$, contrary to the assumption that $T \in \mathcal{SS}(d_{w,p})$. Therefore, (Tz_{n_k}) is weakly Cauchy. Put $x_k = z_{n_{2k}} - z_{n_{2k-1}}$. Then (x_k) is basic and (Tx_k) is weakly null. Passing to a further subsequence of (z_{n_k}) we may assume that (Tx_k) is seminormalized. Also, (x_k) is still equivalent to (f_n), hence is dominated by (f_n).

Case: (z_n) *is weakly Cauchy.* Clearly, (Tz_n) is also weakly Cauchy. Consider the sequence (u_n) in $d_{w,p}$ defined by $u_n = z_{2n} - z_{2n-1}$. Then both (u_n) and (Tu_n) are weakly null. Passing to a subsequence of (z_n), we may assume that (Tu_n) and, hence, (u_n) are seminormalized. Applying Theorem 1.3, we get a subsequence (u_{n_k}) of (u_n) which is basic and equivalent to a block sequence (v_n) of (e_n). Denote $x_k = u_{n_k}$. By Proposition 1.8, (f_n) dominates (v_n) and, hence, (x_k). $\square$

Theorem 4.7. *If* $d_{w,p}$ *has exactly two non-equivalent symmetric basic sequences, then* $\overline{J^j}$ *is the only immediate successor of* $\mathcal{K}(d_{w,p})$.

Proof. Let T be a non-compact operator on $d_{w,p}$. It suffices to show that $J^j \subseteq J_T$. We may assume that T is strictly singular because, otherwise, we have $J^j \subseteq J^{\ell_p} \subseteq J_T$ by Theorem 3.1.

Let (x_n) be a sequence as in Lemma 4.6. Passing to a subsequence and using Theorem 1.3, we may assume that (Tx_n) is basic and equivalent to a block sequence (h_n) of (e_n) such that $Tx_n - h_n \to 0$. We claim that (h_n) has no subsequences equivalent to (f_n). Indeed, otherwise, for such a subsequence (h_{n_k}) of (h_n), we would have $(f_n) \sim (f_{n_k}) \succcurlyeq (x_{n_k}) \succcurlyeq (Tx_{n_k}) \sim (h_{n_k}) \sim (f_n)$, so $(x_{n_k}) \sim (Tx_{n_k})$, contrary to $T \in \mathcal{SS}(d_{w,p})$. By [10, Theorem 19], (h_n) has a subsequence which spans a complemented subspace in $d_{w,p}$ and is equivalent to (e_n). Therefore, by Theorem 1.2, we may assume (by passing to a further subsequence) that $(Tx_n) \sim (e_n)$ and $[Tx_n]$ is complemented in $d_{w,p}$.

We have proved that there exists a sequence (x_n) in $d_{w,p}$ such that $[Tx_n]$ is complemented in $d_{w,p}$ and

$$(f_n) \succcurlyeq (x_n) \succcurlyeq (Tx_n) \sim (e_n).$$

Let $A \in L(\ell_p, d_{w,p})$ and $B \in L([Tx_n], d_{w,p})$ be defined by $Af_n = x_n$ and $B(Tx_n) = e_n$. Let $Q \in L(d_{w,p})$ be a projection onto $[Tx_n]$. Then for all $n \in \mathbb{N}$, we obtain: $BQTAf_n = BQTx_n = BTx_n = e_n$. It follows that $BQTA = j$, so that $J^j \subseteq J_T$. □

In order to prove Conjecture 4.5 without additional conditions on w, it suffices to show that if $T \in \overline{J^j} \setminus \mathcal{K}(d_{w,p})$ then $J^j \subseteq \overline{J_T}$. We will prove a weaker statement: if $T \in J^j \setminus \mathcal{K}(d_{w,p})$ then $J^j \subseteq J_T$.

Recall (see [3, p. 148]) that if $x = (a_n) \in d_{w,p}$ then a block sequence (y_n) of (e_n) is called a ***block of type I generated by*** x if it is of the form $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_{i-p_n} e_i$ for all n. A set $A \subseteq d_{w,p}$ will be said to be ***almost lengthwise bounded*** if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x^*|_{[N,\infty)}\| < \varepsilon$ for all $x \in A$. We will usually use it in the case when $A = \{x_n\}$ for some sequence (x_n) in $d_{w,p}$. We need the following result, which is a slight extension of [3, Theorem 3]. We include the proof for completeness.

Theorem 4.8. *Let (x_n) be a seminormalized block sequence of (e_n) in $d_{w,p}$.*

(i) *If (x_n) is not almost lengthwise bounded then there exists a subsequence (x_{n_k}) such that $(x_{n_k}) \sim (f_n)$.*
(ii) *If (x_n) is almost lengthwise bounded, then there exists a subsequence (x_{n_k}) equivalent to a block of type I generated by a vector $u = \sum_{i=1}^{\infty} b_i e_i \in d_{w,p}$ with $b_i \downarrow 0$. Moreover, if the sequence (x_n) is bounded in ℓ_p then u is in ℓ_p.*[2]

Proof. (i) Without loss of generality, $\|x_n\| \leqslant 1$ for all $n \in \mathbb{N}$. By the assumption, there exists $\varepsilon > 0$ with the property that for each $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that $\|x_{n_k}^*|_{(k,\infty)}\| \geqslant \varepsilon$. Let u_k be a restriction of x_{n_k} such that $u_k^* = x_{n_k}^*|_{[1,k]}$ and $v_k = x_{n_k} - u_k$.

Clearly, each nonzero entry of u_k is greater than or equal to the greatest entry of v_k. By Lemma 3.4, the k-th coordinate of u_k^* is less than or equal to $\frac{1}{s_k^{1/p}}$ where $s_k = \sum_{i=1}^{k} w_i$. It follows that (v_k) is a block sequence of (e_n) such that $\varepsilon \leqslant \|v_k\| \leqslant 1$ and absolute values of the entries of v_k are all at most $\frac{1}{s_k^{1/p}}$. Since $\lim_k s_k = +\infty$ by the definition of $d_{w,p}$, passing to a subsequence and using Remark 1.4 we may assume that (v_k) is equivalent to (f_n). By Proposition 1.8, (f_n) dominates (x_{n_k}). Using also Lemma 1.9, we obtain the following diagram:

$$(f_n) \succcurlyeq (x_{n_k}) \succcurlyeq (v_k) \sim (f_n).$$

Hence (x_{n_k}) is equivalent to (f_n).

(ii) Suppose that $x_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$. Clearly, the sequence (a_i) is bounded. Without loss of generality, $a_{p_n+1} \geqslant \cdots \geqslant a_{p_{n+1}} \geqslant 0$ for each n. Put $y_n = x_n^*$. Using a standard diagonalization argument and passing to a subsequence, we may assume that (y_n) converges coordinate-wise; put $b_i = \lim_{n \to \infty} y_{n,i}$. It is easy to see that $b_i \geqslant b_{i+1}$ for all i. Put $u = (b_i)$.

Case: the sequence $(p_{n+1} - p_n)$ is bounded. Passing to a subsequence, we may assume that $N := p_{n_k+1} - p_{n_k}$ is a constant. Note that $\operatorname{supp} u \subseteq [1, N]$ and $\operatorname{supp} y_{n_k} \subseteq [1, N]$ for all k. Put $u_k = \sum_{i=p_{n_k}+1}^{p_{n_k+1}} b_{i-p_{n_k}} e_i$, then $u = u_k^*$ and (u_k) as a block of type I generated by u. By compactness, $\|x_{n_k} - u_k\| = \|y_{n_k} - u\| \to 0$. Therefore, passing to a further subsequence, we have $(x_{n_k}) \sim (u_k)$. Being a vector with finite support, u belongs to ℓ_p.

Case: the sequence $(p_{n+1} - p_n)$ is unbounded. We will construct the required subsequence (x_{n_k}) and a sequence (N_k) inductively. Put $n_1 = N_1 = 1$ and let $k > 1$. Suppose that $n_1, \ldots, n_{k-1}$ and $N_1, \ldots, N_{k-1}$ have already been selected. Since (x_n) is almost lengthwise bounded, we can find $N_k > N_{k-1}$ such that $\|y_n|_{(N_k,\infty)}\| < \frac{1}{k}$ for all n. Put $v_k = u|_{[1,N_k]}$. Using coordinate-wise convergence, we can find $n_k > n_{k-1}$ such that $\|y_{n_k}|_{[1,N_k]} - v_k\|_{\ell_p} < \frac{1}{k}$ and $p_{n_k} + N_k \leqslant p_{n_k+1}$. Put $u_k = \sum_{i=p_{n_k}+1}^{p_{n_k}+N_k} b_{i-p_{n_k}} e_i$. Then $u_k^* = v_k$, so that

$$\|x_{n_k}|_{(p_{n_k}, p_{n_k}+N_k]} - u_k\|_{\ell_p} = \|y_{n_k}|_{[1,N_k]} - v_k\|_{\ell_p} < \frac{1}{k} \tag{1}$$

and

$$\|x_{n_k}|_{(p_{n_k}+N_k, p_{n_k+1}]}\| = \|y_{n_k}|_{(N_k,\infty)}\| < \frac{1}{k}.$$

It follows that $\|x_{n_k} - u_k\| \to 0$. Passing to a subsequence, we get $(x_{n_k}) \sim (u_k)$.

Next, we show that $u \in d_{w,p}$. Since $\|\cdot\| \leqslant \|\cdot\|_{\ell_p}$, it follows from (1) that

$$\|v_k\| = \|u_k\| \leqslant \|x_{n_k}|_{(p_{n_k}, p_{n_k}+N_k]}\| + \frac{1}{k} \leqslant \|x_{n_k}\| + \frac{1}{k}.$$

---

[2] As a sequence space, ℓ_p is a subset of $d_{w,p}$. That is, we can identify ℓ_p with $\operatorname{Range} j$. More precisely, we claim here that if $(j^{-1}x_n)$ is bounded in ℓ_p then u is in $\operatorname{Range} j$. Being a block sequence of (e_n), (x_n) is contained in $\operatorname{Range} j$.

Since (x_n) is bounded, so is (v_k). Since supp $v_k = N_k \to \infty$, we have $u \in d_{w,p}$. For the "moreover" part, we argue in a similar way. By (1), we have

$$\|v_k\|_{\ell_p} \leqslant \|u_k\|_{\ell_p} \leqslant \|x_{n_k}|_{(p_{n_k}, p_{n_k} + N_k]}\|_{\ell_p} + \frac{1}{k} \leqslant \|x_{n_k}\|_{\ell_p} + \frac{1}{k}.$$

Therefore, if (x_n) is bounded in ℓ_p then so is (v_k), hence $u \in \ell_p$. $\quad\square$

Lemma 4.9. *Suppose that (u_n) is a block of type I in $d_{w,p}$ generated by some $u = \sum_{i=1}^{\infty} b_i e_i$. If $b_i \downarrow 0$ and $u \in \ell_p$ then (u_n) has a subsequence equivalent to (e_n)*

Proof. By Corollary 4 of [3], we may assume that the basic sequence (u_n) is symmetric. It suffices to show that $[u_n]$ is isomorphic to $d_{w,p}$ because all symmetric bases in $d_{w,p}$ are equivalent; see, e.g., Theorem 4 of [3]. Without loss of generality, $\|u\| = 1$. Lemma 4 of [3] asserts that $[u_n]$ is isomorphic to $d_{w,p}$ iff $(s_n^{(u)}) \sim (s_n)$, where $s_n = \sum_{i=1}^{n} w_i$, $s_n^{(u)} = \sum_{i=1}^{\infty} b_i^p (s_{ni} - s_{n(i-1)})$, and $(\alpha_n) \sim (\beta_n)$ means that there exist positive constants A and B such that $A\alpha_n \leqslant \beta_n \leqslant B\alpha_n$ for all n. Let's verify that this condition is, indeed, satisfied. On one hand, taking only the first term in the definition of $s_n^{(u)}$, we get $s_n^{(u)} \geqslant b_1^p s_n$. On the other hand, it follows from $w_i \downarrow$ that $s_{ni} - s_{n(i-1)} \leqslant s_n$ for every i, hence $s_n^{(u)} \leqslant \sum_{i=1}^{\infty} b_i^p s_n = \|u\|_{\ell_p}^p s_n$. $\quad\square$

Lemma 4.10. *Let (x_n) be a block sequence of (f_n) in ℓ_p such that the sequences (x_n) and (jx_n) are seminormalized in ℓ_p and $d_{w,p}$, respectively. Then there exists a subsequence (x_{n_k}) such that $(jx_{n_k}) \sim (e_n)$.*

Proof. Clearly, $(x_n) \sim (f_n)$. It follows that $(jx_n) \nsim (f_n)$ because, otherwise, j would be an isomorphism on $[x_n]$, which is impossible because j is strictly singular by Theorem 4.3. Applying Theorem 4.8 to (jx_n) and passing to a subsequence, we may assume that $(jx_n) \sim (u_n)$, where (u_n) is a block of type I generated by some $u = \sum_{i=1}^{\infty} b_i e_i$ such that $b_i \downarrow 0$ and $u \in \ell_p$. Applying Lemma 4.9 and passing to a subsequence, we get $(u_n) \sim (e_n)$. $\quad\square$

Theorem 4.11. *If $T \in J^j \setminus \mathcal{K}(d_{w,p})$ then $J^j \subseteq J_T$.*

Proof. Write $T = AjB$ where $B : d_{w,p} \to \ell_p$ and $A : d_{w,p} \to d_{w,p}$. Let (x_n) be as in Lemma 4.6. The sequence (Bx_n) is bounded, hence we may assume by passing to a subsequence that it converges coordinate-wise. Since (Tx_n) is weakly null and seminormalized, it has no convergent subsequences. It follows that, after passing to a subsequence of (x_n), we may assume that (Tz_n) is seminormalized, where $z_n = x_{2n} - x_{2n-1}$. In particular, (z_n), (Bz_n), and (jBz_n) are seminormalized. Also, (Bz_n) converges to zero coordinate-wise. Using Theorem 1.3 and passing to a further subsequence, we may assume that (Bz_n) is equivalent to a block sequence (u_n) of (f_n) and $Bz_n - u_n \to 0$. It follows from $(f_n) \succcurlyeq (x_n)$ that $(f_n) \succcurlyeq (z_n) \succcurlyeq (Bz_n) \sim (u_n) \sim (f_n)$. In particular, $(z_n) \sim (f_n)$.
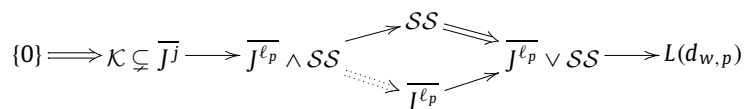
Since $Bz_n - u_n \to 0$ and (jBz_n) is seminormalized, we may assume that the sequence (ju_n) is seminormalized. By Lemma 4.10, passing to a further subsequence, we may assume that (ju_n) and, hence, (jBz_n) are equivalent to (e_n).

Passing to a subsequence and using Theorem 1.3, we may assume that (Tz_n) is equivalent to a block sequence (v_n) of (e_n) such that $Tz_n - v_n \to 0$. Since $T \in \mathcal{SS}(d_{w,p})$, no subsequence of (Tz_n) and, therefore, of (v_n) is equivalent to (f_n). By Proposition 1.8, $(v_n) \succcurlyeq (e_n)$. It follows from $(jBz_n) \sim (e_n)$ that $(e_n) \succcurlyeq (Tz_n)$, hence $(Tz_n) \sim (e_n) \sim (v_n)$.

Write $v_n = \sum_{i=p_n+1}^{p_{n+1}} a_n e_n$. By Remark 1.4, $a_n \nrightarrow 0$. Hence, passing to a subsequence and using [10, Remark 9], we may assume that $[v_n]$ is complemented. By Theorem 1.3, we may assume that $[Tz_n]$ is complemented. Let $P \in L(d_{w,p})$ be a projection onto $[Tz_n]$ and $U \in L(\ell_p, d_{w,p})$ and $V \in L([Tz_n], d_{w,p})$ be defined by $Uf_n = z_n$ and $VTz_n = e_n$. Then we can write $j = VPTU$. Therefore $J^j \subseteq J_T$. $\quad\square$

5. $d_{w,p}$-Strictly singular operators

The ideals in $L(d_{w,p})$ we have obtained so far can be arranged into the following diagram.

$$\{0\} \Longrightarrow \mathcal{K} \subsetneq \overline{J^j} \longrightarrow \overline{J^{\ell_p} \wedge \mathcal{SS}} \begin{array}{c} \nearrow \overline{\mathcal{SS}} \Longrightarrow \\ \dashrightarrow \underset{\overline{J^{\ell_p}}}{\nearrow} \end{array} \overline{J^{\ell_p} \vee \mathcal{SS}} \longrightarrow L(d_{w,p})$$

(see the Introduction for the notations). In this section, we will characterize the greatest ideal in the algebra $L(d_{w,p})$, that is, a proper ideal in $L(d_{w,p})$ that contains all other proper ideals in $L(d_{w,p})$.

If X and Y are two Banach spaces, then an operator $T \in L(X)$ is called Y-***strictly singular*** if for any subspace Z of X isomorphic to Y, the restriction $T|_Z$ is not an isomorphism. The set of all Y-strictly singular operators in $L(d_{w,p})$ will be denoted by $\mathcal{SS}_Y$.

According to this notation, the symbol $\mathcal{SS}_{d_{w,p}}$ stands for the set of all $d_{w,p}$-strictly singular operators in $L(d_{w,p})$ (not to be confused with $\mathcal{SS}(d_{w,p})$).

Lemma 5.1. *Suppose that $T \in SS_{d_{w,p}}$ and (x_n) is a basic sequence in $d_{w,p}$ equivalent to the unit vector basis (e_n). Then $Tx_n \to 0$.*

Proof. Suppose, by way of contradiction, that $Tx_n \nrightarrow 0$. Then there is a subsequence (x_{n_k}) such that (Tx_{n_k}) is seminormalized. Since (x_n) is weakly null (Remark 1.7), we may assume by using Theorem 1.3 and passing to a further subsequence that (Tx_{n_k}) is a basic sequence equivalent to a block sequence (z_k) of (e_n).

By Proposition 1.8, either (z_k) has a subsequence equivalent to (f_n) or $(z_k) \succcurlyeq (e_n)$. Since (Tx_{n_k}) cannot have subsequences equivalent to (f_n) (this would contradict boundedness of T), the former is impossible. Therefore $(z_k) \succcurlyeq (e_n)$. We obtain the following diagram:

$$(e_n) \sim (x_{n_k}) \succcurlyeq (Tx_{n_k}) \sim (z_k) \succcurlyeq (e_n).$$

Therefore $T|_{[x_{n_k}]}$ is an isomorphism. This contradicts T being in $SS_{d_{w,p}}$. $\square$

Corollary 5.2. *Let $T \in SS_{d_{w,p}}$. If $Y \subseteq d_{w,p}$ is a subspace isomorphic to $d_{w,p}$ then there is a subspace $Z \subseteq Y$ such that Z is isomorphic to $d_{w,p}$ and $T|_Z$ is compact.*

Proof. Let (x_n) be a basis of Y equivalent to (e_n). By Lemma 5.1, $Tx_n \to 0$. There is a subsequence (x_{n_k}) of (x_n) such that $\sum_{k=1}^{\infty} \frac{\|Tx_{n_k}\|}{\|x_{n_k}\|}$ is convergent. Let $Z = [x_{n_k}]$. It follows that Z is isomorphic to $d_{w,p}$ and $T|_Z$ is compact (see, e.g., [8, Lemma 5.4.10]). $\square$

Theorem 5.3. *The set $SS_{d_{w,p}}$ of all $d_{w,p}$-strictly singular operators in $L(d_{w,p})$ is the greatest proper ideal in the algebra $L(d_{w,p})$. In particular, $SS_{d_{w,p}}$ is closed.*

Proof. First, let us show that $SS_{d_{w,p}}$ is an ideal. Let $T \in SS_{d_{w,p}}$. If $A \in L(d_{w,p})$ then, trivially, $AT \in SS_{d_{w,p}}$. If $TA \notin SS_{d_{w,p}}$ then there exists a subspace Y of $d_{w,p}$ such that Y and $TA(Y)$ are both isomorphic to $d_{w,p}$. Then $A|_Y$ is bounded below, hence $A(Y)$ is isomorphic to $d_{w,p}$. It follows that T is an isomorphism on a copy of $d_{w,p}$, contrary to $T \in SS_{d_{w,p}}$. So, $SS_{d_{w,p}}$ is closed under two-sided multiplication by bounded operators.

Let $T, S \in SS_{d_{w,p}}$. We will show that $T + S \in SS_{d_{w,p}}$. Let Y be a subspace of $d_{w,p}$ isomorphic to $d_{w,p}$. By Corollary 5.2, there exists a subspace Z of Y such that Z is isomorphic to $d_{w,p}$ and $T|_Z$ is compact. Applying Corollary 5.2 again, we can find a subspace V of Z such that V is isomorphic to $d_{w,p}$ and $S|_V$ is compact. Therefore $(T + S)|_V$ is compact, so that $(T + S)|_Y$ is not an isomorphism. So, $SS_{d_{w,p}}$ is an ideal.

Clearly, the identity operator I does not belong to $SS_{d_{w,p}}$, so $SS_{d_{w,p}}$ is proper. Let us show that $SS_{d_{w,p}}$ is the greatest ideal in $L(d_{w,p})$.

Let $T \notin SS_{d_{w,p}}$. Then there exists a subspace Y of $d_{w,p}$ such that Y and $T(Y)$ are isomorphic to $d_{w,p}$. By [10, Corollary 12], there exists a complemented (in $d_{w,p}$) subspace Z of $T(Y)$ such that Z is isomorphic to $d_{w,p}$. Let $P \in L(d_{w,p})$ be a projection onto Z. Put $H = T^{-1}(Z)$. It follows that H is isomorphic to $d_{w,p}$. Let $U : d_{w,p} \to H$ and $V : Z \to d_{w,p}$ be surjective isomorphisms. Then $S \in L(d_{w,p})$ defined by $S = (VP)TU$ is an invertible operator. Clearly $S \in J_T$, hence $J_T = L(X)$.

The fact that $SS_{d_{w,p}}$ is closed follows from [11, Corollary VII.2.4]. $\square$

The next theorem provides a convenient characterization of $d_{w,p}$-strictly singular operators.

Lemma 5.4. *Let $T \in L(d_{w,p})$ be such that $Te_n \to 0$. Suppose that (x_n) is a bounded block sequence of (e_n) in $d_{w,p}$ such that (x_n) is almost lengthwise bounded. Then $Tx_n \to 0$.*

Proof. Write $x_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$. Since (x_n) is bounded, there is $C > 0$ such that $|a_i| \leqslant C$ for all i and $n \in \mathbb{N}$. Let $\varepsilon > 0$. Find $N \in \mathbb{N}$ such that $\|x_n^*|_{[N,\infty)}\| < \varepsilon$ for all $n \in \mathbb{N}$. Let u_n be a restriction of x_n such that $u_n^* = x_n^*|_{[1,N)}$ and $v_n = x_n - u_n$. It is clear that $\|v_n\| = \|x_n^*|_{[N,\infty)}\| < \varepsilon$. Also, $\|Tu_n\| \leqslant NC \cdot \max_{p_n+1 \leqslant i \leqslant p_{n+1}} \|Te_i\|$.

Pick $M \in \mathbb{N}$ such that $\|Te_k\| < \frac{\varepsilon}{N}$ for all $k \geqslant M$. Then

$$\|Tx_n\| \leqslant \|Tu_n\| + \|Tv_n\| \leqslant NC\frac{\varepsilon}{N} + \varepsilon\|T\| = \varepsilon(C + \|T\|)$$

for all n such that $p_n > M$. It follows that $Tx_n \to 0$. $\square$

Theorem 5.5. *An operator $T \in L(d_{w,p})$ is $d_{w,p}$-strictly singular if and only if $Te_n \to 0$.*

Proof. Suppose that $Te_n \to 0$ but $T \notin SS_{d_{w,p}}$. Then there exists a subspace Y of $d_{w,p}$ such that Y is isomorphic to $d_{w,p}$ and $T|_Y$ is an isomorphism. Let (x_n) be a basis of Y equivalent to (e_n). By Remark 1.7, $x_n \xrightarrow{w} 0$. Using Theorem 1.3 and passing to a subsequence, we may assume that (x_n) is equivalent to a block sequence (z_n) of (e_n) such that $x_n - z_n \to 0$. Since (z_n) is equivalent to (e_n), it is almost lengthwise bounded by Theorem 4.8. By Lemma 5.4, $Tz_n \to 0$. Since $x_n - z_n \to 0$, we obtain $Tx_n \to 0$. This is a contradiction since (x_n) is seminormalized and $T|_{[x_n]}$ is an isomorphism.

The converse implication follows from Lemma 5.1. $\square$

Remark 5.6. In Theorem 5.3 we showed, in particular, that $\mathcal{SS}_{d_{w,p}}$ is closed under addition. Alternatively, we could have deduced this from Theorem 5.5.

Recall that an operator T on a Banach space X is called **_Dunford–Pettis_** if for any sequence (x_n) in X, $x_n \xrightarrow{w} 0$ implies $Tx_n \to 0$. If $1 < p < \infty$ then the class of Dunford–Pettis operators on $d_{w,p}$ coincides with $\mathcal{K}(d_{w,p})$ because $d_{w,p}$ is reflexive. For the case $p = 1$ we have the following result.

Theorem 5.7. *Let* $T \in L(d_{w,1})$. *Then* T *is* $d_{w,1}$*-strictly singular if and only if* T *is Dunford–Pettis.*

Proof. If T is Dunford–Pettis then T is $d_{w,1}$-strictly singular by Theorem 5.5 because (e_n) is weakly null.

Conversely, suppose that T is $d_{w,1}$-strictly singular. Let (x_n) be a weakly null sequence. Suppose that (Tx_n) does not converge to zero. Then, passing to a subsequence, we may assume that (x_n) is a seminormalized weakly null basic sequence equivalent to a block sequence (u_n) of (e_n) such that $x_n - u_n \to 0$. Clearly, (u_n) is weakly null. In particular, (u_n) has no subsequences equivalent to (f_n). By Theorem 4.8, (u_n) is almost lengthwise bounded. Hence, by Lemma 5.4, $Tu_n \to 0$. It follows that $Tx_n \to 0$, contrary to the choice of (x_n). □

6. Strictly singular operators between ℓ_p and $d_{w,p}$

We do not know whether the ideals $\overline{J^j}$, $\mathcal{SS} \wedge \overline{J^{\ell_p}}$, and $\mathcal{SS}$ are distinct. In this section, we discuss some connections between these ideals.

Conjecture 6.1. $\overline{J^j} = \mathcal{SS} \wedge \overline{J_{\ell_p}}$. *In particular, every strictly singular operator in* $L(d_{w,p})$ *which factors through* ℓ_p *can be approximated by operators that factor through* j.

The following statement is a refinement of Lemma 1.9. Recall that $d_{w,p}$ is a Banach lattice with respect to the coordinate-wise order.

Lemma 6.2. *Suppose that* (x_n) *and* (y_n) *are seminormalized sequences in* $d_{w,p}$ *such that* $|x_n| \geqslant |y_n|$ *for all* $n \in \mathbb{N}$ *and* $x_n \to 0$ *coordinate-wise. Then there exists an increasing sequence* (n_k) *in* $\mathbb{N}$ *such that* (x_{n_k}) *and* (y_{n_k}) *are basic and* $(x_{n_k}) \succcurlyeq (y_{n_k})$.

Proof. Clearly, $y_n \to 0$ coordinate-wise. By Theorem 1.3, we can find a sequence (n_k) and two block sequences (u_k) and (v_k) of (e_n) such that (x_{n_k}) and (y_{n_k}) are basic, $(x_{n_k}) \sim (u_k)$, $(y_{n_k}) \sim (v_k)$, $x_{n_k} - u_k \to 0$, $y_{n_k} - v_k \to 0$, and for each $k \in \mathbb{N}$, the vector u_k (v_k, respectively) is a restriction of (x_{n_k}) (of (y_{n_k}), respectively).

For each $k \in \mathbb{N}$, define $h_k \in d_{w,p}$ by putting its i-th coordinate to be equal to $h_k(i) = \text{sign}(v_k(i)) \cdot (|u_k(i)| \wedge |v_k(i)|)$. Then (h_k) is a block sequence of (e_n) such that $|h_k| \leqslant |u_k|$. A straightforward verification shows that $|h_k - v_k| \leqslant |u_k - x_{n_k}|$. It follows that $h_k - v_k \to 0$. By Theorem 1.2, passing to a subsequence, we may assume that (h_k) is basic and $(h_k) \sim (v_k)$. By Lemma 1.9, $(u_k) \succcurlyeq (h_k)$. Hence $(x_{n_k}) \succcurlyeq (y_{n_k})$. □

The next lemma is a version of Theorem 4.8 for the case (x_n) is an arbitrary bounded sequence.

Lemma 6.3. *If the bounded sequence* (x_n) *in* $d_{w,p}$ *is not almost lengthwise bounded, then there is a subsequence* (x_{n_k}) *such that* $(x_{n_{2k}} - x_{n_{2k-1}})$ *is equivalent to the unit vector basis* (f_n) *of* ℓ_p.

Proof. We can assume without loss of generality that no subsequence of (x_n) is equivalent to the unit vector basis of ℓ_1. Indeed, if (x_{n_k}) is equivalent to the unit vector basis of ℓ_1 then $p = 1$. It follows that (x_{n_k}) is equivalent to (f_n) and hence $(x_{n_{2k}} - x_{n_{2k-1}})$ is equivalent to (f_n), as well.

Without loss of generality, $\sup_n \|x_n\| = 1$. Since (x_n) is not almost lengthwise bounded, there exists $c > 0$ such that

$$\forall N \in \mathbb{N} \; \exists n \in \mathbb{N} \quad \big\|x_n^*|_{[N,\infty)}\big\| > c. \tag{2}$$

Let $\frac{c}{4} > \varepsilon_k \downarrow 0$. We will inductively construct increasing sequences (n_k) and (N_k) in $\mathbb{N}$ and a sequence (y_k) in $d_{w,p}$ such that the following conditions are satisfied for each k:

(i) $\|x_{n_k}|_{[N_{k+1},\infty)}\| < \varepsilon_k$;
(ii) y_k is supported on $[N_k, N_{k+1})$;
(iii) y_k is a restriction of x_{n_k};
(iv) $\|y_k\| > \frac{c}{2}$;
(v) $\|y_k\|_\infty \leqslant s_{N_k}^{-1/p}$ where s_N is as in Lemma 3.4.

For $k = 1$, we put $N_1 = 1$, and define n_1 to be the first number n such that $\|x_n\| > c$; such an n exists by (2). Pick $N_2 \in \mathbb{N}$ such that $\|x_{n_1}|_{[N_2,\infty)}\| < \varepsilon_1$. Put $y_1 = x_{n_1}|_{[N_1,N_2)}$. It follows that $1 \geqslant \|y_1\| > c - \varepsilon_1 > \frac{c}{2}$, and the coordinates of y_1 are all at most $1 \,(= s_1^{-1/p})$, hence all the conditions (i)–(v) are satisfied for $k = 1$.

Suppose that appropriate sequences $(n_i)_{i=1}^k$, $(N_i)_{i=1}^{k+1}$, and $(y_i)_{i=1}^k$ have been constructed. Use (2) to find n_{k+1} such that $\|x_{n_{k+1}}^*|_{[2N_{k+1},\infty)}\| > c$. Let z be the vector obtained from $x_{n_{k+1}}$ by replacing its N_{k+1} largest (in absolute value) entries with zeros. Then $\|z|_{[N_{k+1},\infty)}\| \geqslant \|z^*|_{[N_{k+1},\infty)}\| = \|x_{n_{k+1}}^*|_{[2N_{k+1},\infty)}\| > c$. By Lemma 3.4, $\|z\|_\infty \leqslant s_{N_{k+1}}^{-1/p}$. Choose N_{k+2} such that $\|x_{n_{k+1}}|_{[N_{k+2},\infty)}\| < \varepsilon_{k+1}$. It follows that $\|z|_{[N_{k+2},\infty)}\| < \varepsilon_{k+1}$. Put $y_{k+1} = z|_{[N_{k+1},N_{k+2})}$. Then $\|y_{k+1}\| \geqslant c - \varepsilon_{k+1} > \frac{c}{2}$, and the inductive construction is complete.

The sequence (y_k) constructed above is a seminormalized block sequence of (e_n) such that the coordinates of (y_k) converge to zero by condition (v). Using Remark 1.4 and passing to a subsequence, we may assume that (y_k) is equivalent to the unit vector basis (f_n) of ℓ_p.

Since (x_n) contains no subsequences equivalent to the unit vector basis of ℓ_1, using Rosenthal's ℓ_1-theorem and passing to a further subsequence, we may assume that (x_{n_k}) is weakly Cauchy. For all $m > k \in \mathbb{N}$, we have: $\|x_{n_k}|_{[N_m,\infty)}\| \leqslant \|x_{n_k}|_{[N_{k+1},\infty)}\| \leqslant \varepsilon_k$. Therefore $\|x_{n_m} - x_{n_k}\| \geqslant \|(x_{n_m} - x_{n_k})|_{[N_m,\infty)}\| \geqslant \|x_{n_m}|_{[N_m,\infty)}\| - \varepsilon_k \geqslant \|y_m\| - \varepsilon_k \geqslant \frac{c}{2} - \varepsilon_k > \frac{c}{4}$. It follows that the sequence (u_k) defined by $u_k = x_{n_{2k}} - x_{n_{2k-1}}$ is seminormalized and weakly null. Passing to a subsequence of (x_{n_k}), we may assume that (u_k) is equivalent to a block sequence of (e_n). By Proposition 1.8, $(f_n) \succcurlyeq (u_k)$.

Let $v_k = x_{n_{2k}} - (x_{n_{2k-1}}|_{[1,N_{2k})})$. Then $\|u_k - v_k\| = \|x_{n_{2k-1}}|_{[N_{2k},\infty)}\| < \varepsilon_{2k-1} \to 0$. By Theorem 1.2, passing to a subsequence of (x_{n_k}), we may assume that (v_k) is basic and $(v_k) \sim (u_k)$. Also, (v_k) is weakly null. Note that $|y_{2k}| \leqslant |v_k|$ for all $k \in \mathbb{N}$, since $\operatorname{supp} y_{2k} \subseteq [N_{2k}, N_{2k+1})$, so that y_{2k} is a restriction of v_k. By Lemma 6.2, passing to a subsequence, we may assume that $(v_k) \succcurlyeq (y_{2k})$. Therefore we obtain the following diagram:

$$(f_k) \succcurlyeq (u_k) \sim (v_k) \succcurlyeq (y_{2k}) \sim (f_{2k}) \sim (f_n).$$

It follows that (u_k) is equivalent to (f_k). $\square$

Corollary 6.4. *If $T \in \mathcal{SS}(\ell_p, d_{w,p})$ then the sequence (Tf_n) is almost lengthwise bounded.*

Proof. Suppose that (Tf_n) is not almost lengthwise bounded. By Lemma 6.3, there is a subsequence (f_{n_k}) such that $(Tf_{n_{2k}} - Tf_{n_{2k-1}})$ is equivalent to (f_n). It follows that $T|_{[f_{n_{2k}} - f_{n_{2k-1}}]}$ is an isomorphism. $\square$

Remark 6.5. If we view T as an infinite matrix, the vectors (Tf_n) represent its columns.

Theorem 6.6. *If $T \in L(\ell_1, d_{w,1})$ is such that the sequence (Tf_n) is almost lengthwise bounded, then for any $\varepsilon > 0$ there exists $S \in L(\ell_1)$ such that $\|T - jS\| < \varepsilon$, where $j \in L(\ell_1, d_{w,1})$ is the formal identity operator.*

Proof. Let $\varepsilon > 0$ be fixed. Find $N \in \mathbb{N}$ such that $\|(Tf_n)^*|_{[N,\infty)}\| < \varepsilon$ for all n. Let $z_n \in d_{w,1}$ be the vector obtained from Tf_n by keeping its largest N coordinates and replacing the rest of the coordinates with zeros.

Define $S : \ell_1 \to d_{w,1}$ by $Sf_n = z_n$. Note that $\|T - S\| = \sup_n \|(T - S)f_n\| = \sup_n \|Tf_n - z_n\| \leqslant \varepsilon$; in particular, S is bounded. Let $F = \operatorname{span}\{e_1, \ldots, e_N\}$. Since $\dim F < \infty$, there exists $C > 0$ such that

$$\frac{1}{C}\|x\|_{\ell_1} \leqslant \|x\|_{d_{w,1}} \leqslant C\|x\|_{\ell_1}$$

for all $x \in F$. Observe that for each $n \in \mathbb{N}$, the non-increasing rearrangement $(Sf_n)^*$ is in F. Therefore, for all $n \in \mathbb{N}$, we have

$$\|Sf_n\|_{\ell_1} = \|(Sf_n)^*\|_{\ell_1} \leqslant C\|(Sf_n)^*\|_{d_{w,1}} = C\|Sf_n\|_{d_{w,1}} \leqslant C\|S\|.$$

It follows that the operator $\widetilde{S} : \ell_1 \to \ell_1$ defined by $\widetilde{S}f_n = Sf_n$ belongs to $L(\ell_1)$. Obviously, $S = j\widetilde{S}$. So, $\|T - j\widetilde{S}\| < \varepsilon$. $\square$

The next corollary follows immediately from Theorem 6.6 and Corollary 6.4. This corollary can be considered as a support for Conjecture 6.1.

Corollary 6.7. *$\mathcal{SS}(\ell_1, d_{w,1})$ is contained in the closure of $\{jS : S \in L(\ell_1, d_{w,1})\}$.*

Question. Does Corollary 6.7 remain valid for $p > 1$?

The following fact is standard, we include its proof for convenience of the reader.

Proposition 6.8. *If X is a Banach space then $\mathcal{SS}(X, \ell_1) = \mathcal{K}(X, \ell_1)$.*

Proof. Let $T \notin \mathcal{K}(X, \ell_1)$. Pick a bounded sequence (x_n) in X such that (Tx_n) has no convergent subsequences. By Schur's theorem, (Tx_n) and, therefore, (x_n) have no weakly Cauchy subsequences. Applying Rosenthal's ℓ_1-theorem twice, we find a subsequence (x_{n_k}) such that (x_{n_k}) and (Tx_{n_k}) are both equivalent to the unit vector basis of ℓ_1. It follows that T is not strictly singular. $\quad\square$

Proposition 6.9. *For all* $p \in [1, \infty)$, $\mathcal{SS}(d_{w,p}, \ell_p) = \mathcal{K}(d_{w,p}, \ell_p)$.

Proof. By Proposition 6.8, we only have to consider the case $p > 1$. Let $T \notin \mathcal{K}(X, \ell_p)$. Pick a bounded sequence (x_n) in X such that (Tx_n) has no convergent subsequences. Since $d_{w,p}$ contains no copies of ℓ_1, by Rosenthal's ℓ_1-theorem we may assume that (x_n) is weakly Cauchy. Passing to a further subsequence, we may assume that the sequence (Ty_n), where $y_n = x_{2n} - x_{2n-1}$, is seminormalized. It follows that (y_n) is also seminormalized. Also, (y_n) and, therefore, (Ty_n) are weakly null. Passing to a subsequence of (x_n), we may assume that (y_n) and (Ty_n) are both basic, equivalent to block sequences of (e_n) and (f_n), respectively. By [3, Proposition 5] and [17, Proposition 2.a.1], $(f_n) \succcurlyeq (y_n)$ and $(f_n) \sim (Ty_n)$. So, we obtain the diagram

$$(f_n) \succcurlyeq (y_n) \succcurlyeq (Ty_n) \sim (f_n).$$

Hence $[y_n]$ is isomorphic to $[Ty_n]$, so that T is not strictly singular. $\quad\square$

The following lemma is standard.

Lemma 6.10. *Let X be a Banach space. Every seminormalized basic sequence in X is dominated by the unit vector basis of ℓ_1.*

Lemma 6.11. *Let (x_n) and (y_n) be two sequences in a Banach space X such that (x_n) is equivalent to the unit vector basis of ℓ_1 and (y_n) is convergent. Then the sequence (z_n) defined by $z_n = x_n + y_n$ has a subsequence equivalent to the unit vector basis of ℓ_1.*

Proof. Observe that (z_n) cannot have weakly Cauchy subsequences since (x_n) does not have such subsequences. Since (z_n) is bounded, the result follows from Rosenthal's ℓ_1-theorem. $\quad\square$

Recall that an operator A between two Banach lattices X and Y is called **positive** if $x \geqslant 0$ entails $Tx \geqslant 0$.

Conjecture 6.1 asserts, in particular, that if $T \in \mathcal{SS}(d_{w,p})$ and $T = AB$ for some $A : d_{w,p} \to \ell_p$ and $B : \ell_p \to d_{w,p}$ then $T \in \overline{J^j}$. In the next theorem, we prove this under the additional assumptions that $p = 1$ and both A and B are positive.

Theorem 6.12. *Let $T \in \mathcal{SS}(d_{w,1})$ be such that $T = AB$, where $A \in L(\ell_1, d_{w,1})$, $B \in L(d_{w,1}, \ell_1)$, and both A and B are positive. Then $T \in \overline{J^j}$.*

Proof. Define a sequence (A_N) of operators in $L(\ell_1, d_{w,1})$ by the following procedure. For each $n \in \mathbb{N}$, let $A_N f_n$ be obtained from Af_n by keeping the largest N coordinates and replacing the rest of the coordinates with zeros. Since $Af_n \geqslant 0$ for all $n \in \mathbb{N}$, this defines a positive operator $\ell_1 \to d_{w,1}$. Also, $\|A_N f_n\| \leqslant \|Af_n\| \leqslant \|A\|$ for all $n \in \mathbb{N}$, hence $\|A_N\| \leqslant \|A\|$.

Define $A'_N = A - A_N$. It is clear that $0 \leqslant A'_N f_n \leqslant Af_n$ for all $n \in \mathbb{N}$, hence $A'_N \geqslant 0$ and $\|A'_N\| \leqslant \|A\|$. We claim that $A'_N \to 0$ in the strong operator topology (SOT). Indeed, since $A'_N f_n$ is obtained from Af_n by removing the largest N coordinates, the elements of the matrix of A'_N are all smaller than $\frac{\|A\|}{s_N}$ by Lemma 3.4. In particular, if $0 \leqslant x \in \ell_1$, then $A'_N x \downarrow 0$; it follows that $\|A'_N x\| \to 0$ because $d_{w,1}$ has order continuous norm (see Remark 1.5). If $x \in \ell_1$ is arbitrary then $\|A'_N x\| \leqslant \|A'_N |x|\| \to 0$.

We will show that $\|A'_N B\| \to 0$ as $N \to \infty$, so that $\|AB - A_N B\| \to 0$ as $N \to \infty$. Since $(A_N f_n)_{n=1}^\infty$ is almost lengthwise bounded (in fact, the vectors in the sequence $(A_N f_n)_{n=1}^\infty$ all have at most N nonzero entries), the theorem will follow from Theorem 6.6.

Assume, by way of contradiction, that there are $c > 0$ and a sequence (N_k) in $\mathbb{N}$ such that $\|A'_{N_k} B\| > c$. Then there exists a normalized positive sequence (x_k) in $d_{w,p}$ such that $\|A'_{N_k} B x_k\| > c$. By Rosenthal's ℓ_1-theorem, we may assume that (x_k) is either weakly Cauchy or equivalent to (f_n).

Assume that (x_k) is weakly Cauchy. Then (Bx_k) is weakly Cauchy. Since (Bx_k) is a sequence in ℓ_1, it must converge to some $z \in \ell_1$ by the Schur property. Then $\|A'_{N_k} Bx_k - A'_{N_k} z\| \leqslant \|A'_{N_k}\| \cdot \|Bx_k - z\| \leqslant \|A\| \cdot \|Bx_k - z\| \to 0$. Since $A'_{N_k} \to 0$ in SOT, it follows that $A'_{N_k} Bx_k \to 0$, contrary to the assumption. Therefore (x_k) must be equivalent to (f_n).

Since the entries of the matrix of A'_N are all less than $\frac{\|A\|}{s_N}$, the coordinates of the vector $A'_{N_k} Bx_k$ are all less than $\frac{\|A\|}{s_{N_k}} \|B\| \to 0$. Hence, passing to a subsequence, we may assume that $(A'_{N_k} Bx_k)$ is equivalent to a block sequence (u_k) of (e_n) such that each u_k is a restriction of $A'_{N_k} Bx_k$. In particular, the coordinates of (u_k) converge to zero. Passing to a further subsequence, we may assume by Remark 1.4 that $(A'_{N_k} Bx_k) \sim (f_n)$.

The sequence (Tx_k) cannot have subsequences equivalent to (f_n) since T is strictly singular. Therefore, by Rosenthal's ℓ_1-theorem, we may assume that (Tx_k) is weakly Cauchy. Since $d_{w,1}$ is weakly sequentially complete (Remark 1.5), the sequence (Tx_k) weakly converges to a vector $y \in d_{w,1}$. Since the positive cone in a Banach lattice is weakly closed, $y \geqslant 0$.

Note that $Tx_k \geqslant A'_{N_k} Bx_k \geqslant u_k \geqslant 0$ for every k. Since (u_k) is a seminormalized block sequence of (e_n), it follows that (Tx_k) is not norm convergent. Write $Tx_k = y + h_k$; then (h_k) converges to zero weakly but not in norm. Therefore, passing to a subsequence, we may assume that (h_k) is seminormalized and basic (but not, necessarily, positive).

Let $r_k = A'_{N_k} Bx_k - (A'_{N_k} Bx_k \wedge y) \geqslant 0$, $k \in \mathbb{N}$. Observe that $A'_{N_k} Bx_k \wedge y \in [0, y]$ for all k. Since $d_{w,1}$ has order continuous norm and the order in $d_{w,1}$ is defined by a 1-unconditional basis, order intervals in $d_{w,1}$ are compact (see, e.g., [24, Theorem 6.1]). Therefore, passing to a subsequence of (x_{n_k}), we may assume that $(A'_{N_k} Bx_k \wedge y)$ is convergent, hence, passing to a further subsequence, (r_k) is equivalent to (f_n) by Lemma 6.11 and Theorem 1.2.

It follows from $y + h_k \geqslant A'_{N_k} Bx_k \geqslant 0$ that $|h_k| \geqslant r_k$ for all k. Passing to a subsequence, we may assume by Lemma 6.2 that $(h_k) \succcurlyeq (r_k) \sim (f_n)$. By Lemma 6.10, in fact $(h_k) \sim (f_n)$, and, hence, by Lemma 6.11, $(ABx_k) \sim (f_n)$. Since also $(x_k) \sim (f_n)$, this contradicts to $T = AB \in \mathcal{SS}(d_{w,1})$. $\quad\square$

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