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Note

C_3 saturated graphs

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Abstract

The first part of this paper deals with the properties of C_3-saturated graphs. It will be shown that for any C_3-saturated graph, G, $D_2(D_2(G)) = G$, where $D_k(G)$ is the graph with vertex set $V(G)$, with which two vertices are adjacent iff the distance between them, in G, is k. In addition to this, a full description of the set of planar C_3-saturated graphs, $\mathrm{PSAT}(n, C_3)$, will be given. It will be shown that there are only three kinds of such graphs. In the second part of the paper a useful characterization of graphs which are C_3-saturated and C_4-free will be given in terms of the adjacency and incidence matrices A and B.

Keywords: Saturated graphs; C_3; C_4-free

1. Introduction

We study undirected graphs without loops or multiple edges. Given a graph G; $V(G)$, $E(G)$, $v(G)$ and $e(G)$ stands for the set of vertices, the set of edges, the order (number of vertices) and the size (number of edges) of G. By $N(x)$ we denote the set of vertices adjacent to the vertex x, $[N(x)] = N(x) \cup \{x\}$ and $d(x) = |N(x)|$. By $d(x, y)$ we denote the distance between the vertices x and y, and by $\mathrm{diam}(G)$ we denote the diameter of the graph G.

K_n, E_n, K_{r_1,r_2}, C_n and S_n stand for the complete graph of order n, the empty graph of order n, the complete bipartite graph with r_i vertices in the ith class, a cycle of length n and a star of size n.

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Given a set of forbidden graphs $\mathscr{R}$, we say that the graph G is $\mathscr{R}$- *saturated* if it contains no $R \in \mathscr{R}$ but the addition of any new edge gives a forbidden subgraph. We denote the set of $\mathscr{R}$-saturated graphs of order n by $\mathrm{SAT}(n, \mathscr{R})$. By $\mathrm{PSAT}(n, \mathscr{R})$ we denote the set of planar $\mathscr{R}$-saturated graphs of order n. Furthermore, we denote by $ex(n, \mathscr{R})$ and $mx(n, \mathscr{R})$ the maximal and the minimal size of an $\mathscr{R}$-saturated graph of order n. By $EX(n, \mathscr{R})$ and $MX(n, \mathscr{R})$ we denote the sets of all $\mathscr{R}$-saturated graphs of order n and size $ex(n, \mathscr{R})$ or $mx(n, \mathscr{R})$, respectively. A graph $G \in EX(n, \mathscr{R})$ ($G \in MX(n, \mathscr{R})$) will be called a maximal (minimal) $\mathscr{R}$-saturated graph. If $\mathscr{R} = \{H\}$, for a certain graph H, we write H-saturated instead of $\{H\}$-saturated.

The problem of determining the value of $ex(n, \mathscr{R})$ is called *a Turán-type extremal problem*. In 1941 Turán [16] proved that the only maximal K_p-saturated graph of order n is the Turán graph $T_{p-1}(n)$ which is the complete $(p-1)$-partite graph with as equal classes as possible. Turán probably got the idea for this kind of problem from a theorem that Ramsey proved 13 years before [14]. He proved that for every s and t, there exists a sufficiently large n and a graph G of order n such that $K_s \subseteq G$ or $K_t \subseteq \bar{G}$. Ramsey searched for a condition on the order of a graph G that will assure the existence of a complete subgraph (of G or $\bar{G}$). On the other hand, Turán searched for a condition on the size of a graph that will assure the existence of a complete subgraph. Due to Ramsey's and Turán's types of problems, Erdös and Sós presented a new type of problem called *a Ramsey–Turán type extremal problem* [9], that is, finding the largest number of edges in a graph of order n not containing a complete subgraph of order t and no independent set of vertices of order s. Denoting this number by $g(n, t, s)$ they proved that $g(n, t, s) \leqslant ns/2$, and for sufficiently large n, $g(n, 5, s) = \frac{(1+o(1))n^2}{4}$ and $g(n, 4, s) \leqslant \frac{(1+o(1))n^2}{6}$. The last inequality was improved by Szemerédi [15]. He showed that for sufficiently large n, $g(n, 4, s) \leqslant \frac{(1+o(1))n^2}{8}$. In 1976 Bollobás and Erdös showed that the last inequality is actually an equality [6].

Actually, Turán was not the first one to deal with the type of problem that is named after him. In 1907, the theorem of Turán was already proved for the case of K_3 by Mantel [12], and in 1938 Erdös raised the question about the maximal size of a C_4-saturated graph, in connection with the number theory [7]. This problem is only partially solved. It is known that $ex(n, C_4) \leqslant \frac{n}{4}(1 + \sqrt{4n-3})$ and equality holds for infinitely many values of n [5,10]. The first theorem of *Turán-type* is probably: "a graph of order n and size n contains a cycle" namely $ex(n, \mathscr{C}) = n - 1$, where $\mathscr{C}$ is the set of all cycles. Furthermore, it is easy to see that $\mathrm{SAT}(n, \mathscr{C}) = \mathscr{T}_n$. In time another type of problem was raised: finding the minimal size of a $\mathscr{R}$-saturated graph of order n for a set $\mathscr{R}$ of forbidden graphs. The first to deal with this type of problem were Erdös et al. [8]. They proved that the only minimal K_p-saturated graph of order n is the graph $K_{p-2} + E_{n-p+2}$. Ollman proved [13] that $mx(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor$. Barefoot et al. [1] gave an asymptotic value of $mx(n, C_k)$. They proved that for any natural number k and sufficiently large n, $n + \frac{an}{k} \leqslant mx(n, C_k) \leqslant n + \frac{bn}{k}$ for some positive constants a and b. Finally Kászonyi and Tuza [11] found the values of $mx(n, S_m)$, $mx(n, mK_2)$ and $mx(n, P_m)$ for any m and sufficiently large n.

2. C_3-saturated graphs

Theorem 2.1. *A graph G of order $n \geqslant 3$ is C_3-saturated if and only if it is C_3-free and* $\mathrm{diam}(G) = 2$.

Proof. The conditions are clearly sufficient, since every two nonadjacent vertices x and y of a graph G with diameter 2 must be connected by a path of length 2; therefore, $G+xy$ contains a triangle.

Conversely, since G is C_3-saturated, it is clear that $\mathrm{diam}(G)\leqslant 2$, and it cannot be 1 since for $n\geqslant 3$, K_n is not triangle-free. □

Theorem 2.2. *Let G be a C_3-saturated graph. G contains a cutting point if and only if G is a star.*

Proof. Clearly a star is C_3-saturated. Suppose G is C_3-saturated and it contains a cutting point x. Let $\{V_1, V_2\}$ be a partition of $V(G)-\{x\}$ such that there is no edge connecting a vertex from V_1 to a vertex of V_2 in $G-x$. Since $\mathrm{diam}(G)=2$, $d_G(x,t)=1$ for all t in V_1 and V_2. Furthermore, since G is triangle-free, V_1 and V_2 are independent in G. Therefore, G is a star. □

From this theorem we can conclude very easily the validity of the theorem proved by Erdös et al. [8] for the triangle case

$$mx(n, K_3) = n-1 \quad \text{and} \quad MX(n, K_3) = \{S_{n-1}\}.$$

Theorem 2.3. *Let G be a C_3-saturated graph which is not a star. Any two distinct vertices of G are contained in a cycle of length* 4 *or* 5.

Proof. Let x and y be two distinct vertices of G. Since G does not contain a cutting point, there exists a cycle in G containing x and y. If $xy \in E(G)$, let $C = xyt_1 \ldots t_m$ be a cycle containing x and y with a minimal length. By Theorem 2.1, $d=d(x,t_2)\leqslant 2$. If $d=1$, then $C=C_4$ and if $d=2$, then $C=C_5$.

If $xy \notin E(G)$, then $d(x,y)=2$; therefore, x and y have a mutual neighbor s. Since s is not a cutting point, there exists a path between x and y not going through s. Let $xt_1\ldots t_m y$ be a minimal such path. $d(t_1,y)\leqslant 2$, therefore either $d(t_1,y)=1$, which implies that x and y are contained in a C_4, or $d(t_1,y)=2$, which implies that x and y are contained in a C_5. □

It is clear that a bipartite graph is C_3-saturated if and only if it is complete.

Theorem 2.4. *If G is a C_3-saturated graph, which is not bipartite, then it contains a cycle of length* 5.

Proof. Let G be a C_3-saturated graph which is not bipartite, and x a vertex of G. Let $V_1=\{y \in V(G) | d(x,y)=1\}$ and $V_2=\{y\in V(G) | d(x,y)=2\}\cup\{x\}$. By Theorem 2.1 $\{V_1, V_2\}$ is a partition of $V(G)$. Since G is not bipartite, G must contain an edge connecting two vertices of V_1 or two vertices of V_2. Since G is C_3-free there cannot be an edge between two vertices of V_1; therefore, there exists $y_1, y_2 \in V_2-\{x\}$ such that $y_1y_2 \in E(G)$. By the definition of V_2, there exists $s_1, s_2 \in V_1$, $s_1 \neq s_2$, such that xs_1y_1 and xs_2y_2 are paths in G. Therefore, G contains a cycle of length 5 on the vertices x, s_1, y_1, y_2, s_2. □

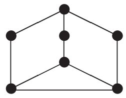

Fig. 1. L.

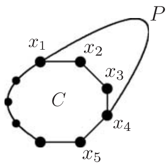

Fig. 2.

Theorem 2.5. *If G is a C_3-saturated graph which is not a complete bipartite graph, then $D_2(D_2(G)) = D_2(\bar{G}) = G$.*

Proof. Since G is C_3-saturated, $\operatorname{diam}(G)=2$; therefore, $xy \notin E(G)$ if and only if $d(x, y)=2$, hence $D_2(G) = \bar{G}$.

If $xy \in E(D_2(\bar{G}))$, then $xy \notin E(\bar{G})$, hence $xy \in E(G)$.

If $xy \in E(G)$, then $xy \notin E(\bar{G})$ and we separate two cases.

(a) There is no $x - y$ path in $\bar{G}$. In that case $N_{\bar{G}}(x) \cap N_{\bar{G}}(y) = \phi$. Furthermore, $[N_{\bar{G}}(x)] \cup [N_{\bar{G}}(y)] = V(\bar{G})$ because if $t \notin [N_{\bar{G}}(x)] \cup [N_{\bar{G}}(y)]$, then x, y and t form a triangle in G. Any two distinct vertices in $N_{\bar{G}}(x)$ are adjacent because otherwise they form a triangle with y (in G). The same is true for $N_{\bar{G}}(y)$. Thus in that case $\bar{G}$ consists of two complete components,$N_{\bar{G}}(x)$ and $N_{\bar{G}}(y)$; therefore, G is a complete bipartite graph, contradicting the assumption.

(b) There is a path between x and y in $\bar{G}$. Let $x, t_1, \ldots, t_m, y$ be a minimal such path.

$m < 2$, since otherwise $t_1t_2 \in E(\bar{G})$, giving $t_1t_2 \notin E(G)$ which imply that $d_G(t_1, t_2) = 2$. In that case there exists a vertex s such that $t_1s, t_2s \in E(G)$. Hence G contains a C_5 on the vertices xyt_1st_2. This cycle is chordless because G is C_3-free. Therefore xsy is an x–y path in $\bar{G}$ with length 2 contradicting the minimality of m.

So $d_{\bar{G}}(x, y) = 2$, implying that $xy \in E(D_2(\bar{G}))$. □

The opposite of the last theorem is not true. For example the graph L in Fig. 1 is not bipartite and it satisfies the conditions $D_2(L) = \bar{L}$ and $D_2(\bar{L}) = L$, but it is not C_3-saturated.

Lemma 2.6. *If G is a planar C_3-saturated graph, which is not a star, then all its faces are C_4 or C_5.*

Proof. Suppose $r \geqslant 6$, and the cycle $C = x_1x_2 \ldots x_r$ is a face of a planar presentation of a C_3-saturated graph G. Since $d(x_1, x_4) \leqslant 2$, there is an $x_1 - x_4$ path, P, of length $\leqslant 2$, such that $E(P) \cap E(C) = \phi$ (see Fig. 2). P separates x_2 from x_5 (i.e. there cannot be an x_2–x_5 path of length $\leqslant 2$ that is not crossing P), contradicting the fact that $d(x_2, x_5) \leqslant 2$. □

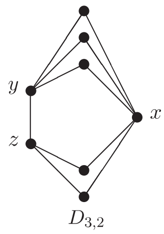

Fig. 3. $D_{3,2}$.

Lemma 2.7. *If G is a planar C_3-saturated graph, in which all the faces are quadrilateral, then G is bipartite.*

Proof. If G is not bipartite then it contains an odd cycle. Let C be an odd cycle, in a planar presentation of G, which contains a minimal amount of faces. Let F be a face of G contained in C such that $E(C) \cap E(F) \neq \phi$. By deleting the common edges of F and C from C, and adding $E(F) - E(C)$ we get another odd cycle in G with less faces inside, contradicting the assumption. □

Denote by $D_{k,m}$ the graph of order $k+m+3$ obtained from k distinct x–y paths of length 2 and m distinct x–z paths of length 2 together with an edge yz (see Fig. 3).

Theorem 2.8. *For $n \geqslant 5$,*

$$\mathrm{PSAT}(n, C_3) = \{S_{n-1}, K_{2,n-2}\} \cup \{D_{l,n-l-3}\}_{1 \leqslant l \leqslant \lfloor \frac{n-3}{2} \rfloor}.$$

Proof. Let $n \geqslant 5$ be a natural number and $G \in \mathrm{PSAT}(n, C_3)$. If G is bipartite, then $G = K_{1,n-1}$ or $G = K_{2,n-2}$, since otherwise $K_{3,3} \subseteq G$, contradicting its planarity. Assume that G is not bipartite and consider a planar representation of G. For the sake of convenience we call it G. By 2.6 and 2.7, G contains a pentagon face. Let $K = \{x_1, \ldots, x_5\}$ be the set of vertices of a pentagon face of G (appearing at this order). Denote $T_i = N(x_i) - K$, $1 \leqslant i \leqslant 5$, $T = \cup_{i=1}^{5} T_i$ and $S = V(G) - (T \cup K)$. Note that for every $s \in S$ and $1 \leqslant i \leqslant 5$ there exists $t_i \in T_i$ such that $st_i \in E(G)$, since $d(s, x_i) = 2$. Furthermore, $G[T_i]$ is an empty graph and $T_i \cap T_i' = \phi$, $\forall i \leqslant 5$, where $T_i' = T_{i-1 \bmod 5} \cup T_{i+1 \bmod 5}$. We separate two cases:

Case a: $T_i \cap T_j = \phi, \forall i \neq j$. In that case $S = \phi$, since otherwise if $s \in S$, then $\forall i, \exists t_i \in T_i, st_i \in E(G)$ and t_1 is separated from x_3. If T is also empty then $G = C_5 = D_{1,1}$. Otherwise,without loss of generality, suppose $t_1 \in T_1$. Since $d(t_1, x_3) = d(t_1, x_4) = 2$, there exists $t_3 \in T_3$ and $t_4 \in T_4$ such that $t_1t_3, t_1t_4 \in E(G)$, but then x_2 is separated from t_4.

Case b: There exists $i \neq j$ such that $T_i \cap T_j \neq \phi$. Without loss of generality, suppose that $t \in T_1 \cap T_3$. It follows that $T_2 = \phi$ since otherwise $\exists t_2 \in T_2$ which is separated from x_4. $S = \phi$, since any vertex $s \in S$ would be separated from x_2. Now, if $T_4 \cup T_5 = \phi$, then we easily get that $G = D_{1,n-4}$. If $T_5 \neq \phi$, then let $t_5 \in T_5$. $t_5 \in T_3$, since otherwise t_5 would be separated from x_2. As before, we have $T_4 = \phi$, and we easily get that $G = D_{l,n-l-3}$ for some l, $1 \leqslant l \leqslant \lfloor \frac{n-3}{2} \rfloor$, as claimed. Finally, by symmetry, If $T_4 \neq \phi$, then $T_5 = \phi$ and $T_4 \subseteq T_1$, which means $G = D_{l,n-l-3}$ for some l, $1 \leqslant l \leqslant \lfloor \frac{n-3}{2} \rfloor$, as claimed. □

3. C_3-saturated graphs that are C_4-free

Denote by $\mathrm{SAT}_4(n, C_3)$ the set of C_3-saturated graphs of order n, that are C_4-free.

For a graph G on the set of vertices $\{v_1, \ldots, v_n\}$; $A = A(G)$, $B = B(G)$ and $D = D(G)$ stand for the adjacency matrix, the incidence matrix and the $n \times n$ diagonal matrix $D = (D_{ij})$ with $D_{ii} = d(v_i)$. Also let $J = (J_{ij})$ be an $n \times n$ matrix with $J_{ij} = 1$ for all i, j, and let I stand for the unit matrix. The next theorem gives us a very easy way to check by a computer if $G \in SAT_4(n, C_3)$ or not.

Theorem 3.1. $G \in \mathrm{SAT}_4(n, C_3)$ *iff* $A^2 + 2A - BB^t = J - I$.

Proof. A necessary and sufficient condition for a graph G, of order n, to be C_3-saturated and C_4-free is that for every two distinct non-adjacent vertices, $v_i, v_j \in V(G)$, G contains exactly one $v_i - v_j$ path with length 2, and for every two adjacent vertices, $v_i, v_j \in V(G)$, G does not contain any $v_i - v_j$ path with length 2.

Consider the matrix A^2. Since, for all $i \neq j$, $(A^2)_{i,j}$ equals to the number of $v_i - v_j$ paths with length 2, and $(A^2)_{i,i} = d(v_i)$, the above condition is equivalent to

$$(A^2)_{i,j} = \begin{cases} d(v_i), & i = j, \\ 1, & i \neq j \text{ and } A_{i,j} = 0, \\ 0, & i \neq j \text{ and } A_{i,j} = 1. \end{cases}$$

Therefore, $G \in SAT_4(n, C_3)$ iff

$$A^2 = J - I - A + D.$$

Hence the assertion is true by the equation $A + D = BB^t$, which is true for every adjacency and incidence matrices A and B. □

Theorem 3.2. *The only planar graph in* $\mathrm{SAT}_4(n, C_3)$ *for* $n \geqslant 5$ *is* S_{n-1} *except the case where* $n = 5$ *in which the graph* C_5 *is added.*

Proof. Immediately from 2.8. □

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