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Duality for Minmax Programs

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A duality theory is derived for minimizing the maximum of a finite set of convex functions subject to a convex constraint set generated by both linear and nonlinear inequalities. The development uses the theory of generalised geometric programming. Further, a particular class of minmax program which has some practical significance is considered and a particularly simple dual program is obtained.

1. INTRODUCTION

We consider minmax programs of the form

$$P_1: \qquad \underset{x}{\text{Minimize}} \underset{i=1,\ldots,I}{\text{Maximize}} \{f_i(x)\} \tag{1}$$

subject to the constraints

$$x \in C, \tag{2}$$

$$g_j(x) \leqslant 0, \qquad j = 1,\ldots, J. \tag{3}$$

$$Ax \geqslant b, \tag{4}$$

Here $f_i(\cdot)$, $i = 1,\ldots, I$ are closed convex functions with domain C, a closed convex set. $g_j(\cdot)$ are closed convex functions defined over the same domain and A and b are given matrices. It is well known that the maximum of a finite set of closed convex functions is both closed and convex. Hence program P_1 is a convex program.

Minmax optimization problems arise in a variety of settings as, for example, in planning production to meet a stochastic demand when the

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distribution function of demand is unknown [3] and in the location of a critical facility such as an ambulance base [1].

In this paper, we derive a complete duality theory for minmax programs of the type P_1. This development uses the theory of generalised geometric programming which is summarised in Section 2. In Section 4, we further particularise the form of program P_1 to a subset which contains location problems of practical significance and for which the dual program is particularly simple.

2. Conjugate Duality Theory

The approach to duality presented in this paper uses the concepts of conjugate functions due to Fenchel [2] and further developed by Rockafellar [4]. Due to its great flexibility, we use the generalised geometric programming version of conjugate duality theory which was developed by Peterson [5]. The theory of generalised geometric programming pairs the following two programs.

$$P: \qquad \text{Minimize } g_0(x^0)$$

subject to explicit constraints

$$g_i(x^i) \leqslant 0, \qquad i \in I,$$

implicit constraints

$$x^0 \in C_0, \qquad x^i \in C_i, \qquad i \in I,$$

and cone condition

$$x \in \chi \subset R^n$$

$$D \qquad \text{(the geometric programming dual of } P\text{):}$$

$$\text{Minimize } g_0^*(x^{0*}) + \sum_I g_i^{*+}(x^{i*}; \lambda_i^*)$$

subject to implicit constraints

$$x^{0*} \in C_0^*, \qquad (x^{i*}; \lambda_i^*) \in C_i^{+*}, \qquad i \in I,$$

$$\text{and cone condition } x^* \in \chi^* \subset R^n$$

The relations between programs P and D are as follows: χ is a closed convex cone. χ^* is the dual cone of χ, i.e., $\chi^* = \{x^* \mid \langle x^*, x\rangle \geqslant 0 \ \forall x \in \chi\}$. I is the index set of explicit constraints in program P. $x = (x^0, x^I)$ is the cartesian product of vectors x^0 and x^I. $x^* = (x^{0*}, x^{I*})$ is similarly defined. x^I is the cartesian product of vectors x^i, $i \in I$. x^{I*} is similarly defined. $\langle \cdot, \cdot \rangle$ denotes a finite dimensional inner product.

$[g_i(x^i), C_i]$, $i \in \{0\} \cup I$ is a pair of closed convex function g_i defined over a convex set $C_i \subset R^{n_i}$. $[g_i^*(x^{i*}), C_i^*]$, $i \in \{0\} \cup I$ is a pair of closed convex function g_i^* defined over the convex set C_i^* and is the conjugate transform of $[g^i(x^i), C_i]$, i.e.,

$$g_i^*(x^{i*}) \triangleq \sup_{x^i \in C_i} (\langle x^{i*}, x^i \rangle - g_i(x^i))$$

and

$$C_i^* = \{x^{i*} \mid \sup_{x^i \in C_i} (\langle x^{i*}, x^i \rangle - g_i(x^i)) < \infty\}.$$

$[g_i^{*+}(x^{i*}; \lambda_i^*), C_i^{*+}]$, $i \in I$ is the positive homogeneous extension of $[g_i^*(x^{i*}), C_i^*]$ with

$$g_i^{*+}(x^{i*}; \lambda_i^*) \triangleq \begin{cases} \sup_{x^i \in C_i} \langle x^{i*}, x^i \rangle \text{ if } \lambda_i^* = 0 \text{ and } \sup_{x^i \in C_i} \langle x^{i*}, x^i \rangle < \infty \\ \lambda_i^* g_i^*(x^{i*}/\lambda_i^*) \text{ if } \lambda_i^* > 0 \text{ and } x_i^*/\lambda_i^* \in C_i, \end{cases}$$

$$C_i^{*+} = \{(x_i^*; \lambda_i^*) \mid \lambda_i^* = 0 \text{ and } \sup_{x^i \in C_i} \langle x^{i*}, x^i \rangle < \infty\}$$
$$\cup \{(x_i^*; \lambda_i^*) \mid \lambda_i^* > 0 \text{ and } x_i^*/\lambda_i^* \in C_i\}.$$

It is interesting to note that all the explicit constraints in the primal (program P) are transferred to the objective function in the dual (program D). Under mild conditions concerning feasibility and relative interiors [5], the primal and dual programs are related at optimality in the following manner:

$$g_0(x^0) + g_0^*(x^{0*}) + \sum_I g_i^{*+}(x^{i*}; \lambda_i^*) = 0,$$

$$x^{0*} \in \partial g_0(x^0),$$

$$x^{i*}/\lambda_i^* \in \partial g_i(x^i), \qquad \lambda_i > 0, i \in I.$$

These optimality conditions allow an optimal point for one program to be calculated from an optimal point of the other. Here $\partial g_0(x^0)$ denotes the subgradient set of g_0 at the point x^0, i.e.,

$$\partial g_i(x^i) \triangleq \{x^{i*} \mid g_i(x^i) + \langle x^{i*}, z^i - x^i \rangle \leqslant g_i(z^i), \forall z^i \in C_i\}.$$

3. Minmax Duality

Our original minmax program P_1 may be written in the equivalent form

$$P_2: \qquad \underset{x,\alpha}{\text{Minimize}}\ \alpha, \tag{5}$$

s.t.

$$x \in C, \tag{6}$$

$$g_j(x) \leqslant 0, \qquad j = 1, 2,..., J, \tag{7}$$

$$f_i(x) \leqslant \alpha, \qquad i = 1, 2,..., I, \tag{8}$$

$$Ax \geqslant b, \tag{9}$$

where the functions $f_i(\cdot)$, $i = 1,..., I$ are transferred from the objective to the constraint set. A new scalar variable α becomes the objective. In order to invoke the duality theory of generalised geometric programming as presented in Section 2, the variables in program P_2 must be separated. This results in the following equivalent program:

$$P_3: \qquad \text{Minimize}\ \alpha \tag{10}$$

subject to explicit constraints

$$g_j(x^j) \leqslant 0, \qquad j = 1, 2,..., J, \tag{11}$$

$$f_i(x^i) + \alpha_i \leqslant 0, \qquad i = J+1,..., J+I, \tag{12}$$

implicit constraints

$$x \in C, \tag{13}$$

$$x^j \in C, \qquad j = 1,..., J, \tag{14}$$

$$x_i \in C, \qquad i = J+1,..., J+I, \tag{15}$$

$$\alpha \in R, \tag{16}$$

$$\beta \in \{b\}, \tag{17}$$

and cone condition

$$Ax \geqslant \beta, \tag{18}$$

$$x = x^1 = \cdots = x^{J+I}, \tag{19}$$

$$\alpha = -\alpha_i, \qquad i = J+1,..., J+I. \tag{20}$$

Using the prescriptions of programs P and D in Section 2, the dual objective is found to be

$$\sum_{i=J+1}^{J+I} f_i^{*+}(x^{i*}; \alpha_i^*) + \sum_{j=1}^{J} g_j^{*+}(x^{j*}; \alpha_j^*) + \langle \beta^*, b \rangle,$$

where

$$\begin{aligned} x^* &= 0, \\ \alpha^* &= 1, \\ (x^{j*}; \alpha_j^*) &\in C_j^*, \qquad j = 1,..., J, \\ (x^{i*}; \alpha_i^*) &\in C_i^*, \qquad i = J+1,..., J+I. \end{aligned}$$

The dual cone corresponding to the cone generated by (18), (19) and (20) is defined by

$$\begin{aligned} x^* &= A^T u + \sum_{j=1}^{J} v^j + \sum_{i=J+1}^{J+I} v^i \\ x^{j*} &= -v^j, \qquad j = 1,..., J, \\ x^{i*} &= -v^i, \qquad i = J+1,..., J+I, \\ \alpha^* &= \sum_{i=J+1}^{J+I} w_i, \\ \alpha^{i*} &= w_i, \qquad i = J+1,..., J+I, \\ \beta^* &= -u, \\ u &\geqslant 0. \end{aligned}$$

Here u, v and w are dual vector variables associated with (18), (19) and (20), respectively.

Combining the above results gives the following dual to program P_3 and hence to program P_1.

D_1(or D_3): Minimize $\sum_{j=1}^{J} g_j^{+*}(-v^j; \alpha_j^*) + \sum_{i=J+1}^{J+I} f_j^{+*}(-v^i; w_i) - b^T u$

subject to implicit constraints

$$\begin{aligned} (-v^j; \alpha_j^*) &\in C_j^{+*}, \qquad j = 1,..., J, \\ (-v^i; w_i) &\in C_i^{+*}, \qquad i = J+1,..., J+I, \end{aligned}$$

and the cone condition

$$A^T u + \sum_{j=1}^{J} v^j + \sum_{i=J+1}^{J+I} v^i = 0,$$

$$\sum_{i=J+1}^{J+I} w_i = 1,$$

$$u \geqslant 0.$$

4. A Particular Class

Consider the following minmax program

$$P_4: \qquad \underset{x}{\text{Minimize}}\ \underset{i}{\text{Maximize}}\ \{a_i f(x) + \langle b_i, x\rangle + c_i\}, \tag{21}$$

where $f(x)$ is a closed convex function defined on a closed convex set C. $a_i > 0$, b_i and c_i are given. A special case of the above formulation arises in the location of a facility to minimize the maximum weighted distance from a finite number of fixed points. In this case, program P_4 assumes the form

$$P_5: \qquad \underset{x}{\text{Minimize}}\ \underset{i}{\text{Maximize}}\ w_i \langle x - a_i, x - a_i\rangle,$$

where $w_i > 0$ are weights and a_i are fixed locations. It has been shown that the dual approach to program P_5 is computationally more efficient than the primal. The theory given in Section 3 allows the extension of minmax location theory to location within a constrained region.

Returning to program P_4, it may be written in the equivalent form

$$P_6: \qquad \underset{x, \alpha}{\text{Minimize}}\ \alpha \text{ over } x \in C \tag{22}$$

s.t.

$$a_i f(x) + \langle b_i, x\rangle + c_i \leqslant \alpha. \tag{23}$$

Inequality (23) may be further written as

$$f(x) + \beta \leqslant 0, \tag{24}$$

$$\beta - \langle \tilde{b}_i, x\rangle - \gamma_i + a_i^{-1}\alpha - \delta_i = 0 \qquad \forall i, \tag{25}$$

$$\delta_i \geqslant 0, \qquad \forall i, \tag{26}$$

$$\gamma_i \in \{\tilde{c}_i\}. \tag{27}$$

Here $\tilde{b}_i = a_i^{-1} b_i$ and $\tilde{c}_i = a_i^{-1} c_i$.

Identifying, the nonlinear inequality (24) as an explicit constraint, (27) as an implicit constraint and (25), (26) as generating a cone, we may use the prescription of program D to generate a dual to program P_6 and hence of program P_4. The dual objective function will be

$$f^{+*}(x^*;\beta^*) + \langle \gamma^*, \tilde{c} \rangle$$

with

$$(x^*;\beta^*) \in C^{+*} \qquad \text{and} \qquad \alpha^* = 1.$$

The dual cone of (25) and (26) is shown to be given by

$$\begin{aligned} \beta^* &= \sum_i u_i, \\ \alpha^* &= \sum_i a_i^{-1} u_i, \\ x^* &= B^T u, \\ \gamma^* &= -u, \\ \delta^* &= -u + w, \\ w &\geqslant 0, \end{aligned} \tag{28}$$

where $B = (\tilde{b}_1, \tilde{b}_2, \ldots)$. Here u and w are dual variables corresponding to the cone constraints (25) and (26), respectively. It is further noted, that since β^* is the dual variable associated with the nonlinear inequality (24) which arises from the minmax objective function, it follows that $\beta^* > 0$ and the dual objective function becomes

$$\beta^* f(x^*/\beta^*) + \langle \gamma^*, \tilde{c} \rangle$$

with

$$x^*/\beta^* \in C^* \qquad \text{and} \qquad \alpha^* = 1, \qquad \delta^* = 0.$$

It should be noted that the nonlinear inequality will always be active since it prevents the objective function from tending to minus infinity.

Combining the above gives the dual program

$$D_4(\text{or } D_6): \qquad \text{Minimize} \sum_i u_i f^* \left(B^T u \Big/ \sum_i u_i \right) - \langle c, u \rangle$$

s.t.

$$\sum_i a_i^{-1} u_i = 1,$$

$$u_i \geqslant 0,$$

$$B^T u \Big/ \sum_i u_i \in C^*.$$

Generally C^* will be R^n or R^n_+ and hence the latter constraint will present no computational difficulty. Further $\beta^* > 0$, implies that $\sum_i u_i > 0$ and D_4 is a straightforward finite dimensional convex program over a linear space.

Finally the primal and dual variables are related by

$$B^T u \Big/ \sum_i u_i \in \partial f(x).$$

References

1. J. Elzinga, D. Hearn, and W. D. Randolph, Minimax multifacility location with Euclidean distances, *Transportation Sci.* **10** (1976), 321–336.
2. W. Fenchel, "Convex Cones, Sets and Functions," Lecture Notes, Princeton Univ. Press, Princeton, N.J., 1953.
3. R. Jagannathan, A Minmax ordering policy for the infinite stage dynamic inventory problem, *Management Sci.* **24** (1978), 1138–1149.
4. T. R. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, N.J., 1970.
5. E. L. Peterson, Geometric programming, *SIAM Rev.* **18** (1976), 1–52.