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A Maximum Principle for Periodic Solutions of the Telegraph Equation

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1. INTRODUCTION

Let $\mathscr{L} = \mathscr{L}u$ be a linear differential operator acting on functions

$$u\colon \Omega \to \mathbb{R}$$

that are defined on a fixed manifold Ω. These functions will belong to a certain family $\mathscr{B} \subset \mathscr{F}(\Omega, \mathbb{R})$, and the definition of $\mathscr{B}$ may include some boundary conditions or other requirements that must be satisfied by any function $u \in \mathscr{B}$. It is said that $\mathscr{L}$ satisfies the maximum principle if the differential inequality

$$\mathscr{L}u \geq 0, \qquad u \in \mathscr{B}$$

implies

$$u \geq 0 \qquad \text{in } \Omega.$$

The book [18] is devoted to the study of the maximum principle for second order operators. Other related results can be found in [17], [5], [12], [8], [2], [14].

In this paper we study the maximum principle for periodic solutions of the telegraph equation. The operator is

$$\mathscr{L}_\lambda u = u_{tt} - u_{xx} + cu_t - \lambda u,$$

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with $c > 0$ and $\lambda \in \mathbb{R}$. The functions u in the class $\mathscr{B}$ are defined in the whole plane

$$u = u(t, x), \qquad (t, x) \in \mathbb{R}^2,$$

and they are doubly periodic in the following sense:

$$u(t + 2\pi, x) = u(t, x + 2\pi) = u(t, x), \qquad (t, x) \in \mathbb{R}^2.$$

The main result of the paper will say that $\mathscr{L}_\lambda$ satisfies the maximum principle if and only if λ belongs to a certain interval of the form

$$[-\nu, 0),$$

where $\nu = \nu(c)$ is a positive quantity that depends on c. The number ν will not be computed explicitly, but it can be estimated. In particular, it satisfies

$$\frac{c^2}{4} < \nu \leq \frac{c^2}{4} + \frac{1}{4}.$$

The positivity of the friction coefficient c is essential. In fact, there is no maximum principle for the case $c = 0$.

Maximum principles are studied because they have many applications in the theory of linear and nonlinear differential equations. Some of the most classical applications can be seen in [18] and [1]. In this paper we present some consequences of the obtained maximum principle. First we apply the theory of linear positive operators (see [11]) to derive some corollaries for the linear telegraph equation with variable coefficients. In particular, the maximum principle will still be valid for the operator

$$\mathscr{L}_\alpha u = u_{tt} - u_{xx} + cu_t - \alpha(t, x)u$$

if α is a doubly periodic function satisfying

$$-\nu < \alpha(t, x) < 0, \qquad (t, x) \in \mathbb{R}^2.$$

As a second application, we find a method of upper and lower solutions for the periodic boundary value problem associated with the nonlinear equation

$$u_{tt} - u_{xx} + cu_t = F(t, x, u). \tag{1.1}$$

This method is valid when the function F verifies

$$\frac{\partial F}{\partial u}(t, x, u) \geq -\nu.$$

The existence of periodic solutions of (1.1) has been considered in many papers. See, for instance, [19], [15], [9], [20], [10] and the references there. Most of them use local techniques or degree theory. The method of upper and lower solutions can be useful for extend to the telegraph equation some results that are well known for ordinary and parabolic equations.

The forced sine-Gordon equation is in the class defined by (1.1). It has the form

$$u_{tt} - u_{xx} + cu_t + a \sin u = f(t, x), \tag{1.2}$$

where $a > 0$ and f is doubly periodic.

When f is small, the existence of small periodic solutions was analyzed in a very precise way in [3]. We shall obtain some existence results without assuming the smallness of f, but imposing the restriction

$$a \leq \nu.$$

They will follow from a combination of the method of upper and lower solutions with some of the techniques developed in [16] for the forced pendulum equation.

The rest of the paper is organized into four sections. The maximum principle is stated in Section 2. This section also contains some preliminary results on the regularity of solutions. Sections 3 and 4 are devoted to applications. In particular, the results for the sine-Gordon equation are obtained in Section 4. Finally, in Section 5, we give the complete proofs of the results stated in Section 2.

2. A MAXIMUM PRINCIPLE ON THE TORUS

In this section we formulate in precise terms the maximum principle for the telegraph equation. First we shall state two preliminary results on the regularity of solutions. All of the results in this section will be proved in Section 5.

2.1. *Regularity of Solutions*

Let $\mathbb{T}^2$ be the torus defined as

$$\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}).$$

A point of $\mathbb{T}^2$ is denoted as $(\bar{t}, \bar{x})$, where (t, x) is a point of $\mathbb{R}^2$ and $\bar{t} = t + 2\pi\mathbb{Z}$, $\bar{x} = x + 2\pi\mathbb{Z}$. Doubly periodic functions will be identified to functions defined on the torus. In particular, the notations

$$L^p(\mathbb{T}^2), C(\mathbb{T}^2), \mathfrak{D}(\mathbb{T}^2) = C^\infty(\mathbb{T}^2), \ldots$$

stand for the spaces of doubly periodic functions with the indicated degree of regularity. The space of distributions on $\mathbb{T}^2$ is $\mathcal{D}'(\mathbb{T}^2)$, and the space of measures (the dual space of $C(\mathbb{T}^2)$) is $M(\mathbb{T}^2)$.

Given $c > 0$, we define the differential operator

$$\mathcal{L}u = u_{tt} - u_{xx} + cu_t,$$

acting on functions on the torus, $u\colon \mathbb{T}^2 \to \mathbb{R}$. The formal adjoint operator is defined as

$$\mathcal{L}^* u = u_{tt} - u_{xx} - cu_t.$$

Given $\lambda \in \mathbb{R}$ and $f \in L^1(\mathbb{T}^2)$, we consider the problem

$$\mathcal{L}u - \lambda u = f(t, x) \qquad \text{in } \mathcal{D}'(\mathbb{T}^2). \tag{2.1}$$

By a solution of (2.1) we understand a function $u \in L^1(\mathbb{T}^2)$ satisfying

$$\int_{\mathbb{T}^2} u(\mathcal{L}^*\phi - \lambda\phi) = \int_{\mathbb{T}^2} f\phi \qquad \forall \phi \in \mathcal{D}(\mathbb{T}^2).$$

The real eigenvalues of $\mathcal{L}$ are the numbers $\lambda \in \mathbb{R}$ such that the homogeneous equation

$$\mathcal{L}u - \lambda u = 0 \qquad \text{in } \mathcal{D}'(\mathbb{T}^2)$$

has nontrivial solutions. Using Fourier analysis as in [19], one can prove that the real eigenvalues are

$$\lambda_m = m^2, \qquad m = 0, 1, 2, \dots .$$

We introduce the notation

$$\sigma_{\mathbb{R}}(\mathcal{L}) = \{m^2 / m \in \mathbb{N}\}.$$

The next proposition is an analogue of a well-known result for the wave equation without friction (see [4], p. 124). It sums up the regularity theory for (2.1).

PROPOSITION 2.1. *Assume that $\lambda \notin \sigma_{\mathbb{R}}(\mathcal{L})$ and $f \in L^1(\mathbb{T}^2)$. Then* (2.1) *has a unique solution. This solution is continuous and satisfies the estimate*

$$\|u\|_{C(\mathbb{T}^2)} \le C_1 \|f\|_{L^1(\mathbb{T}^2)} \tag{2.2}$$

where C_1 is a constant that depends only on c and λ.

In addition, if $f \in L^p(\mathbb{T}^2)$, $1 < p < \infty$, then

$$u \in C^{0,\alpha}(\mathbb{T}^2), \qquad \alpha = 1 - \frac{1}{p},$$

and

$$\|u\|_{C^{0,\alpha}(\mathbb{T}^2)} \leq C_p \|f\|_{L^p(\mathbb{T}^2)}, \tag{2.3}$$

with $C_p = C_p(c, \lambda)$.

When the function f is replaced by a measure, it is still possible to get some regularity. Consider the problem

$$\mathscr{L}u - \lambda u = \mu \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2), \tag{2.4}$$

where $\mu \in M(\mathbb{T}^2)$ is a given measure. A solution of (2.4) is a function $u \in L^1(\mathbb{T}^2)$ such that

$$\int_{\mathbb{T}^2} u(\mathscr{L}^*\phi - \lambda\phi) = \langle \mu, \phi \rangle \qquad \forall \phi \in \mathfrak{D}(\mathbb{T}^2),$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $M(\mathbb{T}^2)$ and $C(\mathbb{T}^2)$.

PROPOSITION 2.2. *Assume that* $\lambda \notin \sigma_{\mathbb{R}}(\mathscr{L})$ *and* $\mu \in M(\mathbb{T}^2)$. *Then* (2.4) *has a unique solution* u *that belongs to* $L^\infty(\mathbb{T}^2)$ *and satisfies*

$$\|u\|_{L^\infty(\mathbb{T}^2)} \leq C_1 \|\mu\|_{M(\mathbb{T}^2)},$$

where C_1 *is the constant appearing in the previous proposition.*

2.2. *Main Results*

Let $\mathscr{L}_\lambda$ be the differential operator

$$\mathscr{L}_\lambda u = u_{tt} - u_{xx} + cu_t - \lambda u.$$

It is said that $\mathscr{L}_\lambda$ satisfies the maximum principle if $\lambda \notin \sigma_{\mathbb{R}}(\mathscr{L})$, and for each $f \in L^1(\mathbb{T}^2)$ satisfying

$$f \geq 0 \qquad \text{a.e. } \mathbb{R}^2,$$

then

$$u(t, x) \geq 0 \qquad \forall (t, x) \in \mathbb{R}^2.$$

Here u denotes the unique solution of (2.1).

The maximum principle is said to be strong if

$$f \geq 0 \quad \text{a.e. } \mathbb{R}^2, \qquad \int_{\mathbb{T}^2} f > 0$$

implies

$$u(t, x) > 0 \qquad \forall (t, x) \in \mathbb{R}^2.$$

THEOREM 2.3. *There exists a function*

$$\nu\colon (0,\infty) \to (0,\infty), \qquad c \mapsto \nu(c)$$

such that $\mathscr{L}_\lambda$ *satisfies the maximum principle if and only if*

$$-\lambda \in (0, \nu(c)].$$

Moreover, the maximum principle is always strong and the function ν *satisfies*

$$\frac{c^2}{4} < \nu(c) \le \frac{c^2}{4} + \frac{1}{4}, \tag{2.5}$$

$$\nu(c) \to 0 \qquad \text{as } c \searrow 0, \tag{2.6}$$

$$\nu(c) - \frac{c^2}{4} \to \frac{j_0^2}{8\pi^2} \qquad \text{as } c \nearrow +\infty, \tag{2.7}$$

where j_0 *is the first positive zero of the Bessel function* J_0.

Remarks. 1. It is not possible to obtain a maximum principle for the equation without friction ($c = 0$). In fact, given any $\lambda \neq 0$, it is possible to find a positive forcing f such that

$$u_{tt} - u_{xx} - \lambda u = f(t,x) \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2)$$

has a solution that changes sign. (Notice that if $\lambda = 0$, there are no solutions for positive f.)

To give an explicit example, let us start with the function

$$u(t,x) = 1 - \cos t \cos x - \varepsilon \cos nt.$$

It changes sign if ε is positive. Next we construct f so that u becomes a solution, that is,

$$f(t,x) = -\lambda(1 - \cos t \cos x) + (n^2 + \lambda)\varepsilon \cos nt.$$

It is now easy to select the parameters ε and n so that f is positive if $\lambda < 0$.

When $\lambda > 0$, one can repeat the process starting with

$$u(t,x) = -1 + \cos t \cos x + \varepsilon \cos nx.$$

2. If we restrict ourselves to functions that depend only on space or time, $u = u(x)$ or $u = u(t)$, we are led to the ordinary differential operators

$$\ell_\lambda u = -\frac{d^2u}{dx^2} - \lambda u,$$

$$L_\lambda u = \frac{d^2u}{dt^2} + c\frac{du}{dt} - \lambda u.$$

The operator $\mathscr{L}_\lambda$ satisfies the maximum principle if and only if $\lambda < 0$. This is just the classical maximum principle for periodic functions (see [18]). The critical value $\lambda = 0$ is precisely the first eigenvalue of $-d^2/dx^2$ acting on periodic functions.

On the other hand, the operator L_λ satisfies the maximum principle if and only if

$$0 < -\lambda \leq \frac{c^2}{4} + \frac{1}{4}.$$

This is well known for $c = 0$ (see, for instance, [5]). For $c > 0$ the proof is similar.

In this case the critical value $c^2/4 + 1/4$ does not seem to be related to eigenvalues, and it depends on the oscillatory properties of the solutions of $L_\lambda u = 0$. This constant explains the estimate $\nu(c) \leq c^2/4 + 1/4$ in the theorem.

3. The constant $\nu(c)$ in the theorem will depend on the oscillatory properties of the Green's function associated with (2.1). This function will be constructed in Section 5, and the Bessel function J_0 will play an important role.

We conclude this section with a variant of the maximum principle for the case of measures. Given $\mu \in M(\mathbb{T}^2)$, we say that μ is nonnegative if

$$\langle \mu, \phi \rangle \geq 0 \qquad \forall \phi \in C_+(\mathbb{T}^2),$$

where $C_+(\mathbb{T}^2)$ is the class of nonnegative functions in $C(\mathbb{T}^2)$.

COROLLARY 2.4. *Assume that* $-\lambda \in (0, \nu]$, *where* ν *is given by Theorem* 2.3, *and let* $\mu \in M(\mathbb{T}^2)$ *be a nonnegative measure. Then the solution of* (2.4) *satisfies*

$$u \geq 0 \qquad \text{a.e. } \mathbb{T}^2.$$

3. THE LINEAR EQUATION WITH VARIABLE COEFFICIENTS

In this section we combine the maximum principle with the theory of linear positive operators (see [11], [1]) to derive some consequences for linear telegraph equations.

Let σ be a fixed constant satisfying

$$-\sigma \in \big(0, \nu(c)\big], \tag{3.1}$$

where $\nu(c)$ is the constant introduced in Theorem 2.3. We consider the eigenvalue problem

$$\mathscr{L}_\sigma u = \lambda m(t,x)u \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2), \tag{3.2}$$

where m is a function in $L^1(\mathbb{T}^2)$ and λ is the parameter.

An eigenvalue of (3.2) is a number $\lambda \in \mathbb{R}$ (or $\mathbb{C}$, if we admit complex eigenvalues) such that (3.2) has nontrivial solutions (eigenfunctions) belonging to $C(\mathbb{T}^2)$. To take advantage of spectral theory, we formulate the problem in an abstract setting. From now on, X is the Banach space $C(\mathbb{T}^2)$, and $T\colon X \to X$ is a linear operator such that (3.2) is equivalent to

$$u = \lambda T u, \qquad u \in X.$$

To define T, we consider the resolvent associated with $\mathscr{L}_\lambda$ given by

$$R_\sigma\colon L^1(\mathbb{T}^2) \to C(\mathbb{T}^2), \qquad f \mapsto u,$$

where u is the solution of

$$\mathscr{L}u - \sigma u = f \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2).$$

Then

$$T\colon X \to X, \qquad Tu = R_\sigma(mu).$$

If $m \in L^p(\mathbb{T}^2)$ for some p satisfying $1 < p < \infty$, then Proposition 2.1 implies that

$$ImT \subset C^{0,\alpha}(\mathbb{T}^2), \qquad \alpha = 1 - \frac{1}{p}.$$

In such a case the operator T is compact and the spectral theory of this class of operators is applicable. In particular, if (3.2) has eigenvalues, then they can be arranged in a sequence $\{\lambda_n\}$ with

$$|\lambda_0| \le |\lambda_1| \le \cdots \le |\lambda_n| \le \cdots.$$

The space X is an ordered Banach space with cone

$$C = C_+(\mathbb{T}^2) = \{u \in X / u \ge 0 \text{ in } \mathbb{R}^2\}.$$

This cone has a nonempty interior, namely

$$\operatorname{int}(C) = \{u \in X / u(t,x) > 0\ \forall (t,x) \in \mathbb{R}^2\}.$$

When m satisfies the additional condition

$$m(t,x) > 0 \qquad \text{a.e. } \mathbb{R}^2, \tag{3.3}$$

the operator T becomes strongly positive in the following sense:

$$T(C - \{0\}) \subset \operatorname{int}(C).$$

This is a consequence of (3.1) and the maximum principle. Furthermore, the strong positivity in $C(\mathbb{T}^2)$ implies that the spectral radius $r(T)$ is positive.

All of these considerations in functional analysis lead to the following result.

PROPOSITION 3.1. *Assume that* (3.1) *holds and let m be a function satisfying* (3.3) *and*

$$m \in L^p(\mathbb{T}^2)$$

for some p with $1 < p < \infty$. Then the first eigenvalue λ_0 of (3.2) *is positive and the first eigenfunction u_0 is simple and can be chosen strictly positive.*

Moreover, if $0 < \lambda < \lambda_0$ and $f \in L^1(\mathbb{T}^2)$, the problem

$$\mathscr{L}_\sigma u = \lambda m(t,x)u + f(t,x) \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2)$$

has a unique solution, which is positive if f satisfies

$$f \geq 0 \quad \text{a.e. } \mathbb{R}^2, \qquad \int_{\mathbb{T}^2} f > 0.$$

The proof is the same as the proof of Theorems 4.3 and 4.4 in [1]. Following along the lines of Theorem 4.5 in [1], one can also prove the monotonicity of λ_0 with respect to the weight. In fact, given two functions m_1, m_2 in the conditions of the previous proposition, the inequality

$$m_1(t,x) < m_2(t,x) \qquad \text{a.e. } \mathbb{R}^2$$

implies

$$\lambda_0(m_1) > \lambda_0(m_2).$$

We are now going to extend the maximum principle to operators of variable coefficients of the type

$$\mathscr{L}_\alpha u = u_{tt} - u_{xx} + cu_t - \alpha(t,x)u,$$

where $\alpha \in L^\infty(\mathbb{T}^2)$. The notion of maximum principle introduced in Section 2 can be extended to this class of operators.

THEOREM 3.2. *Let $\alpha \in L^\infty(\mathbb{T}^2)$ be a function satisfying*

$$-\nu(c) < \alpha(t,x) < 0 \qquad \text{a.e. } \mathbb{R}^2.$$

Then $\mathscr{L}_\alpha$ satisfies the strong maximum principle.

Proof. The equation

$$\mathscr{L}_\alpha u = f(t,x) \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2)$$

is rewritten in the form

$$\mathscr{L}_{-\nu} u = (\alpha(t,x) + \nu)u + f(t,x) \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2).$$

We can apply Proposition 3.1 with $\sigma = -\nu$ to deduce that there is a maximum principle as soon as $\lambda_0(m_1) > 1$, where $m_1 = \alpha + \nu$. To prove this inequality, we compare m_1 with the constant weight $m_2 = \nu$ and obtain

$$\lambda_0(m_1) > \lambda_0(m_2) = 1.$$

4. THE METHOD OF UPPER AND LOWER SOLUTIONS

Let us consider the nonlinear equation

$$\mathscr{L}u \equiv u_{tt} - u_{xx} + cu_t = F(t,x,u) \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2). \tag{4.1}$$

The function $F\colon \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions; that is,

- For a.e. $(t,x) \in \mathbb{R}^2$, the function $u \in \mathbb{R} \mapsto F(t,x,u)$ is continuous.
- For every $u \in \mathbb{R}$, the function $(t,x) \in \mathbb{R}^2 \mapsto F(t,x,u)$ is measurable.
- For each positive constant $R > 0$ there exists a function $\gamma \in L^1(\mathbb{T}^2)$ such that

$$\sup_{|u| \le R} |F(t,x,u)| \le \gamma(t,x) \qquad \text{a.e. } (t,x) \in \mathbb{R}^2.$$

A function $u_* \in L^\infty(\mathbb{T}^2)$ is a lower solution of (4.1) if it satisfies

$$\mathscr{L}u_* \le F(t,x,u_*) \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2).$$

This differential inequality is understood in the sense of distributions, that is,

$$\int_{\mathbb{T}^2} u_* \mathscr{L}^* \phi \le \int_{\mathbb{T}^2} F(t,x,u)\phi \qquad \forall \phi \in \mathfrak{D}_+(\mathbb{T}^2).$$

An upper solution $u^* \in L^\infty(\mathbb{T}^2)$ is a function that satisfies the reversed inequality.

THEOREM 4.1. *Let u^*, u_* be upper and lower solutions of* (4.1) *satisfying*

$$u_* \le u^* \qquad \text{a.e. } \mathbb{R}^2.$$

In addition,

$$F(t,x,u_2) - F(t,x,u_1) \ge -\nu(u_2 - u_1) \tag{4.2}$$

for a.e. $(t,x) \in \mathbb{R}^2$ *and every u_1, u_2, with*

$$u_*(t,x) \le u_1 \le u_2 \le u^*(t,x).$$

(*The constant $\nu = \nu(c)$ was defined by Theorem* 2.3.) *Then* (4.1) *has a solution $u \in C(\mathbb{T}^2)$ satisfying*

$$u_* \le u \le u^* \qquad \text{a.e. } \mathbb{R}^2.$$

The proof will be based in the standard monotone scheme employed in most studies of upper and lower solutions. However, our upper and lower solutions are weak, and this fact introduces some subtleties.

LEMMA 4.2. *Assume that $-\lambda \in (0,\nu]$ and u is a function in $L^1(\mathbb{T}^2)$ satisfying*

$$\mathscr{L}_\lambda u \ge 0 \qquad \textit{in } \mathfrak{D}'(\mathbb{T}^2).$$

Then $u \ge 0$ a.e. $\mathbb{R}^2$.

Proof. The distribution

$$\phi \in \mathfrak{D}(\mathbb{T}^2) \mapsto \int_{\mathbb{T}^2} u \mathscr{L}_\lambda^* \phi$$

is nonnegative. Thus it can be extended to a measure (see [21], p. 22, or [13], p. 151), and u satisfies

$$\mathscr{L}_\lambda u = \mu \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2),$$

with $\mu \in M(\mathbb{T}^2)$. The results follow from Corollary 2.4.

Proof of Theorem 4.1. Consider the iterative scheme

$$\mathscr{L} u_{n+1} + \nu u_{n+1} = F(t,x,u_n) + \nu u_n \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2).$$

Let $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ denote the sequences generated by the initial conditions $\underline{u}_0 = u_*$ and $\overline{u}_0 = u^*$, respectively. The regularity of solutions of the

linear equation implies that $\underline{u}_n$ and $\bar{u}_n$ are continuous if $n \geq 1$. Moreover, the condition (4.2) and the previous lemma lead to the chain of inequalities

$$u_* = \underline{u}_0 \leq \underline{u}_1 \leq \cdots \underline{u}_n \leq \cdots \leq \bar{u}_n \leq \cdots \leq \bar{u}_1 \leq \bar{u}_0 = u^*.$$

In particular, the sequence $\{\underline{u}_n\}$ is nondecreasing and converges pointwise to a function u that satisfies $u_* \leq u \leq u^*$. A passage to the limit based on the dominated convergence theorem shows that u is a solution of

$$\mathscr{L}u + \nu u = F(t, x, u) + \nu u \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2),$$

or, equivalently, of (4.1). Again, the regularity theory allows us to conclude that u belongs to $C(\mathbb{T}^2)$.

Next we shall apply the method of upper and lower solutions to a class of equations that includes the sine-Gordon equation.

Let $\Phi \in C^1(\mathbb{R})$ be a function satisfying

$$\Phi(u + 2\pi) = \Phi(u) \qquad \forall u \in \mathbb{R}$$

and consider the equation

$$u_{tt} - u_{xx} + cu_t + \Phi(u) = f(t, x) + s \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2). \tag{4.3}$$

Since we want to apply Theorem 4.1, we also impose

$$\Phi'(u) \leq \nu(c) \qquad \forall u \in \mathbb{R}.$$

The function f belongs to $L^1(\mathbb{T}^2)$ and has mean value zero, and $s \in \mathbb{R}$ is interpreted as a parameter.

By integrating (4.3) on $\mathbb{T}^2$, we deduce that

$$\min \Phi \leq s \leq \max \Phi$$

is a necessary condition for the solvability of (4.3). The next result describes in qualitative terms the exact conditions for the solvability of (4.3).

THEOREM 4.3. *In the previous assumptions there exists a nonempty closed interval I (depending on f) such that* (4.3) *has solutions if and only if $s \in I$.*

Remark. From the physical point of view, it may be more reasonable to look for solutions of (4.3) that take values in $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$ instead of $\mathbb{R}$. Given a continuous function u: $\mathbb{T}^2 \to \mathbb{T}^1$, there exist a lift $\tilde{u}$: $\mathbb{R}^2 \to \mathbb{R}$ and integers p, q, such that

$$\tilde{u}(t + 2\pi, x) = \tilde{u}(t, x) + 2\pi p, \qquad \tilde{u}(t, x + 2\pi) = \tilde{u}(t, x) + 2\pi q.$$

The change of unknown,

$$v(t,x) = \tilde{u}(t,x) - pt - qx,$$

reduces the search of these solutions to the problem

$$v_{tt} - v_{xx} + cv_t + \Phi(v + pt + qx) = f(t,x) + s - cp \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2).$$

The conclusion of Theorem 4.3 is also valid for this equation.

The proof of the theorem follows along the same lines as the proof of Theorem 3 on the pendulum equation in [16]. The p.d.e. tools are Theorem 4.1 and the following result on the linear equation that will be proved in Section 5.

PROPOSITION 4.4. *Let p be a given number with $1 \le p < \infty$ and let f be a function satisfying*

$$f \in L^p(\mathbb{T}^2), \qquad \int_{\mathbb{T}^2} f = 0.$$

Then the problem

$$\mathscr{L}u = f(t,x) \quad \text{in } \mathfrak{D}'(\mathbb{T}^2), \qquad \int_{\mathbb{T}^2} u = 0$$

has a unique solution. It satisfies

$$u \in C^{0,\alpha}(\mathbb{T}^2), \qquad \alpha = 1 - \frac{1}{p},$$

and the estimates (2.2) *and* (2.3) *of Proposition* 2.1 *still hold.*

(Notice that we are including the case $p = 1$ with the convention $C^{0,0}(\mathbb{T}^2) = C(\mathbb{T}^2)$.)

We finish this section with a quantitative application to the sine-Gordon equation. Let $f \in L^1(\mathbb{T}^2)$ be a function with mean value zero and let U be the solution of $\mathscr{L}U = f$, $\int_{\mathbb{T}^2} U = 0$, given by the previous proposition. The functions $u^* = \pi/2 + U$, $u_* = -\pi/2 + U$ are ordered upper and lower solutions of (1.2) if $\|U\|_\infty \le \pi/2$. Thus, if the previous inequality holds and $a \le \nu$, the equation (1.2) has a doubly periodic solution u satisfying $\|u - U\|_\infty \le \pi/2$.

5. PROOFS

5.1. *The Green's Function*: *Construction and Properties*

Let $M(\mathbb{R}^2)$ be the space of measures on $\mathbb{R}^2$ (see [7] for the precise definition), and let $\delta_0 \in M(\mathbb{R}^2)$ be the Dirac measure concentrated at the origin. More precisely,

$$\langle \delta_0, \phi \rangle = \phi(0,0) \qquad \forall \phi \in C_0(\mathbb{R}^2),$$

where $C_0(\mathbb{R}^2)$ is the space of continuous functions having compact support. Given $d \geq 0$, we define the function

$$U(t,x) = \begin{cases} \frac{1}{2} e^{-(c/2)t} J_0\left(\sqrt{d(t^2 - x^2)}\right), & \text{if } |x| < t, \\ 0, & \text{otherwise.} \end{cases}$$

The function U belongs to $L^\infty(\mathbb{R}^2)$ and is discontinuous on the set $|x| = t$. It satisfies

$$U_{tt} - U_{xx} + cU_t - \lambda U = \delta_0 \qquad \text{in } \mathfrak{D}'(\mathbb{R}^2),$$

with $\lambda = -d - c^2/4$.

An easy way to prove this fact is to observe that $e^{(c/2)t}U(t,x)$ is precisely a well-known fundamental solution of the operator

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \left(\lambda + \frac{c^2}{4}\right).$$

(See [6], p. 186, and [21], p. 223.)

Consider the double series

$$G(t,x) = \sum_{(n,m) \in \mathbb{Z}^2} U(t + 2\pi n, x + 2\pi m), \qquad (t,x) \in \mathbb{R}^2. \quad (5.1)$$

It is uniformly convergent on bounded subsets of the plane. In fact, the Weierstrass test is applicable because U satisfies the estimate

$$|U(t,x)| \leq \tfrac{1}{2} e^{-(c/2)t} \chi_\kappa(t,x),$$

where χ_κ is the characteristic function of the cone

$$\kappa = \{(t,x) \in \mathbb{R}^2 / |x| < t\}.$$

In consequence, the function G is doubly periodic and continuous on the set $\mathscr{D} = \mathbb{R}^2 - \mathscr{C}$, where $\mathscr{C}$ is the family of lines

$$x \pm t = 2\pi N, \qquad N \in \mathbb{Z}.$$

Furthermore, it belongs to $L^\infty(\mathbb{T}^2)$, and the superposition principle implies that it verifies

$$G_{tt} - G_{xx} + cG_t - \lambda G = \delta_S \qquad \text{in } \mathfrak{D}'(\mathbb{R}^2). \tag{5.2}$$

Here $\delta_S \in M(\mathbb{R}^2)$ is the measure defined by

$$\langle \delta_S, \phi \rangle = \sum_{(n,m)\in\mathbb{Z}^2} \phi(2\pi n, 2\pi m), \qquad \forall \phi \in C_0(\mathbb{R}^2).$$

The function G can be interpreted as a doubly periodic function on $\mathbb{R}^2$ or as a function on $\mathbb{T}^2$. Next we shall translate the equation (5.2) to the torus. To do that we consider $\delta_{\bar{0}} \in M(\mathbb{T}^2)$, the Dirac measure concentrated on $\bar{0} \in \mathbb{T}^2$. In precise terms,

$$\langle \delta_{\bar{0}}, \phi \rangle = \phi(\bar{0}, \bar{0}) \qquad \forall \phi \in C(\mathbb{T}^2).$$

LEMMA 5.1. *Assume $\lambda \le -c^2/4$ and define $d = -c^2/4 - \lambda$. Then the function G defined by* (5.1) *is a solution of*

$$\mathscr{L}_\lambda G = \delta_{\bar{0}} \qquad \textit{in } \mathfrak{D}'(\mathbb{T}^2).$$

Remark. Notice that G is the unique solution of this equation because λ is not an eigenvalue of $\mathscr{L}$. It is called the Green's function of the operator $\mathscr{L}_\lambda$.

Proof. Let us consider a periodic partition of unity in the sense of [21, p. 127]. This means that we are considering a function

$$\varphi \in \mathfrak{D}(\mathbb{R}^2) = C_0^\infty(\mathbb{R}^2), \qquad \operatorname{supp}(\varphi) \subset \left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right] \times \left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$$

such that

$$\sum_{(n,m)\in\mathbb{Z}^2} \varphi(t + 2\pi n, x + 2\pi m) = 1 \qquad \forall (t,x) \in \mathbb{R}^2. \tag{5.3}$$

For an arbitrary function $g \in L^1(\mathbb{T}^2)$, the identities below are easily proved:

$$\int_{\mathbb{R}^2} g\varphi = \int_{\mathbb{T}^2} g, \qquad \int_{\mathbb{R}^2} g\partial^\alpha \varphi = 0 \quad \forall \alpha \in \mathbb{N}^2, \quad |\alpha| > 0. \tag{5.4}$$

We are now in a position to prove the lemma.

Given a function $\phi \in \mathfrak{D}(\mathbb{T}^2)$, $\phi\varphi$ is a test function in $\mathfrak{D}(\mathbb{R}^2)$, and one can apply (5.2) to obtain

$$\int_{\mathbb{R}^2} \mathscr{L}_\lambda^*(\phi\varphi)G = \langle \delta_S, \phi\varphi \rangle.$$

It follows from (5.4) that

$$\int_{\mathbb{R}^2} \mathscr{L}_\lambda^*(\phi\varphi)G = \int_{\mathbb{T}^2} (\mathscr{L}_\lambda^*\phi)G,$$

and, from (5.3),

$$\langle \delta_S, \phi\varphi \rangle = \phi(0,0).$$

In consequence, G verifies

$$\int_{\mathbb{T}^2} (\mathscr{L}_\lambda^*\phi)G = \phi(0,0) \qquad \forall \phi \in \mathfrak{D}(\mathbb{T}^2),$$

and the proof is finished.

For each $n \in \mathbb{Z}$ and $(t,x) \in \mathbb{R}^2$, let us define

$$I_n(t,x) = \{m \in \mathbb{Z} / |x + 2\pi m| < t + 2\pi n\}.$$

Let $\mathscr{D}_{ij}$ denote the connected component of $\mathscr{D} = \mathbb{R}^2 - \mathscr{C}$ with center at the point $(i\pi, j\pi)$, where $i + j$ is an odd number. The set $I_n(t,x)$ does not change on each of these components. For instance, for the components $\mathscr{D}_{10}$ and $\mathscr{D}_{01}$, the set I_n is given by

$$\begin{aligned} I_n(t,x) &= \{-n, \dots, n\} \qquad \text{if } n \geq 0, \\ &= \varnothing \qquad \text{if } n < 0, \quad (t,x) \in \mathscr{D}_{10} \end{aligned}$$

and

$$\begin{aligned} I_n(t,x) &= \{-n, \dots, n-1\} \qquad \text{if } n \geq 1, \\ &= \varnothing \qquad \text{if } n \leq 0, \quad (t,x) \in \mathscr{D}_{01}. \end{aligned}$$

This notation allows us to express G on $\mathscr{D}$ in a more explicit way:

$$\begin{aligned} G(t,x) = {} & \frac{e^{-(c/2)t}}{2} \\ & \times \sum_{n > -t/2\pi} e^{-c\pi n} \sum_{m \in I_n(t,x)} J_0\left(\sqrt{d\left[(t+2\pi n)^2 - (x+2\pi m)^2\right]}\right). \end{aligned} \tag{5.5}$$

When $\lambda = -c^2/4$ (i.e., $d = 0$), the identity $J_0(0) = 1$ allows us to compute the series.

LEMMA 5.2. *Assume* $\lambda = -c^2/4$. *Then*

$$G(t,x) = \begin{cases} \gamma_{10} e^{-(c/2)t}, & \text{if } (t,x) \in \mathscr{D}_{10}, \\ \gamma_{01} e^{-(c/2)t}, & \text{if } (t,x) \in \mathscr{D}_{01}, \end{cases}$$

where

$$\gamma_{10} = \frac{1}{2} \frac{1 + e^{-c\pi}}{(1 - e^{-c\pi})^2}, \qquad \gamma_{01} = \frac{e^{-c\pi}}{(1 - e^{-c\pi})^2}.$$

Remark. By periodicity, this result determines the value of G on the whole set $\mathscr{D}$.

When $\lambda < -c^2/4$, it is possible to deduce some properties of the Green's function from formula (5.5). First we shall obtain estimates that are useful when the friction coefficient becomes large, and later we shall prove that G cannot vanish on sets of positive measure.

LEMMA 5.3. *Assume* $c \geq 1$ *and* $\lambda < -c^2/4$. *Then there exist positive constants* k_1, k_2 (*independent of* c *and* λ) *such that*

$$\left| e^{(c/2)t} G(t,x) - \tfrac{1}{2} J_0\left(\sqrt{d(t^2 - x^2)}\right) \right| \leq k_1 e^{-c\pi} \qquad \text{if } (t,x) \in \mathscr{D}_{10},$$

$$\left| e^{(c/2)(t+2\pi)} G(t,x) \right.$$
$$\left. - \tfrac{1}{2} \left\{ J_0\left(\sqrt{d\left[(t+2\pi)^2 - (x-2\pi)^2\right]}\right) + J_0\left(\sqrt{d\left[(t+2\pi)^2 - x^2\right]}\right) \right\} \right|$$
$$\leq k_2 e^{-c\pi} \qquad \text{if } (t,x) \in \mathscr{D}_{01}.$$

The proof follows from (5.5).

PROPOSITION 5.4. *Assume* $\lambda \leq -c^2/4$. *The set*

$$Z = \{(t,x) \in \mathbb{R}^2 / G(t,x) = 0\}$$

has measure zero.

It is sufficient to prove that the intersection of Z with $\mathscr{D}_{10}$ and $\mathscr{D}_{01}$ has measure zero. To do that, we shall prove that G is a nonzero analytic function on each of these domains.

Let ω be an open and connected subset of $\mathbb{R}^2$, and let $h = h(t, x), (t, x) \in \omega$ be a given function. We shall say that h is analytic in the closure of ω if there exist an open subset Ω of $\mathbb{C}^2$ and a holomorphic function $\tilde{h} = \tilde{h}(t, x)$, $(t, x) \in \Omega$, such that $\overline{\omega} \subset \Omega$ and $\tilde{h}$ extends h.

LEMMA 5.5. *The function G is analytic in the closure of $\mathscr{D}_{10}$ (resp. $\mathscr{D}_{01}$).*

Proof. We prove the result for $\mathscr{D}_{10}$. For each $n = 0, 1, \ldots, |m| \leq n$ and $(t, x) \in \mathbb{C}^2$, define

$$z_{nm}(t, x) = (t + 2\pi n)^2 - (x + 2\pi m)^2.$$

The Green's function on $\mathscr{D}_{10}$ is given by the series

$$G(t, x) = \frac{e^{-(c/2)t}}{2} \sum_{n=0}^{\infty} e^{-c\pi n} \sum_{m=-n}^{n} J_0\left(\sqrt{dz_{nm}}\right).$$

The function $J_0(\sqrt{dz_{nm}})$ is holomorphic in the whole space $(t, x) \in \mathbb{C}^2$ because J_0 is an entire function. Notice that the square root does not create any trouble, since the odd coefficients in the power series of J_0 are zero. To prove the lemma, it is sufficient to show the existence of a domain $\Omega \subset \mathbb{C}^2$ with $\overline{\mathscr{D}_{10}} \subset \Omega$, such that the series of G converges uniformly on Ω.

Let $\delta > 0$ be a number satisfying

$$\pi^2 - \delta^2 > \tfrac{1}{8}, \qquad 16\pi\delta\sqrt{d} < c\pi. \tag{5.6}$$

Define Ω as the set of points $(t, x) \in \mathbb{C}^2$ such that

$$\Re e(t - x) > -\pi, \qquad \Re e(t + x) > -\pi,$$
$$|\Re e\, t| < 7, \qquad |\Re e x| < 7, \quad |\Im m\, t| < \delta, \quad |\Im m x| < \delta.$$

It is clear that $\overline{\mathscr{D}_{10}} \subset \Omega$. To prove the convergence of the series, we shall use the following inequalities:

$$\left|J_0(z)\right| \leq e^{|\Im m\, z|} \qquad \forall z \in \mathbb{C} \tag{5.7}$$

$$|\Im m \sqrt{z}| \leq \frac{|\Im m z|}{\sqrt{2|z|}} \qquad \forall z \in \mathbb{C}, \quad \Re e z > 0. \tag{5.8}$$

The first inequality is classical, whereas the second is easily proved by using polar coordinates. The identities below will also be useful:

$$\Re e z_{nm} = (\Re e(t + x) + 2\pi(n + m))(\Re e(t - x) + 2\pi(n - m))$$
$$-(\Im m\, t)^2 + (\Im m x)^2$$
$$\Im m z_{nm} = 2(2\pi n + \Re e\, t)\Im m\, t - 2(2\pi m + \Re e x)\Im m x.$$

We distinguish two cases:

Case (*i*). $|m| < n$. The definition of Ω and (5.6) leads to

$$\Re e\, z_{nm} > \pi^2 - \delta^2 > \tfrac{1}{8}, \qquad |\Im m\, z_{nm}| < 4(2\pi n + 7)\delta.$$

In particular, $|z_{nm}| > \frac{1}{8}$, and we can apply (5.8) to obtain

$$|\Im m \sqrt{z_{nm}}| \le 2|\Im m\, z_{nm}| < 8(2\pi n + 7)\delta.$$

From (5.7),

$$\sum_{|m|<n} \left| J_0\left(\sqrt{dz_{nm}}\right)\right| \le \sum_{|m|<n} e^{|\Im m \sqrt{dz_{nm}}|} < (2n-1)e^{8(2\pi n+7)\delta\sqrt{d}},$$

and the convergence of this part of the series follows from (5.6).

Case (*ii*). $|m| = n$. The estimate for $\Im m\, z_{nm}$ of the previous case is still valid. For the real part, one obtains

$$|\Re e\, z_{nm}| \le 14(14 + 4\pi n) + 2\delta^2.$$

In consequence, $|z_{nm}| = O(n)$ as $n \to \infty$, and there exist positive numbers A, B such that

$$|\Im m \sqrt{dz_{nm}}| \le \sqrt{d}\,|z_{nm}|^{1/2} \le An^{1/2} + B.$$

The convergence of the terms $m = \pm n$ is now easily proved.

LEMMA 5.6. *The function G is not identically zero in any of the components of $\mathscr{D}$.*

Proof. The function G can be extended in a smooth way to the closure $\overline{\mathscr{D}_{ij}}$ of each component of $\mathscr{D}$. Of course, the value of this extension at points in $\mathscr{C}$ depends on the chosen component. We introduce the notation

$$G_{ij}(t,x) = \lim_{\substack{(\tau,\xi)\to(t,x)\\(\tau,\xi)\in\mathscr{D}_{ij}}} G(\tau,\xi), \qquad (t,x) \in \mathscr{C}.$$

A direct computation based on (5.5) allows us to find explicitly the jump of G at the boundary of two adjacent components. Namely, given two adjacent components $\mathscr{D}_{ij}$ and $\mathscr{D}_{hk}$, with $(h,k) = (i-1, j-1)$ or $(i-1, j+1)$, and a point $(t,x) \in \partial\mathscr{D}_{ij} \cap \partial\mathscr{D}_{hk}$ in the common side,

$$G_{ij}(t,x) - G_{hk}(t,x) = \frac{e^{-(c/2)t}}{2(1-e^{-c\pi})}.$$

Now we prove the lemma by a contradiction argument. Let us assume first that $G \equiv 0$ in $\mathscr{D}_{01}$. Thus, by periodicity, G also vanishes in $\mathscr{D}_{21}$, and

$$G_{01}(\pi, \pi) = G_{21}(\pi, \pi) = 0.$$

A jump from $\mathscr{D}_{01}$ to $\mathscr{D}_{10}$ shows that

$$G_{10}(\pi, \pi) = \frac{c^{-(c/2)\pi}}{2(1 - e^{-c\pi})} > 0.$$

On the other hand, a jump from $\mathscr{D}_{10}$ to $\mathscr{D}_{21}$ shows that

$$G_{10}(\pi, \pi) = -\frac{e^{-(c/2)\pi}}{2(1 - e^{-c\pi})} < 0.$$

This is the required contradiction.

Let us now assume that $G \equiv 0$ in $\mathscr{D}_{10}$. The same argument works if one replaces (π, π) with $(2\pi, 0)$.

5.2. *Regularity of Solutions*

In the previous subsection we found the Green's function when $\lambda \leq -c^2/4$. The use of Fourier analysis provides an alternative method of computing G that is valid for any $\lambda \notin \sigma_{\mathbb{R}}(\mathscr{L})$. In fact, the formal solution of

$$\mathscr{L}_\lambda G = \delta_{\bar{0}} \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2) \tag{5.9}$$

is

$$G(t, x) = \frac{1}{4\pi^2} \sum_{m, n = -\infty}^{\infty} \frac{1}{m^2 - n^2 - \lambda + icn} e^{i(nt + mx)}.$$

Since the double series

$$\sum_{n, m} |m^2 - n^2 - \lambda + icn|^{-2}$$

is convergent, the function G belongs to $L^2(\mathbb{T}^2)$ and is a solution of (5.9) in the sense of distributions. Given $f \in L^1(\mathbb{T}^2)$, the function $u = G * f$ is the solution of (2.1). The convolution must be understood on the torus, that is,

$$u(t, x) = \int_{\mathbb{T}^2} G(t - \tau, x - \xi) f(\tau, \xi)\, d\tau\, d\xi. \tag{5.10}$$

In this way, we arrive at a preliminary result (already obtained in [10]). Namely, given $\lambda \notin \sigma_{\mathbb{R}}(\mathscr{L})$ and $f \in L^1(\mathbb{T}^2)$, (2.1) has a unique solution $u \in L^2(\mathbb{T}^2)$ that satisfies

$$\|u\|_{L^2(\mathbb{T}^2)} \leq \|G\|_{L^2(\mathbb{T}^2)} \|f\|_{L^1(\mathbb{T}^2)}. \tag{5.11}$$

Proof of Proposition 2.1. First we assume $\lambda = -c^2/4$. In this case G is explicitly given by Lemma 5.2. Let $\mathscr{C}$ be the family of lines $x \pm t = 2N\pi$ that we already considered in the definition of G. For each $\varepsilon > 0$, let $\mathscr{C}_\varepsilon$ be the ε-neighborhood defined by

$$\mathscr{C}_\varepsilon = \left\{(t,x) \in \mathbb{R}^2 / \operatorname{dist}\left[(t,x), \mathscr{C}\right] < \varepsilon\right\}.$$

The characteristic function of $\mathscr{C}_\varepsilon$ is a doubly periodic function that will be denoted by χ_ε. It satisfies

$$\|\chi_\varepsilon\|_{L^1(\mathbb{T}^2)} \leq k\varepsilon, \tag{5.12}$$

where k is a fixed constant.

The function G given by Lemma 5.2 is discontinuous on $\mathscr{C}$, but it satisfies, for each $h = (h_1, h_2) \in \mathbb{R}^2$,

$$\begin{aligned} &\left|G(t+h_1, x+h_2) - G(t,x)\right| \\ &\qquad \leq L\left[|h| + \chi_{|h|}(t,x)\right] \qquad \text{a.e. } (t,x) \in \mathbb{R}^2, \end{aligned} \tag{5.13}$$

where L is a fixed constant.

Given $f \in L^1(\mathbb{T}^2)$, the solution given by (5.10) belongs to $L^\infty(\mathbb{T}^2)$ and satisfies

$$\|u\|_{L^\infty(\mathbb{T}^2)} \leq \|G\|_{L^\infty(\mathbb{T}^2)} \|f\|_{L^1(\mathbb{T}^2)}.$$

Let us show that u is continuous. From (5.13) we deduce

$$\begin{aligned} &\left|u(t+h_1, x+h_2) - u(t,x)\right| \\ &\qquad \leq L|h|\,\|f\|_{L^1(\mathbb{T}^2)} + L\int_{\mathbb{T}^2} \chi_{|h|}(t-\tau, x-\xi)\left|f(\tau,\xi)\right| d\tau\, d\xi. \end{aligned}$$

The continuity of u follows from the dominated convergence theorem.

When $f \in L^p(\mathbb{T}^2)$, $p > 1$, it is possible to improve the previous estimate. Applying Hölder inequality and (5.12),

$$\begin{aligned} \int_{\mathbb{T}^2} \chi_{|h|}(t-\tau, x-\xi)\left|f(\tau,\xi)\right| d\tau\, d\xi &\leq \|f\|_{L^p} \|\chi_{|h|}\|_{L^{p'}} \\ &= \|f\|_{L^p} \|\chi_{|h|}\|_{L^1}^\alpha \leq A^\alpha \|f\|_{L^p} |h|^\alpha, \end{aligned}$$

where $1/p + 1/p' = 1$.

This proves that u is Hölder-continuous. The estimate (2.3) also follows from the previous reasoning.

Let us now consider the case $\lambda \neq -c^2/4$. The solution u of (2.1) is also a solution of

$$\mathscr{L}u + \frac{c^2}{4}u = g(t, x) \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2),$$

with $g = f + (\lambda + c^2/4)u$.

If $f \in L^1$, we know from (5.11) that $u \in L^2$, and therefore $g \in L^1$. From the previous case we know that u is continuous. Once we know $u \in C(\mathbb{T}^2)$, the equivalence $f \in L^p \Leftrightarrow g \in L^p$ holds. Thus $u \in C^{0,\alpha}$ if $f \in L^p$. The estimates (2.2), (2.3) follow from the inequality

$$\|g\|_{L^p} \leq k\|f\|_{L^p}.$$

Remark. The constant $C_p = C_p(\lambda)$ in (2.2), (2.3) can be chosen as a continuous function of $\lambda \in \mathbb{R} - \sigma_{\mathbb{R}}(\mathscr{L})$.

Proof of Proposition 2.2. The uniqueness is a consequence of $\lambda \notin \sigma_{\mathbb{R}}(\mathscr{L})$. Let $f_n \in L^1(\mathbb{T}^2)$ be a sequence of functions such that

$$\int_{\mathbb{T}^2} f_n \phi \to \langle \mu, \phi \rangle \quad \forall \phi \in C(\mathbb{T}^2), \qquad \|f_n\|_{L^1(\mathbb{T}^2)} \leq \|\mu\|_{M(\mathbb{T}^2)}.$$

Proposition 2.1 can be applied to the approximate problem

$$\mathscr{L}u - \lambda u = f_n(t, x) \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2)$$

to obtain a sequence $\{u_n\}$ in $C(\mathbb{T}^2)$ with

$$\|u_n\|_{L^\infty(\mathbb{T}^2)} \leq C_1\|\mu\|_{M(\mathbb{T}^2)}.$$

Let $\{u_k\}$ be a subsequence of $\{u_n\}$ converging to $u \in L^\infty(\mathbb{T}^2)$ in the weak* sense; that is,

$$\int_{\mathbb{T}^2} u_k \phi \to \int_{\mathbb{T}^2} u\phi \qquad \forall \phi \in L^1(\mathbb{T}^2).$$

It is easy to verify that u is a solution of (2.4). Moreover,

$$\|u\|_{L^\infty(\mathbb{T}^2)} \leq \liminf \|u_k\|_{L^\infty(\mathbb{T}^2)} \leq C_1\|\mu\|_{M(\mathbb{T}^2)}.$$

This already finishes the proof, but it is convenient to notice that the whole sequence $\{u_n\}$ must converge to u.

To prove Proposition 4.4, it is convenient to use the following:

LEMMA 5.7. *Assume $p \in [1, \infty)$, and let $\varepsilon_n \in \mathbb{R}$, $u_n \in C^{0,\alpha}(\mathbb{T}^2)$, $f_n \in L^p(\mathbb{T}^2)$ be three sequences such that*

$$\varepsilon_n \to 0, \quad \int_{\mathbb{T}^2} u_n = 0, \quad \sup \|f_n\|_{L^p} < \infty,$$

and

$$\mathscr{L} u_n - \varepsilon_n u_n = f_n(t, x) \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2).$$

Then $\|u_n\|_{C^{0,\alpha}(\mathbb{T}^2)}$ is bounded.

Proof. Let us assume $p = 1$ (the case $p > 1$ is similar). Define $\eta_n = \|u_n\|_{L^\infty}$ and assume, by a contradiction argument, that a subsequence $\{\eta_k\}$ satisfies

$$\eta_k > 0, \quad \eta_k \to \infty.$$

Then $v_k = \eta_k^{-1} u_k$ is a solution of

$$\mathscr{L} v_k - \varepsilon_k v_k = \eta_k^{-1} f_k(t, x) \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2). \tag{5.14}$$

Let us choose a number $\lambda \notin \sigma_{\mathbb{R}}(\mathscr{L})$. Then v_k is also the unique solution of

$$\mathscr{L} v - \lambda v = g_k(t, x) \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2),$$

with $g_k = (\varepsilon_k - \lambda) v_k + \eta_k^{-1} f_k$. This fact allows us to decompose v_k in the form

$$v_k = w_k + z_k,$$

with

$$\mathscr{L} w_k - \lambda w_k = (\varepsilon_k - \lambda) v_k, \qquad \mathscr{L} z_k - \lambda z_k = \eta_k^{-1} f_k.$$

The estimates (2.2) and (2.3) imply that $\|z_k\|_{L^\infty} \to 0$, and $\|w_k\|_{C^{0,\alpha}}$ is bounded for any $\alpha \in (0, 1)$. In consequence, it is possible to extract a new subsequence v_r converging uniformly to $v \in C(\mathbb{T}^2)$. A passage to the limit in (5.14) shows that v is identically zero, but this is not consistent with the value of the norm

$$\|v\|_{L^\infty} = \|v_r\|_{L^\infty} = 1.$$

Proof of Proposition 4.4. Let $f \in L^1(\mathbb{T}^2)$, $\int_{\mathbb{T}^2} f = 0$ be given. We prove that $\mathscr{L} u = f$, $\int_{\mathbb{T}^2} u = 0$ has a solution.

Let $\varepsilon_n \nearrow 0$ be a sequence with $\varepsilon_n \notin \sigma_{\mathbb{R}}(\mathscr{L})$, and let u_n be the solution of

$$\mathscr{L} u_n - \varepsilon_n u_n = f(t, x) \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2).$$

The triplet ε_n, u_n, f is in the conditions of the previous lemma. Thus $\|u_n\|_{L^\infty}$ is bounded, and it is possible to extract a subsequence u_k converging to $u \in L^\infty(\mathbb{T}^2)$ in the weak* sense. The function u is the searched solution. The rest of the proof is immediate.

5.3. *Proof of the Maximum Principle*

We are now ready to prove the main result of the paper. It follows from (5.10) that $\mathscr{L}_\lambda$ satisfies the maximum principle if and only if

$$G \geq 0 \qquad \text{a.e. } \mathbb{T}^2.$$

Moreover, this maximum principle is strong if

$$G > 0 \qquad \text{a.e. } \mathbb{T}^2.$$

We can apply Lemma 5.2 and conclude that the strong maximum principle at least holds for $\lambda = -c^2/4$. The result stated below implies that it also holds for the values of λ that are close to $-c^2/4$.

LEMMA 5.8. *Assume that, for some $\lambda \notin \sigma_{\mathbb{R}}(\mathscr{L})$, the Green's function satisfies*

$$\operatorname*{ess\,inf}_{\mathbb{T}^2} G > 0.$$

Then there exists $\varepsilon_0 > 0$ such that the strong maximum principle holds for $\mathscr{L}_{\lambda+\varepsilon}$ if $|\varepsilon| \leq \varepsilon_0$.

Proof. Given $f \in L^1$, $f \geq 0$, let u and u_0 be the solutions of $\mathscr{L}_\lambda u_0 = f$, $\mathscr{L}_{\lambda+\varepsilon} u = f$, respectively. From (5.10),

$$u_0(t,x) \geq \underline{G}\|f\|_{L^1}, \qquad \underline{G} = \operatorname{ess\,inf} G.$$

On the other hand, if ε is small, we can find a constant C independent of ε such that

$$\|u\|_{L^\infty} \leq C\|f\|_{L^1}.$$

This is a consequence of the remark after the proof of Proposition 2.1. The function $w = u - u_0$ is a solution of

$$\mathscr{L}_\lambda w = \varepsilon u(t,x) \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2)$$

and

$$\|u - u_0\|_{L^\infty} \leq C|\varepsilon| 4\pi^2 \|u\|_{L^\infty} \leq C^2 4\pi^2 |\varepsilon| \, \|f\|_{L^1}.$$

Thus,

$$u(t,x) \geq u_0(t,x) - \|u - u_0\|_{L^\infty} \geq \left(\underline{G} - C^2 4\pi^2 |\varepsilon|\right)\|f\|_{L^1}$$

is positive if ε is small.

The problem (2.1) for f constant shows that if the maximum principle holds, then λ is negative. Let us define the set

$$\mathfrak{J}_c = \{\lambda \in (-\infty, 0) / \mathscr{L}_\lambda \text{ satisfies the maximum principle}\}.$$

We already know that $\mathfrak{J}_c$ is nonempty. In fact, $-c^2/4$ is an interior point. The same technique of the proof of Proposition 2.2 can be employed to prove that $\mathfrak{J}_c$ is closed relative to $(-\infty, 0)$. The following property of $\mathfrak{J}_c$ is a consequence of well-known results in the theory of positive linear operators, namely,

$$\lambda \in \mathfrak{J}_c \Rightarrow [\lambda, 0) \subset \mathfrak{J}_c.$$

Moreover, if the maximum principle is strong for $\mathscr{L}_\lambda$, then it is also strong for each $\mathscr{L}_\Lambda$, $\Lambda \in [\lambda, 0)$.

Let us review the arguments to prove these facts. The equation $\mathscr{L}_{\lambda+\varepsilon} u = f$ is equivalent to

$$(I - \varepsilon R_\lambda) u = R_\lambda f,$$

where R_λ is the resolvent of $\mathscr{L}_\lambda$ introduced in Section 3. If we restrict R_λ to $C(\mathbb{T}^2)$, then we can look at it as an endomorphism of this space with spectral radius $1/|\lambda|$. Therefore, $I - \varepsilon R_\lambda$ has an inverse if $|\varepsilon| < |\lambda|$, and u can be expressed as

$$u = \sum_{n=0}^{\infty} \varepsilon^n R_\lambda^{n+1} f.$$

This formula shows the positivity of u if $\varepsilon \in (0, |\lambda|)$ and $f \geq 0$.

Define

$$\nu(c) = -\min \mathfrak{J}_c.$$

The previous discussions show that $\nu(c) > c^2/4$, and the maximum principle holds if and only if

$$\lambda \in [-\nu(c), 0).$$

We can now apply Proposition 5.4 to deduce that, for $\lambda = -\nu(c)$, the Green's function G is positive almost everywhere. In consequence, the maximum principle is strong for each $\lambda \in [-\nu(c), 0)$.

The estimate (2.5) is a consequence of the previous discussions, together with the second remark after Theorem 2.3.

To prove (2.6) let us consider a fixed number $\lambda < 0$, and let u and f be smooth functions on the torus such that

$$u_{tt} - u_{xx} - \lambda u = f;$$

f is positive and u changes sign. An example of such situation was presented in remark 1 after theorem 2.3. The function u also satisfies

$$\mathscr{L}_\lambda u = g$$

with $g = f + cu_t$. When c is small g is positive, and therefore $\lambda \notin \mathfrak{I}_c$ if c is small enough. Finally, to prove (2.7) we fix any number d with $0 < d < j_0^2/8\pi^2$ and define $\lambda_c = -d - c^2/4$. It is clear from Lemma 5.3 that, for large c, G is positive. Thus, $\nu(c) > d + c^2/4$ and this proves

$$\liminf_{c \to \infty} \left\{ \nu(c) - \frac{c^2}{4} \right\} \geq \frac{j_0^2}{8\pi^2}.$$

To prove the other inequality, we proceed in a similar way for $d > j_0^2/8\pi^2$ and observe that, in this case, G changes sign if c is large.

This finishes the proof of Theorem 2.3. The proof of Corollary 2.4 is an easy approximation argument in the line of the proof of Proposition 2.2.

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