ON THE EULER CHARACTERISTIC OF ANALYTIC AND ALGEBRAIC SETS

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Let \( f_1, \ldots, f_s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0) \) be germs of real analytic functions, and let \( S_r \) be a small sphere centered at the origin. The Euler characteristic of the set \( \{ x \in S_r \mid f_1(x) = \ldots = f_s(x) = 0 \} \) is an interesting local invariant.

Define

\[
g(x) = f_1^2(x) + \ldots + f_s^2(x) - c(x_1^2 + \ldots + x_n^2)^k,
\]

where \( c > 0 \) is a small constant and \( k \geq 0 \) is an integer. We prove that \( g \) has an isolated singular point at the origin and

\[
\chi(\{ x \in S_r \mid f_1(x) = \ldots = f_s(x) = 0 \}) = 1 - \deg(dg),
\]

where \( \deg(dg) \) is the degree of the mapping

\[
x \mapsto \frac{\text{grad} g(x)}{\|\text{grad} g(x)\|}
\]

from a small sphere \( S_r \) to the unit sphere of \( \mathbb{R}^n \).

We can apply this fact to real polynomials and see how to compute the Euler characteristic of any algebraic subset of \( \mathbb{R}^n \).

Let \( f : (U, 0) \rightarrow (\mathbb{R}, 0) \) be an analytic function defined in an open subset of \( \mathbb{R}^n \). Set

\[
\omega(x) = x_1^2 + \ldots + x_n^2.
\]

Define

\[
V = \{(x, r, y) \in U \times \mathbb{R} \times \mathbb{R} \mid \omega(x) = r^2, \ \text{rank} (d\omega(x), df(x)) \leq 1, \ y = f(x)\}.
\]

Then \( V \) is an analytic subset of \( \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \).

Let \( \pi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R} \) be the natural projection. Of course \( \pi : V \rightarrow \pi(V) \) is proper in some neighbourhood of the origin. Hence \( \pi(V) \) is closed, semianalytic in some neighbourhood of the origin (see [3], p. 127).

Denote \( Y_1 = \mathbb{R} \times \{0\}, Y_2 = \mathbb{R} \times \{0\} \). From [3] \( Y_2 \) is semianalytic. If \( r \neq 0 \) then

\[
\pi(V) \cap \{r \times \mathbb{R} \} = \{r \times \{\text{the set of critical values of } f_{i,j}\}\},
\]

where \( S_r = \{x \in \mathbb{R}^n \mid \|x\|^2 = r^2\} \). Then \( \pi(V) \cap \{r \times \mathbb{R} \} \) is finite (see [4], p. 16). Hence \( \dim \pi(V) = \dim Y_2 = 1 \). Then \( \{0\} \) is an isolated point of \( Y_1 \cap Y_2 \). From ([3], p. 85) there exist constants \( C > 0, \sigma > 0 \) such that

\[
d((r, y), Y_1) \geq C \cdot d((r, y), \{0\})^{2\sigma}
\]

for each \( (r, y) \in Y_2 \) (\( d \) denoting the euclidean distance in \( \mathbb{R}^2 \)). Of course \( d((r, y), Y_1) = |y| \) and \( d((r, y), \{0\}) \geq |r| \). So

\[
|y| \geq Cr^{2\sigma}
\]

for each \( (r, y) \in Y_2 \) sufficiently close to the origin (if \( f \) is a homogeneous polynomial of degree
p then $C = 2$ and $x = [p/2] + 1$. Let $c \in (0, C)$, and let $k \geq x$ be an integer. Define $g(x) = f(x) - c\omega^k(x)$.

Set

$$V' = \{(x, r, y) \in U \times \mathbb{R} \times \mathbb{R} | \omega(x) = r^2, \quad \text{rank } (d\omega(x), d\omega(x)) \leq 1, \quad y = g(x)\}.$$

We have rank $(d\omega(x), d\omega(x)) = \text{rank } (df(x), dg(x))$. Then

$$V' = \{(x, r, y) | \omega(x) = r^2, \quad \text{rank } (d\omega(x), df(x)) \leq 1, \quad y = f(x) - cr^{2k}\}.$$

Define $G(r, y) = (r, y - cr^{2k})$. Then $\pi(V') = G(\pi(V))$. By (1) we have

$$\pi(V') \cap \mathbb{R} \times \{0\} = \{(0, 0)\}$$

in some neighbourhood of the origin. Hence, if $r \neq 0$ is sufficiently close to the origin then $\{0\}$ is a regular value of $g|_{S'}$.

**Lemma 1.** Let $f: (U, 0) \rightarrow (\mathbb{R}, 0)$ be a real analytic function defined in an open subset of $\mathbb{R}^n$. Then there exist constants $C > 0, \alpha > 0$ such that: if $c \in (0, C)$, $k \geq \alpha$ is an integer, $r \neq 0$ is sufficiently close to the origin and $g = f - c\omega^k$ then $\{0\}$ is a regular value of $g|_{S'}$. In particular $g$ has an isolated singular point at the origin. Moreover the Euler characteristic

$$\chi(\{x \in S, |f(x) \leq 0\}) = 1 - \deg (dg).$$

**Proof.** Let $r \neq 0$ be close to the origin. Define $N_f = \{x \in S, |f(x) \leq 0\}, \quad N_g = \{x \in S, \quad g(x) \leq 0\}$. Then $N_f \subset \text{int } N_g$. By (1) and (2) the function $g|_{S'}$ does not have critical points in $N_f - N'_f$. The set $N_f$ is closed, semianalytic and hence, according to [2], can be triangulated. We have $g|_{S'} = f|_{S'} - cr^{2k}$. So $N_f$ is a deformation retraction of $N_g$. Then $\chi(N_f) = \chi(N_g)$. The function $g$ has an isolated singular point at the origin. By [1, 5] we have $\chi(N_g) = 1 - \deg (dg)$. This ends the proof.

**Corollary 1.** If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a homogeneous polynomial of degree $d$ then

$$\chi(\{x \in S, |f(x) \leq 0\}) = 1 - \deg (dg),$$

where $g = f - \omega^{(d/2)+1}$.

**Theorem 1.** Let $f_1, \ldots, f_s: (U, 0) \rightarrow (\mathbb{R}, 0)$ be real analytic functions defined in an open subset of $\mathbb{R}^n$. Then there exist constants $C > 0, \alpha > 0$ such that: if $c \in (0, C), \quad k \geq \alpha$ is an integer, $r \neq 0$ is sufficiently close to the origin and

$$g = f_1^2 + \ldots + f_s^2 - c\omega^k$$

then $g$ has an isolated singular point at the origin and

$$\chi(\{x \in S, |f_1(x) = \ldots = f_s(x) = 0\}) = 1 - \deg (dg).$$

**Proof.** We have

$$\{x \in S, |f_1(x) = \ldots = f_s(x) = 0\} = \{x \in S, |f_1^2(x) + \ldots + f_s^2(x) \leq 0\}.$$

The rest is a consequence of Lemma 1.

**Corollary 2.** If $f_1, \ldots, f_s: \mathbb{R}^n \rightarrow \mathbb{R}$ are homogeneous polynomials of degree $d$ then

$$\chi(\{x \in S, |f_1(x) = \ldots = f_s(x) = 0\}) = 1 - \deg (dg),$$

where $g = f_1^2 + \ldots + f_s^2 - \omega^{d+1}$.
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LEMMA 2. Let \( f_1, \ldots, f_l : \mathbb{R}^n \to \mathbb{R} \) be polynomials of degree \( \leq d \). Assume that \( \{ x \in \mathbb{R}^n | f_1(x) = \ldots = f_l(x) = 0 \} \) is compact. Define
\[
\begin{align*}
g_i(x_0, x_1, \ldots, x_n) & = x_0^d \frac{f_i}{x_0} (x_1/x_0, \ldots, x_n/x_0), \\
g & = g_1^2 + \ldots + g_l^2 - (x_0^d + \ldots + x_n^d)^{d+2},
\end{align*}
\]
Then \( g \) has an isolated singular point at the origin and
\[
\chi(\{ x \in \mathbb{R}^n | f_1(x) = \ldots = f_l(x) = 0 \}) = \frac{1}{2}((-1)^n - \deg(dg)).
\]

Proof. Each \( g_i \) is a homogeneous polynomial in \( \mathbb{R}^{n+1} \) of degree \( d + 1 \). Of course
\[
S^{-1}_n = \{ x \in S^*_n | x_0 = 0 \} \subset \{ x \in S^*_n | g_1(x) = \ldots = g_l(x) = 0 \}
\]
and the sets:
\[
\begin{align*}
\{ x \in \mathbb{R}^n | f_1(x) = \ldots = f_l(x) = 0 \}, \\
\{ x \in S^*_n | x_0 > 0, \ g_1(x) = \ldots = g_l(x) = 0 \}
\end{align*}
\]
are homeomorphic.

By compactness of \( \{ x \in \mathbb{R}^n | f_1(x) = \ldots = f_l(x) = 0 \} \) we have
\[
\chi(\{ x \in S^*_n | g_1(x) = \ldots = g_l(x) = 0 \}) = 2 \chi(\{ x \in \mathbb{R}^n | f_1(x) = \ldots = f_l(x) = 0 \}) + \chi(S^{-1}_n)
\]
Of course \( \chi(S^{-1}_n) = (-1)^{n+1} + 1 \).
The rest is a consequence of Corollary 2.

THEOREM 2. Let \( f_1, \ldots, f_l : \mathbb{R}^n \to \mathbb{R} \) be polynomials. Then there exist polynomials \( g : \mathbb{R}^{n+1} \to \mathbb{R}, h : \mathbb{R}^n \to \mathbb{R} \) with isolated singular points at the origin such that
\[
\chi(\{ x \in \mathbb{R}^n | f_1(x) = \ldots = f_l(x) = 0 \}) = \frac{1}{2}((-1)^n - \deg(dg)) - \deg(dh).
\]

Proof. Define \( f = f_1^2 + \ldots + f_l^2 \). Of course
\[
\{ x \in \mathbb{R}^n | f(x) = 0 \} = \{ x \in \mathbb{R}^n | f_1(x) = \ldots = f_l(x) = 0 \}.
\]
We may assume that \( f(0) \neq 0 \). Let \( d = \deg(f) \). Set
\[
\Gamma(x_1, \ldots, x_n) = (x_1^2 + \ldots + x_n^2)^{d+1} f(x/(x_1^2 + \ldots + x_n^2)).
\]
Then:
\[
\begin{align*}
(\text{i}) & \quad F \in \mathbb{R}[x_1, \ldots, x_n], \quad \Gamma(0) = 0, \\
(\text{ii}) & \quad V(F) = \{ x \in \mathbb{R}^n | F(x) = 0 \} \text{ is the single point (Alexandroff) compactification of} \\
& \quad \{ x \in \mathbb{R}^n | f(x) = 0 \}.
\end{align*}
\]
From Lemma 2 there exists a polynomial \( g : \mathbb{R}^{n+1} \to \mathbb{R} \) with an isolated singular point at the origin such that
\[
\chi(V(F)) = \frac{1}{2}((-1)^n - \deg(dg)).
\]
Let \( r > 0 \). Then
\[
\chi(V(F)) = \chi(\{ x \in V(F) | ||x|| \leq r \}) + \chi(\{ x \in V(F) | ||x|| \geq r \}) - \chi(V(F) \cap S_r).
\]
If \( r \) is sufficiently close to the origin then:
\[
\begin{align*}
(\text{iii}) & \quad \chi(\{ x \in V(F) | ||x|| \leq r \}) = 1, \\
(\text{iv}) & \quad \chi(\{ x \in V(F) | ||x|| \geq r \}) = \chi(V(F) \setminus \{ 0 \}) = \chi(\{ x \in \mathbb{R}^n | f(x) = 0 \}).
\end{align*}
\]
(v) from Theorem 1 there exists a polynomial $h: \mathbb{R}^n \to \mathbb{R}$ with an isolated singular point at the origin such that $\chi(V(F) \cap S_r) = 1 - \deg(dh)$.

This ends the proof.

REFERENCES


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