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Towards the critical problem: on the coalgebraic relation between sets and multisets

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Abstract

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We introduce a generalization of the coalgebra of sets which we call the coalgebra of multisets. After studying this new coalgebra (in fact, a Hopf algebra), we characterize the coalgebra of sets within the coalgebra of multisets by its coalgebraic properties. This gives rise to a new understanding of the relations between sets and multisets.

1. Introduction

One of the critical problems of combinatorics as enumerated by Prof. G.-C. Rota is that of discovering the relation between sets and multisets. Many combinatorial concepts possess a 'multiset analog,' for which solving the critical problem completely would lead to a better understanding. The present work is a small step in that direction.

We recall in Section 2 the definition of an incidence coalgebra $\mathscr{C}(P)$ of a poset P: $\mathscr{C}(P)$ is the vector space spanned by the intervals of P (or, in the case of a reduced coalgebra, equivalence classes of intervals of P) and equipped with a comultiplication defined by convolution. Next, we consider the particular example of the Boolean incidence Hopf algebra $\mathscr{C}(\mathscr{B}_{\sim})$ defined as the reduced incidence coalgebra over the lattice $\mathscr{B}$ of subsets of a set $\mathscr{U}$ with respect to the equivalence $[S,T]\sim[U,V]$ if and only if $T-S=V-U$. Operations in this Hopf algebra are equivalent to operations involving subsets of $\mathscr{U}$.

We then define the coalgebra of multisets in terms of the Boolean incidence Hopf algebra. This coalgebra is also a reduced incidence coalgebra but with respect to

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a coarser equivalence. Operations in this coalgebra (actually a Hopf algebra) correspond to operations involving multisets. Finally, we classify the ideals and coideals of the Hopf algebra of multisets. In particular, the multisets which are not sets correspond to a biideal $\overline{\mathrm{ENS}}$ and are thus characterized by their coalgebraic properties.

2. Background and terminology

2.1. Coalgebras

A *coalgebra* is a vector space C over a field K equipped with two linear maps — a *comultiplication* $\Delta : C \to C \otimes C$ and a *counit* $\varepsilon : C \to K$ — subject to the following two conditions:

(1) *Coassociativity.* For all $x \in C$, $\Delta(x) \in C \otimes C$. Thus, we can apply to $\Delta(x)$ either $\Delta \otimes I$ or $I \otimes \Delta$, where I is the identity map. The coassociative property claims that the result is the same irrespective of which map is applied next: $(\Delta \otimes I) \circ \Delta = (I \otimes \Delta) \circ \Delta$.

(2) *Counitary property.* There is a canonical isomorphism of the tensors $K \otimes C$ and $C \otimes K$ with C itself. It is given by the identification of x with $x \otimes 1_K$ and $1_K \otimes x$. The counitary property claims that maps $\varepsilon \otimes I$ and $I \otimes \varepsilon$ composed with the comultiplication Δ are identity maps modulo these canonical isomorphisms, i.e., $x \otimes 1_K = (I \otimes \varepsilon)(\Delta(x))$ and $1_K \otimes x = (\varepsilon \otimes I)(\Delta(x))$.

We say that a linear map f between two coalgebras C and C' is a *coalgebra map* if it is well behaved under comultiplication and the counitary map, i.e., $(f \otimes f)(\Delta(x)) = \Delta'(f(x))$ and $\varepsilon'(f(x)) = \varepsilon(x)$ for all $x \in C$.

A space B which is simultaneously an algebra and a coalgebra is said to be a *bialgebra* if Δ and ε are algebra maps or, equivalently, if the multiplication $\mu : B \otimes B \to B$ and the unit map $\eta : K \to B$ are coalgebra maps.

In an ordinary algebra A, a (proper) ideal is a proper subspace $I \subseteq A$ closed under multiplication on both sides, i.e., $\mu(A \otimes I)$ and $\mu(I \otimes A)$ are subspaces of I and $1 \notin I$ or, equivalently, $\mu(A \otimes I + I \otimes A)$ is a subspace of I, and $\eta^{-1}(I) = 0$. This is exactly the condition we need in order for the multiplication on the quotient algebra A/I to be well defined.

For proper coideals of coalgebras, we require the dual condition, i.e., a subspace I of a coalgebra C is a *coideal* if $\Delta(I)$ is a subspace of $I \otimes C + C \otimes I$ and if $\varepsilon(I) = 0$. We can then define the quotient algebra C/I via $\Delta(x + I) = \Delta(x) + C \otimes I + I \otimes C$ and $\varepsilon(x + I) = \varepsilon(x)$.

A bialgebra is called a *Hopf algebra* when it is equipped with a (necessarily unique) involution S called the *antipode* such that $\mu \circ (S \otimes I) \circ \Delta = \eta \circ \varepsilon$.

See [5] for basic results and examples of co-, bi- and Hopf-algebras.

2.2. Reduced incidence coalgebra

Let P be a locally finite poset and consider the collection $\mathrm{Seg}(P)$ of all segments of P. Given an equivalence relation on these segments, we will be interested in the coalgebra defined over the vector space $\mathscr{C}(P_\sim)$ whose basis $X^{\mathrm{cl}\sim[x,y]}$ is indexed by equivalence classes of $\mathrm{Seg}(P)$. We define the comultimplication and counit of $\mathscr{C}(P_\sim)$ as follows:

$$\Delta(X^{\mathrm{cl}_\sim[x,y]})=\sum_{z\in[x,y]}(X^{\mathrm{cl}_\sim[x,z]}\otimes X^{\mathrm{cl}_\sim[z,y]})$$

and

$$\varepsilon(X^{\mathrm{cl}_\sim[x,y]}=\begin{cases}1 & \text{if } x=y,\\ 0 & \text{otherwise.}\end{cases}$$

Only when $\sim$ is *order-compatible* are these comultiplication and counit maps well defined and form a coalgebra (which we will call *the reduced incidence coalgebra of P*). That is, whenever $[x,y]\sim[u,v]$ there must be a bijection (not necessarily an isomorphism) $\phi:[x,y]\to[u,v]$ such that, for all $z\in[x,y]$, we have

$$[x,z]\sim[\phi(x)=u,\phi(x)],\tag{1}$$

$$[z,y]\sim[\phi(x),\phi(y)=v].\tag{2}$$

The strongest equivalence relation is that of equality. Clearly, it is order-compatible. The resulting coalgebra is called the (*full* or *unreduced*) *incidence coalgebra of P*. Another standard choice of equivalence relation is that of isomorphism. By definition, isomorphism is order-compatible. We call this coalgebra the *standard reduced coalgebra of P*.

Another result of [4] is the following: Suppose $\sim$ is an equivalence relation on $\mathrm{Seg}(P)$ and $J_\sim$ is the linear subspace of the full incidence coalgebras spanned by all combinations of the form $X^\alpha-X^\beta$, where $\alpha\sim\beta$ in $\mathrm{Seg}(P)$. Then $\sim$ is order-compatible if and only if $J_\sim$ is a coideal. Furthermore, in that case, the reduced incidence coalgebra is the quotient of the full incidence coalgebra with the coideal $J_\sim$.

3. Hopf algebra of subsets of $\mathscr{U}$

3.1. Coalgebraic properties

Choose a universe $\mathscr{U}$. Let $\mathscr{B}$ be the lattice of finite subsets of $\mathscr{U}$. Then the full incidence algebra $\mathscr{C}(\mathscr{B})$ is the vector space spanned by the base $X^{[S,T]}$, where $[S,T]$ is an interval of $\mathscr{B}$, i.e. where $S\subseteq T\subseteq\mathscr{U}$. $\mathscr{C}(\mathscr{B})$ is equipped with the following coalgebra structure:

$$\Delta(X^{[S,T]})=\sum_{S\subseteq U\subseteq T}X^{[S,U]}\otimes X^{[U,T]},$$

$$\varepsilon(X^{[S,T]})=\begin{cases}1 & \text{if } S=T,\\ 0 & \text{if } S\subset T.\end{cases}$$

Consider the equivalence relation $\sim$ defined by $[S,T]\sim[U,V]$ if and only if $T-S=V-U$. Now, the function $f:[S,T]\to[U,V]$ defined by $f(W)=U\cup(W-S)$ is a bijection which obeys equations (1) and (2). Thus, $\sim$ is order-compatible.

The resulting reduced incidence coalgebra will be called the *coalgebra of subsets of* $\mathscr{U}$ and will be denoted $\mathscr{S}$ or $\mathscr{S}(\mathscr{U})$. To simplify notation, $X^{\mathrm{cl}_\sim[S,T]}$ will be denoted by $X_\sim^{T-S}$.

To proceed further let us set our field of constants to be the two element field $\boldsymbol{Z}/2\boldsymbol{Z}$.

Proposition 3.1. *$\mathscr{S}(\mathscr{U})$ is a bialgebra when equipped with the multiplication* μ

$$\mu(X_\sim^S\otimes X_\sim^T)=\begin{cases}X_\sim^{S\cup T} & \text{if } S\cap T=\emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

along with the unit $\eta(1_K)=X_\sim^\emptyset$.

Proof. We must verify that the multiplication μ and unit η are coalgebra maps. Let S and U be two disjoint subsets of U:

$$\begin{aligned}(\mu\otimes\mu)\Delta\otimes\Delta(X_\sim^S\otimes X_\sim^U)&=\sum_{\substack{T\subseteq S\\ V\subseteq U}}\mu(X_\sim^T\otimes X_\sim^V)\otimes\mu(X_\sim^{S-T}\otimes X_\sim^{U-V})\\ &=\sum_{\substack{T\subseteq S\\ V\subseteq U}}(X_\sim^{T\cup V}\otimes X_\sim^{(S\cup U)-(T\cup V)})\\ &=\sum_{W\subseteq S\cup U}(X_\sim^W\otimes X_\sim^{S\cup U)-W})\\ &=\Delta(X_\sim^{S\cup U})\\ &=\Delta(\mu(X_\sim^S\otimes X_\sim^U)).\end{aligned}$$

If S and U are not disjoint, then it is easy to see that $(\mu\otimes\mu)$ $(\Delta\otimes\Delta(X_\sim^S\otimes X_\sim^U))$ and $\Delta(\mu(X_\sim^S\otimes X_\sim^U))$ are both zero. Thus, the multiplication respects the comultiplication.

Now, we must check if it respects the counit map, i.e., does $\varepsilon(\mu(X_\sim^S\otimes X_\sim^T))=\varepsilon(X_\sim^S)\varepsilon(X_\sim^T)$? The right-hand side is zero unless both S and T are nonempty, i.e., it is zero unless $S\cup T$ is nonempty which is just the condition for the left-hand side to be zero. Thus, multiplication respects the counit map.

Finally, we must observe that the unit map η is a coalgebra map. First, $\Delta(\eta(1_K))=X_\sim^\emptyset\otimes X_\sim^\emptyset=(\eta\otimes\eta)(1_K\otimes 1_K)$, so that η respects comultiplication. Second, $\mu(\eta(1_K))=\mu(X_\sim^\emptyset)=1_K$, so that η respects the counit map. Hence, $\mathscr{S}(\mathscr{U})$ is a bialgebra. $\square$

Corollary 3.2. *$\mathscr{S}(\mathscr{U})$ is a Hopf algebra with antipode* $f: X_\sim^S\mapsto(-1)^{|S|}X_\sim^S$.

Proof. We compute

$$\begin{aligned}
\mu((f\otimes I)(\Delta(X^S_\sim))) &= \mu\Big((f\otimes I)\Big(\sum_{T\subseteq S}(X^T_\sim\otimes X^{S-T}_\sim)\Big)\Big)\\
&= \mu\Big(\sum_{T\subseteq S}((-1)^{|T|}X^T_\sim\otimes X^{S-T}_\sim)\Big)\\
&= \sum_{T\subseteq S}(-1)^{|T|}X^S_\sim\\
&= \begin{cases} X^\emptyset_\sim & \text{if } S=\emptyset\\ 0 & \text{otherwise}\end{cases}\\
&= \begin{cases} \eta(1_K) & \text{if } S=\emptyset\\ \eta(0) & \text{otherwise}\end{cases}\\
&= \eta(\varepsilon(X^S_\sim)).
\end{aligned}$$

Thus, $\mathscr{S}(\mathscr{U})$ is a Hopf algebra.

3.2. Combinatorial interpretation

Here, we have characteristic two, so the antipode defined above is the identity map.

Moreover, the Hopf algebra of sets now has a combinatorial interpretation: each element of $\mathscr{S}(\mathscr{U})$ represents a collection of subsets of $\mathscr{U}$.

Certain operations on sets, for example, the involutions studied in [2], can be more easily expressed using this notation.

(1) The Graham/Mycielski operation σ is defined in [2] as the following map σ on collections of subsets of $\mathscr{U}$: A subset $S\subseteq\mathscr{U}$ is a member of $\sigma(X)$ if and only if it is a superset of an odd number of members of the collection X. Using the Hopf algebra notation, we have more simply $\sigma(X)=B\cdot X$, where $B=\sum_{S\subseteq\mathscr{U}}X^S_\sim$. The fact that σ is an involution now reduces to the fact that $B\cdot B=X^\emptyset_\sim$.

(2) The outer complementation $\rho(X)$ is defined as the collection of subsets $S\subseteq\mathscr{U}$ which are not members of X. $\rho(X)$ can be rewritten in our notation as $X+B$. Since $B+B=0$, ρ is an involution.

4. Hopf algebra of multisets

Let f be a map from the universe $\mathscr{U}$ to some set X.

Consider the linear transformation f^* of $\mathscr{S}(\mathscr{U})$ defined by

$$f^*X^S_\sim = Y^{f(S)},$$

where $f(\{a_1, a_2, \ldots, a_n\})$ is the (multi)set $\{f(a_1), f(a_2), \ldots, f(a_n)\}$. Obviously, if f is a bijection between $\mathscr{U}$ and $\mathscr{U}'$ then f^* is a coalgebra isomorphism between $\mathscr{S}(\mathscr{U})$ and $\mathscr{S}(\mathscr{U}')$. Similarly, if f is injective, then f^* is a coalgebra monomorphism. However, what happens if f is not injective?

Let $\mathscr{M}$ be the multiset $f(\mathscr{U})$ and denote the image of f^* by $\mathscr{S}(\mathscr{M})$.[1] Clearly, $\mathscr{S}(\mathscr{M})$ is a coalgebra. In fact, $\mathscr{S}(\mathscr{M})$ is a reduced incidence coalgebra. The equivalence classes of intervals in the Boolean lattice $\mathscr{B}$ is given by $[S, T] \cong [U, W]$ if and only if $f(T-S) = f(W-U)$ as multisets.

For example,

$$\Delta(Y^{\{a^n\}}) = \sum_{k=0}^{n} \binom{n}{i} (Y^{\{a^{n-i}\}} \otimes Y^{\{a^i\}}),$$

where $\{a^n\}$ represents a multiset with a as an element repeated n times. We thus recognize $\mathscr{S}(\{a^\infty\})$ as the coalgebra of polynomials in the 'variable' a. Similarly, it can be shown that $\mathscr{S}(a^i)$ (for i finite) is the coalgebra of polynomials of degree at most i in the variable a.

Similarly,

$$\Delta(Y^{\{a^n b^m\}}) = \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{j}{m} (Y^{\{a^i b^j\}} \otimes Y^{\{a^{n-i} b^{m-j}\}}).$$

5. Coideals

Let us consider maximal coideals of the coalgebra of multisets. These coideals are very easy to calculate.

Proposition 5.1. *The space* $\overline{\text{VIDE}}$ *spanning* Y^M *for* M *nonempty is the unique maximal coideal of* $\mathscr{S}(\mathscr{M})$. *The quotient coalgebra* $\mathscr{S}/\overline{\text{VIDE}}$ *is isomorphic to the base field* $\mathbf{Z}/2\mathbf{Z}$.

Proof. $\overline{\text{VIDE}}$ is a coideal, since ΔY^M for M nonempty contains no terms of the form $Y^\emptyset \otimes Y^\emptyset$. However, at the same time $\overline{\text{VIDE}}$ is equal to the inverse image $\varepsilon^{-1}(0)$. Since coideals are required to have their image under the counitary map equal to zero, we see that $\overline{\text{VIDE}}$ is the unique maximal coideal. □

To continue along this line of reasoning, let us consider the spaces which do not contain any atoms $Y^{\{a\}}$. We will call such spaces *nonatomic*.[2] What are the maximal nonatomic coideals?

[1] Note that $\mathscr{S}(\mathscr{M})$ does *not* depend on the choice of f and $\mathscr{U}$ but only on $\mathscr{M}$.

[2] Spaces which contain at least one atom $Y^{\{a\}}$ will be called *atomic*.

Lemma 5.2. *Let I be a nonatomic coideal and $x \in I$. Then x is the sum of terms of the form Y^M where M is a multiset which is not a set.*

Proof. Can we have $Y^S \in I$ where S is a set (as opposed to merely a multiset)? Obviously, we cannot have $|S|=0$ since, then, $\varepsilon(Y^S)=1$. Similarly, we cannot have $|S|=1$ since, then, I is atomic. Thus, $|S| \geqslant 2$. Now, choose S minimal subject to $Y^S \in I$. Since S can be bipartitioned into $S=T \cup U$ where T and U are proper subsets of S, $\Delta(S)$ contains at least one term which is not a member of $I \otimes \mathscr{S} + \mathscr{S} \otimes I$, a contradiction.

We have thus proved that all nonatomic coideals contain only sums of terms Y^M where M is a multiset which is not a set. $\square$

Consider now the vector space $\overline{\text{ENS}}$ generated by all such terms.

Theorem 5.3. $\overline{\text{ENS}}$ *is the unique maximal nonatomic coideal of $\mathscr{S}(\mathscr{M})$.*

Proof. By Lemma 5.2, it will suffice to show that $\overline{\text{ENS}}$ is a coideal. Therefore, let M be a multiset which is not a set. Is ΔY^M a member of $\overline{\text{ENS}} \otimes \mathscr{S} + \mathscr{S} \otimes \overline{\text{ENS}}$?

Clearly, M contains some element a with multiplicity greater than one. First, suppose a has multiplicity greater than two. In this case, for any 'partition' of M into $M=M_1+M_2$, either M_1 or M_2 is not a set (or both). Hence, $\Delta Y^M \in \overline{\text{ENS}} \otimes \mathscr{S} + \mathscr{S} \otimes \overline{\text{ENS}}$.

Conversely, suppose a has multiplicity two. Consider now the various terms $X^{M_1} \otimes X^{M_2}$ in the coproduct ΔX^M. The terms in which either M_1 or M_2 (or both) is not a set are included in $\overline{\text{ENS}} \otimes \mathscr{S} + \mathscr{S} \otimes \overline{\text{ENS}}$. The other terms occur in cancelling pairs. Remember that we are doing all calculations over $\boldsymbol{Z}/2\boldsymbol{Z}$.

In conclusion, we have shown that there is a unique maximal nonatomic coideal $\overline{\text{ENS}}$ of the coalgebra of multisets $\mathscr{S}(\mathscr{M})$ defined as the span of all terms Y^M where M is a multiset which is not a set. $\square$

6. Biideals

We can equip $\mathscr{S}(\mathscr{M})$ with an algebra structure as follows:

$$\mu(Y^{M_1} \otimes Y^{M_2}) = \begin{cases} Y^{M_1+M_2} & \text{if } M_1+M_2 \subseteq M, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

$$\eta(1_K) = Y^{\emptyset}. \tag{4}$$

This multiplication μ is consistent with the multiplication defined on the bialgebra of sets in the case that $\mathscr{M}$ is a set.

Moreover, this multiplication is a coalgebra map, i.e., $\mathscr{S}(\mathscr{M})$ is a bialgebra with respect to this multiplication and the coproduct defined earlier. In fact, $\mathscr{S}(\mathscr{M})$ is now a Hopf algebra with antipode given by the identity map.

Finally, note that if $\mathcal{M}$ has all of its multiplicities equal to zero or infinity, then $\mathcal{M}$ is closed under addition which means that the 'otherwise' of equation (3) is no longer needed.

It is reasonable to consider what the ideals are with respect to this multiplication. Clearly, if $Y^M \in I$ then $Y^T \in N$ for all $M \subseteq N \subseteq \mathcal{M}$. Thus, algebraic ideals correspond closely to the combinatorial concept of a filter or an 'up-set'.[3]

The following result reduces the question of ideals to that of nonatomic ideals.

Theorem 6.1. *Any I decomposes into a certain number (possibly infinite) of principal atomic ideals $(Y^{\{a_i\}})$ and one nonatomic ideal J. The computation of the quotient algebra $\mathcal{S}(\mathcal{M})/I$ then reduces to the computation of $\mathcal{S}(\mathcal{M}/a_1/a_2/\cdots)/J^*$, where J^* is the result of removing from members of J any terms involving $a_1, a_2 \ldots$*

Proof. Consider the principal (atomic) ideal generated by $Y^{\{a\}}$. It is the vector space $(Y^{\{a\}})$ spanned by Y^M for all M containing a. Thus, $\mathcal{S}(\mathcal{M})/(Y^{\{a\}})$ is isomorphic to $\mathcal{S}(\mathcal{M}/a)$, where $\mathcal{M}/a$ is the multiset $\mathcal{M}$ where the multiplicity of a has been set to zero. □

A *biideal* is a subspace of a bialgebra which is at the same time an ideal and a coideal. Note that $\overline{\text{VIDE}}$ and $\overline{\text{ENS}}$ are both in fact biideals.

By the above theorem, the question of biideals reduces itself to the question of nonatomic biideals. We thus have a completely (bi)algebraic characterization of *sets* among *multisets*. That is, given a multiset $\mathcal{M}$ and the maximal nonatomic biideal $\overline{\text{ENS}}$, $\mathcal{S}(\mathcal{M})/\overline{\text{ENS}}$ is isomorphic to $\mathcal{S}(S)$, where S is the underlying set of $\mathcal{M}$.

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[3] A *filter* or an *up-set* is a subset S of a poset P for which all $x \in P$ if $x \geqslant y \in S$; then $x \in S$ as well. A (*combinatorial*) *ideal* or a *down-set* is the dual concept. Here if $x \leqslant y \in S$ then $x \in S$.