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Differential equations for generalized Jacobi polynomials

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Abstract

We look for differential equations of the form

$$M \sum_{i=0}^{\infty} a_i(x)y^{(i)}(x) + N \sum_{i=0}^{\infty} b_i(x)y^{(i)}(x) + MN \sum_{i=0}^{\infty} c_i(x)y^{(i)}(x) + (1-x^2)y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]y'(x) + n(n + \alpha + \beta + 1)y(x) = 0$$

satisfied by the generalized Jacobi polynomials $\{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty}$ which are orthogonal on the interval $[-1, 1]$ with respect to the weight function

$$\frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha + \beta + 1} \Gamma(\alpha + 1) \Gamma(\beta + 1)} (1-x)^{\alpha} (1+x)^{\beta} + M\delta(x+1) + N\delta(x-1),$$

where $\alpha > -1$, $\beta > -1$, $M \geq 0$ and $N \geq 0$. We give explicit representations for the coefficients $\{a_i(x)\}_{i=0}^{\infty}$, $\{b_i(x)\}_{i=0}^{\infty}$ and $\{c_i(x)\}_{i=0}^{\infty}$ and we show that this differential equation is uniquely determined. For $M^2 + N^2 > 0$ the order of this differential equation is infinite, except for $\alpha \in \{0, 1, 2, \dots\}$ or $\beta \in \{0, 1, 2, \dots\}$. Moreover, the order equals

$$\begin{cases} 2\beta + 4 & \text{if } M > 0, N = 0 \text{ and } \beta \in \{0, 1, 2, \dots\}, \\ 2\alpha + 4 & \text{if } M = 0, N > 0 \text{ and } \alpha \in \{0, 1, 2, \dots\}, \\ 2\alpha + 2\beta + 6 & \text{if } M > 0, N > 0 \text{ and } \alpha, \beta \in \{0, 1, 2, \dots\}. \end{cases}$$

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1. Introduction

In [15] Koornwinder introduced the polynomials $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^\infty$ which are orthogonal on the interval $[-1, 1]$ with respect to the weight function

$$\frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha+\beta+1}\Gamma(\alpha + 1)\Gamma(\beta + 1)}(1 - x)^\alpha(1 + x)^\beta + M\delta(x + 1) + N\delta(x - 1),$$

where $\alpha > -1$, $\beta > -1$, $M \geq 0$ and $N \geq 0$. We call these polynomials the generalized Jacobi polynomials, but sometimes they are also referred to as the Jacobi-type polynomials. As a limit case he also found the generalized Laguerre (or Laguerre-type) polynomials $\{L_n^{\alpha,M}(x)\}_{n=0}^\infty$ which are orthogonal on the interval $[0, \infty)$ with respect to the weight function

$$\frac{1}{\Gamma(\alpha + 1)}x^\alpha e^{-x} + M\delta(x),$$

where $\alpha > -1$ and $M \geq 0$. These generalized Jacobi polynomials and generalized Laguerre polynomials are related by the limit

$$L_n^{\alpha,M}(x) = \lim_{\beta \rightarrow \infty} P_n^{\alpha,\beta,0,M}\left(1 - \frac{2x}{\beta}\right). \tag{1}$$

In [9] we proved that for $M > 0$ the generalized Laguerre polynomials satisfy a unique differential equation of the form

$$M \sum_{i=0}^\infty a_i(x)y^{(i)}(x) + xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0,$$

where $\{a_i(x)\}_{i=0}^\infty$ are continuous functions on the real line and $\{a_i(x)\}_{i=1}^\infty$ are independent of the degree n . In [2] Bavinck found a new method to obtain the main result of [9]. This inversion method was found in a similar way as was done in [6] in the case of generalizations of the Charlier polynomials. See also [11] for more details. In [12] we used this inversion method to find all differential equations of the form

$$M \sum_{i=0}^\infty a_i(x)y^{(i)}(x) + N \sum_{i=0}^\infty b_i(x)y^{(i)}(x) + MN \sum_{i=0}^\infty c_i(x)y^{(i)}(x) + xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0,$$

where the coefficients $\{a_i(x)\}_{i=1}^\infty$, $\{b_i(x)\}_{i=1}^\infty$ and $\{c_i(x)\}_{i=1}^\infty$ are independent of n and the coefficients $a_0(x)$, $b_0(x)$ and $c_0(x)$ are independent of x , satisfied by the Sobolev-type Laguerre polynomials $\{L_n^{\alpha,M,N}(x)\}_{n=0}^\infty$ which are orthogonal with respect to the inner product

$$\langle f, g \rangle = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty x^\alpha e^{-x} f(x)g(x) dx + Mf(0)g(0) + Nf'(0)g'(0),$$

where $\alpha > -1$, $M \geq 0$ and $N \geq 0$. These Sobolev-type Laguerre polynomials $\{L_n^{\alpha,M,N}(x)\}_{n=0}^\infty$ are generalizations of the generalized Laguerre polynomials $\{L_n^{\alpha,M}(x)\}_{n=0}^\infty$. In fact, we have

$$L_n^{\alpha,M,0}(x) = L_n^{\alpha,M}(x).$$

In this paper we will use the inversion formula found in [11] to find differential equations of the form

$$M \sum_{i=0}^{\infty} a_i(x)y^{(i)}(x) + N \sum_{i=0}^{\infty} b_i(x)y^{(i)}(x) + MN \sum_{i=0}^{\infty} c_i(x)y^{(i)}(x) + (1-x^2)y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]y'(x) + n(n + \alpha + \beta + 1)y(x) = 0, \tag{2}$$

where the coefficients $\{a_i(x)\}_{i=1}^{\infty}$, $\{b_i(x)\}_{i=1}^{\infty}$ and $\{c_i(x)\}_{i=1}^{\infty}$ are independent of n and the coefficients $a_0(x)$, $b_0(x)$ and $c_0(x)$ are independent of x , satisfied by the generalized Jacobi polynomials $\{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty}$.

For $\alpha = \beta = 0$, $M > 0$ and $N > 0$ the generalized Jacobi polynomials reduce to the Krall polynomials studied by Littlejohn [20]. These Krall polynomials are generalizations of the Legendre-type polynomials ($\alpha = \beta = 0$ and $N = M > 0$) found by Krall [17,18]. See also [16]. In [20] it is shown that the Krall polynomials satisfy a sixth-order differential equation of the form (2). For $\alpha > -1$, $\beta = 0$, $M > 0$ and $N = 0$ or for $\alpha = 0$, $\beta > -1$, $M = 0$ and $N > 0$ the generalized Jacobi polynomials reduce to the Jacobi-type polynomials which satisfy a fourth-order differential equation of the form (2); see also [16–18].

We emphasize that the case $\beta = \alpha$ and $N = M$ is special in the sense that we can also find differential equations of the form

$$M \sum_{i=0}^{\infty} d_i(x)y^{(i)}(x) + (1-x^2)y''(x) - 2(\alpha + 1)xy'(x) + n(n + 2\alpha + 1)y(x) = 0, \tag{3}$$

where the coefficients $\{d_i(x)\}_{i=1}^{\infty}$ are independent of n and $d_0(x)$ is independent of x , satisfied by the symmetric generalized ultraspherical polynomials $\{P_n^{\alpha, \alpha, M, M}(x)\}_{n=0}^{\infty}$. The Legendre type polynomials for instance satisfy a fourth-order differential equation of the form (3). See [16–18]. In [13] we found all differential equations of the form (3) satisfied by the polynomials $\{P_n^{\alpha, \alpha, M, M}(x)\}_{n=0}^{\infty}$ for $\alpha > -1$ and $M \geq 0$. In [10] we applied the special case $\beta = \alpha$ of the Jacobi inversion formula to solve the systems of equations obtained in [13].

2. The main results

We look for all differential equations of the form (2) satisfied by the generalized Jacobi polynomials $\{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty}$. A representation of these orthogonal polynomials will be given in Section 5. We emphasize that we demand that the coefficients $\{a_i(x)\}_{i=1}^{\infty}$, $\{b_i(x)\}_{i=1}^{\infty}$ and $\{c_i(x)\}_{i=1}^{\infty}$ are independent of the degree n and that $a_0(x)$, $b_0(x)$ and $c_0(x)$ do not depend on x . Therefore we will use the following notations:

$$a_0(x) = a_0(n, \alpha, \beta), \quad b_0(x) = b_0(n, \alpha, \beta), \quad c_0(x) = c_0(n, \alpha, \beta), \quad n = 0, 1, 2, \dots$$

and

$$a_i(x) = a_i(\alpha, \beta, x), \quad b_i(x) = b_i(\alpha, \beta, x), \quad c_i(x) = c_i(\alpha, \beta, x), \quad i = 1, 2, 3, \dots$$

We will apply a general theorem by Bavinck to prove that for $\alpha > -1$, $\beta > -1$, $M \geq 0$ and $N \geq 0$ the polynomials $\{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty}$ satisfy a unique differential equation of the form (2), where

$$a_0(0, \alpha, \beta) = 0, \quad a_0(n, \alpha, \beta) = (\alpha + \beta + 2) \frac{(\beta + 3)_{n-1}(\alpha + \beta + 3)_{n-1}}{(\alpha + 1)_{n-1}(n - 1)!}, \quad n = 1, 2, 3, \dots, \tag{4}$$

$$b_0(0, \alpha, \beta) = 0, \quad b_0(n, \alpha, \beta) = (\alpha + \beta + 2) \frac{(\alpha + 3)_{n-1} (\alpha + \beta + 3)_{n-1}}{(\beta + 1)_{n-1} (n - 1)!}, \quad n = 1, 2, 3, \dots, \quad (5)$$

$$c_0(0, \alpha, \beta) = c_0(1, \alpha, \beta) = 0$$

and

$$c_0(n, \alpha, \beta) = \frac{(\alpha + \beta + 2)^2 (\alpha + \beta + 3)}{(\alpha + 1)(\beta + 1)} \frac{(\alpha + \beta + 4)_{n-1}}{(n - 1)!} \frac{(\alpha + \beta + 4)_{n-2}}{(n - 2)!}, \quad n = 2, 3, 4, \dots \quad (6)$$

Further we will show that

$$a_i(\alpha, \beta, x) = -(\alpha + \beta + 2) 2^i \sum_{\ell=0}^{i-1} (-1)^\ell \frac{(\beta + 3)_{i-\ell-1} (-\beta - 2)_{i-\ell-1}}{(\alpha + 1)_{i-\ell-1} (i - \ell)! (i - \ell - 1)! \ell!} \\ \times {}_3F_2 \left(\begin{matrix} -\ell, \alpha + \beta + 3, \beta + i - \ell + 2 \\ \alpha + i - \ell, i - \ell + 1 \end{matrix} \middle| 1 \right) \left(\frac{x + 1}{2} \right)^{\ell+1} \quad (7)$$

and

$$b_i(\alpha, \beta, x) = (\alpha + \beta + 2) (-2)^i \sum_{\ell=0}^{i-1} \frac{(\alpha + 3)_{i-\ell-1} (-\alpha - 2)_{i-\ell-1}}{(\beta + 1)_{i-\ell-1} (i - \ell)! (i - \ell - 1)! \ell!} \\ \times {}_3F_2 \left(\begin{matrix} -\ell, \alpha + \beta + 3, \alpha + i - \ell + 2 \\ \beta + i - \ell, i - \ell + 1 \end{matrix} \middle| 1 \right) \left(\frac{x - 1}{2} \right)^{\ell+1} \quad (8)$$

for $i = 1, 2, 3, \dots$ and that

$$c_1(\alpha, \beta, x) = 0 \quad \text{and} \quad c_i(\alpha, \beta, x) = c_i^{(1)}(\alpha, \beta, x) + c_i^{(2)}(\alpha, \beta, x), \quad i = 2, 3, 4, \dots, \quad (9)$$

where for $i = 2, 3, 4, \dots$

$$c_i^{(1)}(\alpha, \beta, x) = - \frac{(\alpha + \beta + 2)^2 (\alpha + \beta + 3) (\alpha + \beta + 4)}{(\alpha + 1)(\beta + 1)i} (x^2 - 1) 2^{i-2} \\ \times \sum_{\ell=0}^{i-2} (-1)^\ell \frac{(\beta + 3)_{i-\ell-2} (-\alpha - \beta - 3)_{i-\ell-2}}{(i - \ell - 1)! (i - \ell - 2)! \ell! (i - \ell - 1)!} \\ \times {}_4F_3 \left(\begin{matrix} -\ell, \alpha + \beta + 5, \alpha + \beta + 4, \beta + i - \ell + 1 \\ \beta + 3, i - \ell, i - \ell \end{matrix} \middle| 1 \right) \left(\frac{x + 1}{2} \right)^{\ell+1} \quad (10)$$

and

$$c_i^{(2)}(\alpha, \beta, x) = \frac{(\alpha + \beta + 2)^2 (\alpha + \beta + 3) (\alpha + \beta + 4)}{(\alpha + 1)(\beta + 1)i} (x^2 - 1) (-2)^{i-2} \\ \times \sum_{\ell=0}^{i-2} \frac{(\alpha + 3)_{i-\ell-2} (-\alpha - \beta - 3)_{i-\ell-2}}{(i - \ell - 1)! (i - \ell - 2)! \ell! (i - \ell - 1)!} \\ \times {}_4F_3 \left(\begin{matrix} -\ell, \alpha + \beta + 5, \alpha + \beta + 4, \alpha + i - \ell + 1 \\ \alpha + 3, i - \ell, i - \ell \end{matrix} \middle| 1 \right) \left(\frac{x - 1}{2} \right)^{\ell+1}. \quad (11)$$

Note that we have

$$a_i(\alpha, \beta, x) = (-1)^i b_i(\beta, \alpha, -x), \quad i = 1, 2, 3, \dots \quad (12)$$

and

$$c_i^{(1)}(\alpha, \beta, x) = (-1)^i c_i^{(2)}(\beta, \alpha, -x), \quad i = 2, 3, 4, \dots \tag{13}$$

Finally we will show that for $\alpha > -1$, $\beta > -1$ and $M^2 + N^2 > 0$ the order of the differential equation (2) will be infinite in general. Only for nonnegative integer values of α or β finite order can occur. Moreover, the order of the differential equation equals

$$\begin{cases} 2\beta + 4 & \text{if } M > 0, N = 0 \text{ and } \beta \in \{0, 1, 2, \dots\}, \\ 2\alpha + 4 & \text{if } M = 0, N > 0 \text{ and } \alpha \in \{0, 1, 2, \dots\}, \\ 2\alpha + 2\beta + 6 & \text{if } M > 0, N > 0 \text{ and } \alpha, \beta \in \{0, 1, 2, \dots\}. \end{cases}$$

In fact, we will show that

$$a_i(\alpha, \beta, x) = 0, \quad i > 2\beta + 4 \text{ if } \beta \in \{0, 1, 2, \dots\}, \tag{14}$$

$$b_i(\alpha, \beta, x) = 0, \quad i > 2\alpha + 4 \text{ if } \alpha \in \{0, 1, 2, \dots\} \tag{15}$$

and

$$c_i(\alpha, \beta, x) = 0, \quad i > 2\alpha + 2\beta + 6 \text{ if } \alpha, \beta \in \{0, 1, 2, \dots\}. \tag{16}$$

Further we have

$$a_{2\beta+4}(\alpha, \beta, x) = -\frac{1}{(\alpha + 1)_{\beta+1}} \frac{(x^2 - 1)^{\beta+2}}{(\beta + 2)!}, \quad \beta \in \{0, 1, 2, \dots\}, \tag{17}$$

$$b_{2\alpha+4}(\alpha, \beta, x) = -\frac{1}{(\beta + 1)_{\alpha+1}} \frac{(x^2 - 1)^{\alpha+2}}{(\alpha + 2)!}, \quad \alpha \in \{0, 1, 2, \dots\} \tag{18}$$

and

$$c_{2\alpha+2\beta+6}(\alpha, \beta, x) = -\frac{\alpha + \beta + 2}{(\alpha + 1)(\beta + 1)} \frac{(x^2 - 1)^{\alpha+\beta+3}}{(\alpha + \beta + 1)!(\alpha + \beta + 3)!}, \quad \alpha, \beta \in \{0, 1, 2, \dots\}. \tag{19}$$

3. The classical Jacobi polynomials

In this section we list the definitions and some properties of the classical Jacobi polynomials which we will use in this paper. For details the reader is referred to [7,14,23].

The classical Jacobi polynomials $\{P_n^{(\alpha, \beta)}(x)\}_{n=0}^\infty$ can be defined by

$$P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n \frac{(n + \alpha + \beta + 1)_k}{k!} \frac{(\alpha + k + 1)_{n-k}}{(n - k)!} \left(\frac{x - 1}{2}\right)^k, \quad n = 0, 1, 2, \dots \tag{20}$$

$$= (-1)^n \sum_{k=0}^n \frac{(-n - k - \alpha - \beta)_k}{k!} \frac{(-n - \alpha)_{n-k}}{(n - k)!} \left(\frac{x - 1}{2}\right)^k, \quad n = 0, 1, 2, \dots \tag{21}$$

for all α and β . The Jacobi polynomials satisfy the symmetry relation

$$P_n^{(\alpha, \beta)}(x) = (-1)^n P_n^{(\beta, \alpha)}(-x), \quad n = 0, 1, 2, \dots \tag{22}$$

From (20) and (22) we easily find for $n = 0, 1, 2, \dots$

$$P_n^{(\alpha, \beta)}(1) = \frac{(\alpha + 1)_n}{n!} \quad \text{and} \quad P_n^{(\alpha, \beta)}(-1) = (-1)^n \frac{(\beta + 1)_n}{n!} \tag{23}$$

and

$$D^i P_n^{(\alpha, \beta)}(x) = \frac{(n + \alpha + \beta + 1)_i}{2^i} P_{n-i}^{(\alpha+i, \beta+i)}(x), \quad i = 0, 1, 2, \dots, n, \tag{24}$$

where $D = d/dx$ denotes the differentiation operator. These Jacobi polynomials satisfy the linear second-order differential equation

$$(1 - x^2)y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]y'(x) + n(n + \alpha + \beta + 1)y(x) = 0. \tag{25}$$

By using the definition (20) and the symmetry relation (22) it is not very difficult to derive the following relations:

$$P_n^{(\alpha+1, \beta)}(x) - P_n^{(\alpha, \beta+1)}(x) = P_{n-1}^{(\alpha+1, \beta+1)}(x), \quad n = 1, 2, 3, \dots, \tag{26}$$

$$nP_n^{(\alpha, \beta)}(x) - (n + \alpha)P_{n-1}^{(\alpha, \beta+1)}(x) = (x - 1)DP_n^{(\alpha, \beta)}(x), \quad n = 1, 2, 3, \dots, \tag{27}$$

$$nP_n^{(\alpha, \beta)}(x) + (n + \beta)P_{n-1}^{(\alpha+1, \beta)}(x) = (x + 1)DP_n^{(\alpha, \beta)}(x), \quad n = 1, 2, 3, \dots, \tag{28}$$

$$(n + \alpha + 1)P_n^{(\alpha, \beta)}(x) - (\alpha + 1)P_n^{(\alpha+1, \beta)}(x) = (n + \beta) \left(\frac{x - 1}{2} \right) P_{n-1}^{(\alpha+2, \beta)}(x), \quad n = 1, 2, 3, \dots \tag{29}$$

and

$$(n + \beta + 1)P_n^{(\alpha, \beta)}(x) - (\beta + 1)P_n^{(\alpha, \beta+1)}(x) = (n + \alpha) \left(\frac{x + 1}{2} \right) P_{n-1}^{(\alpha, \beta+2)}(x), \quad n = 1, 2, 3, \dots \tag{30}$$

Note that the differential equation (25) implies that

$$\begin{aligned} n(n + \alpha + \beta + 1)P_n^{(\alpha, \beta)}(x) - [(\beta + 1)(x - 1) + (\alpha + 1)(x + 1)]DP_n^{(\alpha, \beta)}(x) \\ = (x^2 - 1)D^2P_n^{(\alpha, \beta)}(x), \quad n = 0, 1, 2, \dots \end{aligned} \tag{31}$$

By using Leibniz' rule we also have for $n = 0, 1, 2, \dots$ and $i = 0, 1, 2, \dots$

$$\begin{aligned} (1 - x^2)D^{i+2}P_n^{(\alpha, \beta)}(x) + [\beta - \alpha - (\alpha + \beta + 2i + 2)x]D^{i+1}P_n^{(\alpha, \beta)}(x) \\ + (n - i)(n + \alpha + \beta + i + 1)D^iP_n^{(\alpha, \beta)}(x) = 0. \end{aligned} \tag{32}$$

4. Some inversion, summation and transformation formulas

In this section we will give some inversion formulas which we will need in this paper. Further we derive some summation formulas which we will use. Finally, we give two transformation formulas which will be used in Section 8 of this paper.

Let $\alpha > -1$ and $\beta > -1$.

In this paper we have to deal with systems of equations of the form

$$\sum_{i=1}^{\infty} A_i(x)D^i P_n^{(\alpha, \beta)}(x) = F_n(x), \quad n = 1, 2, 3, \dots, \tag{33}$$

where the coefficients $\{A_i(x)\}_{i=1}^\infty$ are independent of n . In [11] we have shown that this system of equations has a unique solution given by

$$A_i(x) = 2^i \sum_{j=1}^i \frac{\alpha + \beta + 2j + 1}{(\alpha + \beta + j + 1)_{i+1}} P_{i-j}^{(-\alpha-i-1, -\beta-i-1)}(x) F_j(x), \quad i = 1, 2, 3, \dots \tag{34}$$

We will also need a variant of this inversion formula. In a similar way we may also conclude that a system of equations of the form

$$\sum_{i=0}^\infty B_i(x) D^i P_n^{(\alpha, \beta)}(x) = G_n(x), \quad n = 0, 1, 2, \dots, \tag{35}$$

where the coefficients $\{B_i(x)\}_{i=0}^\infty$ are independent of n has a unique solution given by

$$B_i(x) = 2^i \sum_{j=0}^i \frac{\alpha + \beta + 2j + 1}{(\alpha + \beta + j + 1)_{i+1}} P_{i-j}^{(-\alpha-i-1, -\beta-i-1)}(x) G_j(x), \quad i = 0, 1, 2, \dots \tag{36}$$

The case $\alpha + \beta + 1 = 0$ must be understood by continuity.

Let N denote a positive integer. Now we consider the $(N \times N)$ -matrix A defined by

$$A = (a_{ij})_{i,j=1}^N \quad \text{with} \quad a_{ij} = \begin{cases} i, & i = j, \\ z, & i = j + 1, \\ 0, & \text{otherwise.} \end{cases} \tag{37}$$

Since $\det(A) = N! \neq 0$ this matrix is invertible for every z . One can show that its inverse is given by

$$A^{-1} = B = (b_{ij})_{i,j=1}^N \quad \text{with} \quad b_{ij} = \begin{cases} \frac{(-1)^{i-j} (j-1)! z^{i-j}}{i!}, & i \geq j, \\ 0, & i < j. \end{cases} \tag{38}$$

We also need the following matrix inverse. Let N denote a positive integer again and consider the $(N \times N)$ -matrix A defined by

$$A = (a_{ij})_{i,j=1}^N \quad \text{with} \quad a_{ij} = \begin{cases} i(i+1), & i = j, \\ 2ix, & i = j + 1, \\ x^2 - 1, & i = j + 2, \\ 0, & \text{otherwise.} \end{cases} \tag{39}$$

Since $\det(A) = N!(N+1)! \neq 0$ this matrix is invertible for every x . It can be shown that its inverse is given by

$$A^{-1} = B = (b_{ij})_{i,j=1}^N$$

with

$$b_{ij} = \begin{cases} \frac{(-1)^{i-j} (j-1)! [(x+1)^{i-j+1} - (x-1)^{i-j+1}]}{2(i+1)!}, & i \geq j, \\ 0, & i < j. \end{cases} \tag{40}$$

We will also need the well-known Vandermonde summation formula

$${}_2F_1 \left(\begin{matrix} -n, a \\ b \end{matrix} \middle| 1 \right) = \frac{(b-a)_n}{(b)_n}, \quad (b)_n \neq 0, \quad n = 0, 1, 2, \dots, \tag{41}$$

which can be found in [1,22] for instance. We also need the following summation formulas:

$$F_n(a, b) = \sum_{k=0}^n \frac{(a)_k (b)_k}{(b-a+1)_k k!} (b+2k) \quad (42)$$

$$= \frac{(a+1)_n (b)_{n+1}}{(b-a+1)_n n!}, \quad n = 0, 1, 2, \dots \quad (43)$$

and

$$\sum_{k=0}^n \frac{(-n)_k (a)_k (b)_k (c)_k}{(b+n+1)_k (b-a+1)_k (b-c+1)_k k!} (b+2k) \\ = \frac{(b)_{n+1} (b-a-c+1)_n}{(b-a+1)_n (b-c+1)_n}, \quad n = 0, 1, 2, \dots \quad (44)$$

Formula (43) can easily be proved by using mathematical induction. Formula (44) can be proved by using the well-known summation formula for a terminating well-poised ${}_5F_4$:

$${}_5F_4 \left(\begin{matrix} -n, a, b, c, \frac{1}{2}b+1 \\ b+n+1, b-a+1, b-c+1, \frac{1}{2}b \end{matrix} \middle| 1 \right) = \frac{(b+1)_n (b-a-c+1)_n}{(b-a+1)_n (b-c+1)_n}, \quad n = 0, 1, 2, \dots$$

This formula can be found in [1,22] for instance. Note that (43) follows from (44) by setting $c = b + n + 1$.

Finally, we will need the following transformation formula (see, for instance, [21, Section 9.1, formula (34)]):

$${}_4F_3 \left(\begin{matrix} a, b, c, p \\ d, e, q \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} (-1)^n \frac{(a)_n (b)_n (c)_n (q-p)_n}{(d)_n (e)_n (q)_n n!} z^n \\ \times {}_3F_2 \left(\begin{matrix} n+a, n+b, n+c \\ n+d, n+e \end{matrix} \middle| z \right), \quad \operatorname{Re}(z) < \frac{1}{2}. \quad (45)$$

As a special case we also have

$${}_3F_2 \left(\begin{matrix} a, b, p \\ c, q \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} (-1)^n \frac{(a)_n (b)_n (q-p)_n}{(c)_n (q)_n n!} z^n {}_2F_1 \left(\begin{matrix} n+a, n+b \\ n+c \end{matrix} \middle| z \right), \quad \operatorname{Re}(z) < \frac{1}{2}. \quad (46)$$

5. The generalized Jacobi polynomials

Let $\alpha > -1$, $\beta > -1$, $M \geq 0$ and $N \geq 0$. In [15] it is shown that the generalized Jacobi polynomials $\{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty}$ can be written as

$$P_n^{\alpha, \beta, M, N}(x) = P_n^{(\alpha, \beta)}(x) + MQ_n^{(\alpha, \beta)}(x) + NR_n^{(\alpha, \beta)}(x) + MNS_n^{(\alpha, \beta)}(x), \quad n = 0, 1, 2, \dots, \quad (47)$$

where

$$Q_0^{(\alpha, \beta)}(x) = R_0^{(\alpha, \beta)}(x) = S_0^{(\alpha, \beta)}(x) = 0$$

and for $n = 1, 2, 3, \dots$

$$Q_n^{(\alpha, \beta)}(x) = \frac{(\beta + 2)_{n-1}(\alpha + \beta + 2)_{n-1}}{(\alpha + 1)_n n!} \times [n(n + \alpha + \beta + 1)P_n^{(\alpha, \beta)}(x) - (\beta + 1)(x - 1)DP_n^{(\alpha, \beta)}(x)], \tag{48}$$

$$R_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 2)_{n-1}(\alpha + \beta + 2)_{n-1}}{(\beta + 1)_n n!} \times [n(n + \alpha + \beta + 1)P_n^{(\alpha, \beta)}(x) - (\alpha + 1)(x + 1)DP_n^{(\alpha, \beta)}(x)] \tag{49}$$

and

$$S_n^{(\alpha, \beta)}(x) = \frac{1}{(\alpha + 1)(\beta + 1)} \frac{(\alpha + \beta + 2)_n(\alpha + \beta + 2)_{n-1}}{n!(n - 1)!} \times [n(n + \alpha + \beta + 1)P_n^{(\alpha, \beta)}(x) - \{(\beta + 1)(x - 1) + (\alpha + 1)(x + 1)\}DP_n^{(\alpha, \beta)}(x)]. \tag{50}$$

First of all we remark that the generalized Jacobi polynomials satisfy the symmetry relation (see [15])

$$P_n^{\alpha, \beta, M, N}(x) = (-1)^n P_n^{\beta, \alpha, N, M}(-x), \quad n = 0, 1, 2, \dots, \tag{51}$$

which implies that

$$Q_n^{(\alpha, \beta)}(x) = (-1)^n R_n^{(\beta, \alpha)}(-x), \quad n = 0, 1, 2, \dots \tag{52}$$

and

$$S_n^{(\alpha, \beta)}(x) = (-1)^n S_n^{(\beta, \alpha)}(-x), \quad n = 0, 1, 2, \dots$$

From (48) and (49) it follows that

$$Q_n^{(\alpha, \beta)}(1) = \frac{(\beta + 2)_{n-1}(\alpha + \beta + 2)_n}{(\alpha + 1)_n (n - 1)!} P_n^{(\alpha, \beta)}(1), \quad n = 1, 2, 3, \dots \tag{53}$$

and

$$R_n^{(\alpha, \beta)}(-1) = \frac{(\alpha + 2)_{n-1}(\alpha + \beta + 2)_n}{(\beta + 1)_n (n - 1)!} P_n^{(\alpha, \beta)}(-1), \quad n = 1, 2, 3, \dots \tag{54}$$

These two formulas will be used in the next section.

Now we use (26)–(28) to obtain for $n = 1, 2, 3, \dots$

$$\begin{aligned} & n(n + \alpha + \beta + 1)P_n^{(\alpha, \beta)}(x) - (\beta + 1)(x - 1)DP_n^{(\alpha, \beta)}(x) \\ &= (n + \alpha)[nP_n^{(\alpha, \beta)}(x) + (\beta + 1)P_{n-1}^{(\alpha, \beta+1)}(x)] \\ &= (n + \alpha)[nP_n^{(\alpha-1, \beta+1)}(x) + (n + \beta + 1)P_{n-1}^{(\alpha, \beta+1)}(x)] \\ &= (n + \alpha)(x + 1)DP_n^{(\alpha-1, \beta+1)}(x). \end{aligned}$$

Hence from (48) we obtain the following representations for $Q_n^{(\alpha, \beta)}(x)$:

$$Q_n^{(\alpha, \beta)}(x) = \frac{(\beta + 2)_{n-1}(\alpha + \beta + 2)_{n-1}}{(\alpha + 1)_{n-1} n!} [nP_n^{(\alpha, \beta)}(x) + (\beta + 1)P_{n-1}^{(\alpha, \beta+1)}(x)] \tag{55}$$

$$= \frac{(\beta + 2)_{n-1}(\alpha + \beta + 2)_{n-1}}{(\alpha + 1)_{n-1} n!} (x + 1)DP_n^{(\alpha-1, \beta+1)}(x) \tag{56}$$

for $n = 1, 2, 3, \dots$. In a similar way from (49) or by using the symmetry relation (52) we find the following representations for $R_n^{(\alpha, \beta)}(x)$:

$$R_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 2)_{n-1}(\alpha + \beta + 2)_{n-1}}{(\beta + 1)_{n-1} n!} [nP_n^{(\alpha, \beta)}(x) - (\alpha + 1)P_{n-1}^{(\alpha+1, \beta)}(x)] \quad (57)$$

$$= \frac{(\alpha + 2)_{n-1}(\alpha + \beta + 2)_{n-1}}{(\beta + 1)_{n-1} n!} (x - 1)DP_n^{(\alpha+1, \beta-1)}(x) \quad (58)$$

for $n = 1, 2, 3, \dots$.

And if we use (31) we easily find from (50) that

$$S_n^{(\alpha, \beta)}(x) = \frac{1}{(\alpha + 1)(\beta + 1)} \frac{(\alpha + \beta + 2)_n(\alpha + \beta + 2)_{n-1}}{n!(n-1)!} (x^2 - 1)D^2P_n^{(\alpha, \beta)}(x), \quad n = 1, 2, 3, \dots \quad (59)$$

Note that the representations (55) and (57) imply that for $n = 1, 2, 3, \dots$ we have

$$Q_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n q_{n,k}^{(\alpha, \beta)} P_k^{(\alpha, \beta)}(x) \quad \text{with} \quad q_{n,n}^{(\alpha, \beta)} = \frac{(\beta + 2)_{n-1}(\alpha + \beta + 2)_{n-1}}{(\alpha + 1)_{n-1} (n-1)!} \quad (60)$$

and

$$R_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n r_{n,k}^{(\alpha, \beta)} P_k^{(\alpha, \beta)}(x) \quad \text{with} \quad r_{n,n}^{(\alpha, \beta)} = \frac{(\alpha + 2)_{n-1}(\alpha + \beta + 2)_{n-1}}{(\beta + 1)_{n-1} (n-1)!}. \quad (61)$$

By using (27) and (28) we also find from (50) that for $n = 1, 2, 3, \dots$ we have

$$S_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n s_{n,k}^{(\alpha, \beta)} P_k^{(\alpha, \beta)}(x)$$

with

$$s_{n,n}^{(\alpha, \beta)} = \frac{n(n-1)}{(\alpha + 1)(\beta + 1)} \frac{(\alpha + \beta + 2)_n(\alpha + \beta + 2)_{n-1}}{n!(n-1)!}. \quad (62)$$

6. The existence and uniqueness of the differential equation and the ‘eigenvalue’ coefficients

First of all we set

$$\lambda_n = n(n + \alpha + \beta + 1), \quad n = 0, 1, 2, \dots,$$

which implies that $\lambda_0 = 0$ and

$$\lambda_n - \lambda_{n-1} = 2n + \alpha + \beta, \quad n = 1, 2, 3, \dots \quad (63)$$

In [4] Bavinck proved a theorem concerning differential or difference equations satisfied by certain orthogonal polynomials. This result can be applied to the generalized Jacobi polynomials $\{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty}$. In that case for $\alpha > -1$, $\beta > -1$, $M \geq 0$ and $N \geq 0$ his result reads as follows:

Theorem (Bavinck [4]). *If*

$$P_n^{(\alpha, \beta)}(1) \neq 0 \quad \text{and} \quad P_n^{(\alpha, \beta)}(-1) \neq 0, \quad n = 0, 1, 2, \dots \quad (64)$$

and

$$P_n^{(\alpha, \beta)}(1) + MQ_n^{(\alpha, \beta)}(1) \neq 0 \quad \text{and} \quad P_n^{(\alpha, \beta)}(-1) + NR_n^{(\alpha, \beta)}(-1) \neq 0, \quad n = 0, 1, 2, \dots, \quad (65)$$

then the generalized Jacobi polynomials given by (47) satisfy a unique differential equation of the form (2), where

$$a_0(0, \alpha, \beta) = b_0(0, \alpha, \beta) = c_0(0, \alpha, \beta) = 0,$$

$$a_0(n, \alpha, \beta) = \sum_{j=1}^n (\lambda_j - \lambda_{j-1}) q_{j,j}^{(\alpha, \beta)}, \quad n = 1, 2, 3, \dots,$$

$$b_0(n, \alpha, \beta) = \sum_{j=1}^n (\lambda_j - \lambda_{j-1}) r_{j,j}^{(\alpha, \beta)}, \quad n = 1, 2, 3, \dots$$

and

$$c_0(n, \alpha, \beta) = \sum_{j=1}^n (\lambda_j - \lambda_{j-1}) s_{j,j}^{(\alpha, \beta)}, \quad n = 1, 2, 3, \dots,$$

with $q_{j,j}^{(\alpha, \beta)}$, $r_{j,j}^{(\alpha, \beta)}$ and $s_{j,j}^{(\alpha, \beta)}$ given by (60), (61) and (62), respectively.

Since $\alpha > -1$, $\beta > -1$, $P_0^{(\alpha, \beta)}(x) = 1$ and $Q_0^{(\alpha, \beta)}(x) = R_0^{(\alpha, \beta)}(x) = 0$ it easily follows from (23) that condition (64) is satisfied. Since also $M \geq 0$ and $N \geq 0$ we conclude, by using (53) and (54), that condition (65) is satisfied too.

By using (60)–(63) we find that

$$a_0(n, \alpha, \beta) = \sum_{j=1}^n \frac{(\beta + 2)_{j-1} (\alpha + \beta + 2)_{j-1}}{(\alpha + 1)_{j-1} (j - 1)!} (2j + \alpha + \beta), \quad n = 1, 2, 3, \dots,$$

$$b_0(n, \alpha, \beta) = \sum_{j=1}^n \frac{(\alpha + 2)_{j-1} (\alpha + \beta + 2)_{j-1}}{(\beta + 1)_{j-1} (j - 1)!} (2j + \alpha + \beta), \quad n = 1, 2, 3, \dots,$$

$c_0(1, \alpha, \beta) = 0$ and

$$c_0(n, \alpha, \beta) = \frac{(\alpha + \beta + 2)^2}{(\alpha + 1)(\beta + 1)} \sum_{j=2}^n \frac{(\alpha + \beta + 3)_{j-1} (\alpha + \beta + 3)_{j-2}}{(j - 1)! (j - 2)!} (2j + \alpha + \beta), \quad n = 2, 3, 4, \dots$$

Note that

$$a_0(n, \alpha, \beta) = \sum_{k=0}^{n-1} \frac{(\beta + 2)_k (\alpha + \beta + 2)_k}{(\alpha + 1)_k k!} (2k + \alpha + \beta + 2) = F_{n-1}(\beta + 2, \alpha + \beta + 2),$$

$$n = 1, 2, 3, \dots,$$

$$b_0(n, \alpha, \beta) = \sum_{k=0}^{n-1} \frac{(\alpha + 2)_k (\alpha + \beta + 2)_k}{(\beta + 1)_k k!} (2k + \alpha + \beta + 2) = F_{n-1}(\alpha + 2, \alpha + \beta + 2),$$

$$n = 1, 2, 3, \dots$$

and since $(k+1)! = (2)_k$, $k = 0, 1, 2, \dots$

$$\begin{aligned} c_0(n, \alpha, \beta) &= \frac{(\alpha + \beta + 2)^2(\alpha + \beta + 3)}{(\alpha + 1)(\beta + 1)} \sum_{k=0}^{n-2} \frac{(\alpha + \beta + 3)_k(\alpha + \beta + 4)_k}{(2)_k k!} (2k + \alpha + \beta + 4) \\ &= \frac{(\alpha + \beta + 2)^2(\alpha + \beta + 3)}{(\alpha + 1)(\beta + 1)} F_{n-2}(\alpha + \beta + 3, \alpha + \beta + 4), \quad n = 2, 3, 4, \dots, \end{aligned}$$

where $F_n(a, b)$ is given by (42). Now we use the summation formula (43) to obtain (4)–(6).

7. The computation of the other coefficients

First of all we remark that the symmetry relation (51) implies that

$$a_0(n, \alpha, \beta) = b_0(n, \beta, \alpha) \quad \text{and} \quad c_0(n, \alpha, \beta) = c_0(n, \beta, \alpha), \quad n = 0, 1, 2, \dots \quad (66)$$

and

$$a_i(\alpha, \beta, x) = (-1)^i b_i(\beta, \alpha, -x) \quad \text{and} \quad c_i(\alpha, \beta, x) = (-1)^i c_i(\beta, \alpha, -x), \quad i = 1, 2, 3, \dots$$

Hence we have (12). Note that in the preceding section we have already determined the ‘eigenvalue’ coefficients $a_0(n, \alpha, \beta)$, $b_0(n, \alpha, \beta)$ and $c_0(n, \alpha, \beta)$. From (4)–(6) it is clear that (66) is satisfied.

In order to compute the other coefficients $\{a_i(x)\}_{i=1}^{\infty}$, $\{b_i(x)\}_{i=1}^{\infty}$ and $\{c_i(x)\}_{i=1}^{\infty}$ we set $y(x) = P_n^{\alpha, \beta, M, N}(x)$ in the differential equation (2) and use (47) and the fact that the classical Jacobi polynomials satisfy the differential equation (25) to obtain for $n = 0, 1, 2, \dots$

$$\begin{aligned} M \sum_{i=0}^{\infty} a_i(x) D^i [P_n^{(\alpha, \beta)}(x) + MQ_n^{(\alpha, \beta)}(x) + NR_n^{(\alpha, \beta)}(x) + MNS_n^{(\alpha, \beta)}(x)] \\ + N \sum_{i=0}^{\infty} b_i(x) D^i [P_n^{(\alpha, \beta)}(x) + MQ_n^{(\alpha, \beta)}(x) + NR_n^{(\alpha, \beta)}(x) + MNS_n^{(\alpha, \beta)}(x)] \\ + MN \sum_{i=0}^{\infty} c_i(x) D^i [P_n^{(\alpha, \beta)}(x) + MQ_n^{(\alpha, \beta)}(x) + NR_n^{(\alpha, \beta)}(x) + MNS_n^{(\alpha, \beta)}(x)] \\ + M[(1-x^2)D^2 Q_n^{(\alpha, \beta)}(x) + [\beta - \alpha - (\alpha + \beta + 2)x]DQ_n^{(\alpha, \beta)}(x) + n(n + \alpha + \beta + 1)Q_n^{(\alpha, \beta)}(x)] \\ + N[(1-x^2)D^2 R_n^{(\alpha, \beta)}(x) + [\beta - \alpha - (\alpha + \beta + 2)x]DR_n^{(\alpha, \beta)}(x) + n(n + \alpha + \beta + 1)R_n^{(\alpha, \beta)}(x)] \\ + MN[(1-x^2)D^2 S_n^{(\alpha, \beta)}(x) + [\beta - \alpha - (\alpha + \beta + 2)x]DS_n^{(\alpha, \beta)}(x) \\ + n(n + \alpha + \beta + 1)S_n^{(\alpha, \beta)}(x)] = 0. \end{aligned} \quad (67)$$

Since $P_0^{(\alpha, \beta)}(x) = 1$, $Q_0^{(\alpha, \beta)}(x) = R_0^{(\alpha, \beta)}(x) = S_0^{(\alpha, \beta)}(x) = 0$ and $a_0(0, \alpha, \beta) = b_0(0, \alpha, \beta) = c_0(0, \alpha, \beta) = 0$ this is trivially true for $n = 0$.

Now we use (55), (57) and (26) to find for $n = 1, 2, 3, \dots$

$$Q_n^{(\alpha, \beta)}(x) = \frac{(\beta + 2)_{n-1}(\alpha + \beta + 2)_{n-1}}{(\alpha + 1)_{n-1} n!} [(n + \beta + 1)P_n^{(\alpha, \beta)}(x) - (\beta + 1)P_n^{(\alpha-1, \beta+1)}(x)]$$

and

$$R_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 2)_{n-1}(\alpha + \beta + 2)_{n-1}}{(\beta + 1)_{n-1} n!} [(n + \alpha + 1)P_n^{(\alpha, \beta)}(x) - (\alpha + 1)P_n^{(\alpha+1, \beta-1)}(x)].$$

By using these representations, (32) and (24) we find that

$$\begin{aligned} & (1 - x^2) D^2 Q_n^{(\alpha, \beta)}(x) + [\beta - \alpha - (\alpha + \beta + 2)x] D Q_n^{(\alpha, \beta)}(x) + n(n + \alpha + \beta + 1) Q_n^{(\alpha, \beta)}(x) \\ &= - \frac{(\beta + 1)_n (\alpha + \beta + 2)_{n-1}}{(\alpha + 1)_{n-1} n!} [(1 - x^2) D^2 P_n^{(\alpha-1, \beta+1)}(x) \\ &\quad + [\beta - \alpha - (\alpha + \beta + 2)x] D P_n^{(\alpha-1, \beta+1)}(x) \\ &\quad + n(n + \alpha + \beta + 1) P_n^{(\alpha-1, \beta+1)}(x)] \\ &= \frac{(\beta + 1)_n (\alpha + \beta + 2)_{n-1}}{(\alpha + 1)_{n-1} n!} 2 D P_n^{(\alpha-1, \beta+1)}(x) \\ &= \frac{(\beta + 1)_n (\alpha + \beta + 2)_n}{(\alpha + 1)_{n-1} n!} P_{n-1}^{(\alpha, \beta+2)}(x), \quad n = 1, 2, 3, \dots \end{aligned}$$

and similarly

$$\begin{aligned} & (1 - x^2) D^2 R_n^{(\alpha, \beta)}(x) + [\beta - \alpha - (\alpha + \beta + 2)x] D R_n^{(\alpha, \beta)}(x) + n(n + \alpha + \beta + 1) R_n^{(\alpha, \beta)}(x) \\ &= - \frac{(\alpha + 1)_n (\alpha + \beta + 2)_n}{(\beta + 1)_{n-1} n!} P_{n-1}^{(\alpha+2, \beta)}(x), \quad n = 1, 2, 3, \dots \end{aligned}$$

Finally we have

$$D[(x^2 - 1) D^2 P_n^{(\alpha, \beta)}(x)] = (x^2 - 1) D^3 P_n^{(\alpha, \beta)}(x) + 2x D^2 P_n^{(\alpha, \beta)}(x), \quad n = 0, 1, 2, \dots$$

and

$$D^2[(x^2 - 1) D^2 P_n^{(\alpha, \beta)}(x)] = (x^2 - 1) D^4 P_n^{(\alpha, \beta)}(x) + 4x D^3 P_n^{(\alpha, \beta)}(x) + 2 D^2 P_n^{(\alpha, \beta)}(x), \quad n = 0, 1, 2, \dots,$$

which implies by using (32) that

$$\begin{aligned} & (1 - x^2) D^2 [(x^2 - 1) D^2 P_n^{(\alpha, \beta)}(x)] + [\beta - \alpha - (\alpha + \beta + 2)x] D [(x^2 - 1) D^2 P_n^{(\alpha, \beta)}(x)] \\ &\quad + n(n + \alpha + \beta + 1) (x^2 - 1) D^2 P_n^{(\alpha, \beta)}(x) \\ &= (x^2 - 1) [(1 - x^2) D^4 P_n^{(\alpha, \beta)}(x) + [\beta - \alpha - (\alpha + \beta + 6)x] D^3 P_n^{(\alpha, \beta)}(x) \\ &\quad + (n - 2)(n + \alpha + \beta + 3) D^2 P_n^{(\alpha, \beta)}(x)] \\ &\quad + 2[(\beta + 1)(x - 1) - (\alpha + 1)(x + 1)] D^2 P_n^{(\alpha, \beta)}(x) \\ &= 2[(\beta + 1)(x - 1) - (\alpha + 1)(x + 1)] D^2 P_n^{(\alpha, \beta)}(x), \quad n = 0, 1, 2, \dots \end{aligned}$$

Hence by using (59) we find that

$$\begin{aligned} & (1 - x^2) D^2 S_n^{(\alpha, \beta)}(x) + [\beta - \alpha - (\alpha + \beta + 2)x] D S_n^{(\alpha, \beta)}(x) + n(n + \alpha + \beta + 1) S_n^{(\alpha, \beta)}(x) \\ &= \frac{2}{(\alpha + 1)(\beta + 1)} \frac{(\alpha + \beta + 2)_n (\alpha + \beta + 2)_{n-1}}{n! (n - 1)!} \\ &\quad \times [(\beta + 1)(x - 1) - (\alpha + 1)(x + 1)] D^2 P_n^{(\alpha, \beta)}(x), \quad n = 1, 2, 3, \dots \end{aligned}$$

Since we demand that the differential equation (2) must hold for all $M \geq 0$ and $N \geq 0$ we view the left-hand side of (67) as a polynomial in M and N and conclude that all coefficients of this polynomial must be equal to zero, hence we derive the following eight systems of equations:

$$\begin{aligned} M: S_1 &= 0, & MN: S_5 &= 0, \\ M^2: S_2 &= 0, & M^2N: S_6 &= 0, \\ N: S_3 &= 0, & MN^2: S_7 &= 0, \\ N^2: S_4 &= 0, & M^2N^2: S_8 &= 0, \end{aligned}$$

where $n = 1, 2, 3, \dots$ and

$$S_1 = \sum_{i=0}^{\infty} a_i(x) D^i P_n^{(\alpha, \beta)}(x) + \frac{(\beta + 1)_n (\alpha + \beta + 2)_n}{(\alpha + 1)_{n-1} n!} P_{n-1}^{(\alpha, \beta+2)}(x),$$

$$S_2 = \sum_{i=0}^{\infty} a_i(x) D^i Q_n^{(\alpha, \beta)}(x),$$

$$S_3 = \sum_{i=0}^{\infty} b_i(x) D^i P_n^{(\alpha, \beta)}(x) - \frac{(\alpha + 1)_n (\alpha + \beta + 2)_n}{(\beta + 1)_{n-1} n!} P_{n-1}^{(\alpha+2, \beta)}(x),$$

$$S_4 = \sum_{i=0}^{\infty} b_i(x) D^i R_n^{(\alpha, \beta)}(x),$$

$$\begin{aligned} S_5 &= \sum_{i=0}^{\infty} a_i(x) D^i R_n^{(\alpha, \beta)}(x) + \sum_{i=0}^{\infty} b_i(x) D^i Q_n^{(\alpha, \beta)}(x) + \sum_{i=0}^{\infty} c_i(x) D^i P_n^{(\alpha, \beta)}(x) \\ &+ \frac{2}{(\alpha + 1)(\beta + 1)} \frac{(\alpha + \beta + 2)_n (\alpha + \beta + 2)_{n-1}}{n! (n-1)!} \\ &\times [(\beta + 1)(x - 1) - (\alpha + 1)(x + 1)] D^2 P_n^{(\alpha, \beta)}(x), \end{aligned}$$

$$S_6 = \sum_{i=0}^{\infty} a_i(x) D^i S_n^{(\alpha, \beta)}(x) + \sum_{i=0}^{\infty} c_i(x) D^i Q_n^{(\alpha, \beta)}(x),$$

$$S_7 = \sum_{i=0}^{\infty} b_i(x) D^i S_n^{(\alpha, \beta)}(x) + \sum_{i=0}^{\infty} c_i(x) D^i R_n^{(\alpha, \beta)}(x)$$

and

$$S_8 = \sum_{i=0}^{\infty} c_i(x) D^i S_n^{(\alpha, \beta)}(x).$$

By using (55) it follows from $S_1 = 0$ and $S_2 = 0$ that

$$\sum_{i=0}^{\infty} a_i(x) D^i P_{n-1}^{(\alpha, \beta+1)}(x) = \frac{(\beta + 2)_{n-1} (\alpha + \beta + 2)_n}{(\alpha + 1)_{n-1} (n-1)!} P_{n-1}^{(\alpha, \beta+2)}(x), \quad n = 1, 2, 3, \dots$$

In view of (4) this is trivial for $n = 1$. Hence, by shifting n and using (4) and (30) we obtain

$$\begin{aligned} & \sum_{i=1}^{\infty} a_i(x) D^i P_n^{(\alpha, \beta+1)}(x) \\ &= \frac{(\beta + 2)_n (\alpha + \beta + 2)_{n+1}}{(\alpha + 1)_n n!} P_n^{(\alpha, \beta+2)}(x) - a_0(n + 1, \alpha, \beta) P_n^{(\alpha, \beta+1)}(x) \\ &= \frac{(\beta + 2)_n (\alpha + \beta + 2)_{n+1}}{(\alpha + 1)_n n!} P_n^{(\alpha, \beta+2)}(x) - \frac{(\beta + 3)_n (\alpha + \beta + 2)_{n+1}}{(\alpha + 1)_n n!} P_n^{(\alpha, \beta+1)}(x) \\ &= \frac{(\beta + 3)_{n-1} (\alpha + \beta + 2)_{n+1}}{(\alpha + 1)_n n!} [(\beta + 2) P_n^{(\alpha, \beta+2)}(x) - (n + \beta + 2) P_n^{(\alpha, \beta+1)}(x)] \\ &= - \frac{(\beta + 3)_{n-1} (\alpha + \beta + 2)_{n+1}}{(\alpha + 1)_n n!} (n + \alpha) \left(\frac{x + 1}{2}\right) P_{n-1}^{(\alpha, \beta+3)}(x) \\ &= - (\alpha + \beta + 2) \left(\frac{x + 1}{2}\right) \frac{(\beta + 3)_{n-1} (\alpha + \beta + 3)_n}{(\alpha + 1)_{n-1} n!} P_{n-1}^{(\alpha, \beta+3)}(x), \quad n = 1, 2, 3, \dots \end{aligned}$$

Note that this system of equations has the form (33). Hence by using (34) we conclude that

$$\begin{aligned} a_i(\alpha, \beta, x) &= -(\alpha + \beta + 2) \left(\frac{x + 1}{2}\right) 2^i \\ &\quad \times \sum_{j=1}^i \frac{\alpha + \beta + 2j + 2}{(\alpha + \beta + j + 2)_{i+1}} \frac{(\beta + 3)_{j-1} (\alpha + \beta + 3)_j}{(\alpha + 1)_{j-1} j!} \\ &\quad \times P_{i-j}^{(-\alpha-i-1, -\beta-i-2)}(x) P_{j-1}^{(\alpha, \beta+3)}(x), \quad i = 1, 2, 3, \dots \end{aligned}$$

In the same way we obtain from $S_3 = 0$ and $S_4 = 0$ by using (57), (5), (29), (33) and (34)

$$\begin{aligned} b_i(\alpha, \beta, x) &= -(\alpha + \beta + 2) \left(\frac{x - 1}{2}\right) 2^i \\ &\quad \times \sum_{j=1}^i \frac{\alpha + \beta + 2j + 2}{(\alpha + \beta + j + 2)_{i+1}} \frac{(\alpha + 3)_{j-1} (\alpha + \beta + 3)_j}{(\beta + 1)_{j-1} j!} \\ &\quad \times P_{i-j}^{(-\alpha-i-2, -\beta-i-1)}(x) P_{j-1}^{(\alpha+3, \beta)}(x), \quad i = 1, 2, 3, \dots \end{aligned}$$

but this is not really necessary in view of (12).

In order to prove (8) we apply the definition (21) to $P_{i-j-1}^{(-\alpha-i-2, -\beta-i-1)}(x)$ and the definition (20) to $P_j^{(\alpha+3, \beta)}(x)$ to find by changing the order of summations and by using the summation formula (44):

$$\begin{aligned} & \sum_{j=1}^i \frac{\alpha + \beta + 2j + 2}{(\alpha + \beta + j + 2)_{i+1}} \frac{(\alpha + 3)_{j-1} (\alpha + \beta + 3)_j}{(\beta + 1)_{j-1} j!} P_{i-j}^{(-\alpha-i-2, -\beta-i-1)}(x) P_{j-1}^{(\alpha+3, \beta)}(x) \\ &= \sum_{j=0}^{i-1} \frac{\alpha + \beta + 2j + 4}{(\alpha + \beta + j + 3)_{i+1}} \frac{(\alpha + 3)_j (\alpha + \beta + 3)_{j+1}}{(\beta + 1)_j (j + 1)!} \end{aligned}$$

$$\begin{aligned}
& \times (-1)^{i-j-1} \sum_{k=0}^{i-j-1} \frac{(\alpha + \beta + i + j - k + 4)_k (\alpha + j + 3)_{i-j-k-1}}{k! (i-j-k-1)!} \left(\frac{x-1}{2}\right)^k \\
& \times \sum_{m=0}^j \frac{(\alpha + \beta + j + 4)_m (\alpha + m + 4)_{j-m}}{m! (j-m)!} \left(\frac{x-1}{2}\right)^m \\
= & \sum_{j=0}^{i-1} \sum_{k=0}^{i-j-1} \sum_{m=0}^j (\alpha + \beta + 2j + 4) (-1)^{i-j-1} \left(\frac{x-1}{2}\right)^{k+m} \\
& \times \frac{(\alpha + 3)_{i-k-1} (\alpha + \beta + 3)_{j+m+1} (\alpha + m + 4)_{j-m}}{(\alpha + \beta + j + 3)_{i-k+1} (\beta + 1)_j (j+1)! k! (i-j-k-1)! m! (j-m)!} \\
= & \sum_{k=0}^{i-1} \sum_{m=0}^{i-k-1} \sum_{j=0}^{i-k-m-1} (\alpha + \beta + 2j + 2m + 4) (-1)^{i-j-m-1} \left(\frac{x-1}{2}\right)^{k+m} \\
& \times \frac{(\alpha + 3)_{i-k-1} (\alpha + \beta + 3)_{j+2m+1} (\alpha + m + 4)_j}{(\alpha + \beta + j + m + 3)_{i-k+1} (\beta + 1)_{j+m} (j+m+1)! k! (i-j-k-m-1)! m! j!} \\
= & \sum_{k=0}^{i-1} \sum_{m=0}^{i-k-1} \frac{(\alpha + 3)_{i-k-1} (\alpha + \beta + 3)_{2m+1}}{(\alpha + \beta + m + 3)_{i-k+1} (\beta + 1)_m (m+1)! k! (i-k-m-1)! m!} \\
& \times (-1)^{i-m-1} \left(\frac{x-1}{2}\right)^{k+m} \\
& \times \sum_{j=0}^{i-k-m-1} \frac{(-i+k+m+1)_j (\alpha + \beta + 2m + 4)_j (\alpha + m + 4)_j (\alpha + \beta + m + 3)_j}{(\alpha + \beta + i - k + m + 4)_j (\beta + m + 1)_j (m+2)_j j!} \\
& \times (\alpha + \beta + 2j + 2m + 4) \\
= & \sum_{k=0}^{i-1} \sum_{m=0}^{i-k-1} \frac{(\alpha + 3)_{i-k-1} (\alpha + \beta + 3)_{2m+1}}{(\alpha + \beta + m + 3)_{i-k+1} (\beta + 1)_m (m+1)! k! (i-k-m-1)! m!} \\
& \times \frac{(\alpha + \beta + 2m + 4)_{i-k-m} (-\alpha - 2)_{i-k-m-1}}{(\beta + m + 1)_{i-k-m-1} (m+2)_{i-k-m-1}} (-1)^{i-m-1} \left(\frac{x-1}{2}\right)^{k+m} \\
= & \sum_{k=0}^{i-1} \sum_{m=0}^{i-k-1} (-1)^{i-m-1} \frac{(\alpha + 3)_{i-k-1} (\alpha + \beta + 3)_m (-\alpha - 2)_{i-k-m-1}}{(\beta + 1)_{i-k-1} (i-k)! k! (i-k-m-1)! m!} \left(\frac{x-1}{2}\right)^{k+m} \\
= & \sum_{m=0}^{i-1} \sum_{\ell=m}^{i-1} (-1)^{i-m-1} \frac{(\alpha + 3)_{i-\ell-1+m} (\alpha + \beta + 3)_m (-\alpha - 2)_{i-\ell-1}}{(\beta + 1)_{i-\ell-1+m} (i-\ell+m)! (\ell-m)! (i-\ell-1)! m!} \left(\frac{x-1}{2}\right)^\ell
\end{aligned}$$

$$\begin{aligned}
 &= (-1)^{i-1} \sum_{\ell=0}^{i-1} \frac{(\alpha+3)_{i-\ell-1}(-\alpha-2)_{i-\ell-1}}{(\beta+1)_{i-\ell-1}(i-\ell)!\ell!(i-\ell-1)!} \left(\frac{x-1}{2}\right)^\ell \\
 &\quad \times \sum_{m=0}^{\ell} \frac{(-\ell)_m(\alpha+i-\ell+2)_m(\alpha+\beta+3)_m}{(\beta+i-\ell)_m(i-\ell+1)_m m!}, \quad i = 1, 2, 3, \dots
 \end{aligned}$$

Hence

$$\begin{aligned}
 \beta_i(\alpha, \beta, x) &= (\alpha + \beta + 2)(-2)^i \sum_{\ell=0}^{i-1} \frac{(\alpha+3)_{i-\ell-1}(-\alpha-2)_{i-\ell-1}}{(\beta+1)_{i-\ell-1}(i-\ell)!(i-\ell-1)!\ell!} \\
 &\quad \times {}_3F_2 \left(\begin{matrix} -\ell, \alpha+i-\ell+2, \alpha+\beta+3 \\ \beta+i-\ell, i-\ell+1 \end{matrix} \middle| 1 \right) \left(\frac{x-1}{2}\right)^{\ell+1}, \quad i = 1, 2, 3, \dots,
 \end{aligned}$$

which proves (8). The proof of (7) is similar, but it is easier to use (12) since then (7) follows easily from (8).

The computation of the coefficients $\{c_i(x)\}_{i=1}^\infty$ is more difficult. First we set $n = 1$ into $S_6 = 0$ or $S_7 = 0$. Since we have $S_1^{(\alpha, \beta)}(x) = 0$ from (59) and $c_0(1, \alpha, \beta) = 0$ we conclude that

$$c_1(x)DQ_1^{(\alpha, \beta)}(x) = 0 \quad \text{and} \quad c_1(x)DR_1^{(\alpha, \beta)}(x) = 0.$$

By using (56), (58) and (24) we find that

$$Q_1^{(\alpha, \beta)}(x) = (\alpha + \beta + 2) \left(\frac{x+1}{2}\right) \quad \text{and} \quad R_1^{(\alpha, \beta)}(x) = (\alpha + \beta + 2) \left(\frac{x-1}{2}\right).$$

Hence

$$DQ_1^{(\alpha, \beta)}(x) = DR_1^{(\alpha, \beta)}(x) = \frac{\alpha + \beta + 2}{2} \neq 0,$$

which implies that $c_1(x) = c_1(\alpha, \beta, x) = 0$.

Now we consider the system of equations $S_8 = 0$. Since $S_1^{(\alpha, \beta)}(x) = 0$ the case $n = 1$ is trivial. Now we use (59) and (24) to find that

$$S_n^{(\alpha, \beta)}(x) = \frac{1}{(\alpha+1)(\beta+1)} \frac{(\alpha+\beta+2)_n(\alpha+\beta+2)_{n+1}}{4n!(n-1)!} (x^2-1)P_{n-2}^{(\alpha+2, \beta+2)}(x), \quad n = 2, 3, 4, \dots$$

By using the fact that

$$\frac{1}{(\alpha+1)(\beta+1)} \frac{(\alpha+\beta+2)_n(\alpha+\beta+2)_{n+1}}{4n!(n-1)!} \neq 0, \quad n = 2, 3, 4, \dots$$

we conclude that

$$\sum_{i=0}^\infty c_i(x)D^i[(x^2-1)P_{n-2}^{(\alpha+2, \beta+2)}(x)] = 0, \quad n = 2, 3, 4, \dots$$

Now we use the fact that $c_1(x) = 0$ to obtain by shifting n

$$\sum_{i=2}^{\infty} c_i(x) D^i [(x^2 - 1) P_n^{(\alpha+2, \beta+2)}(x)] = c_0(n + 2, \alpha, \beta) (1 - x^2) P_n^{(\alpha+2, \beta+2)}(x), \quad n = 0, 1, 2, \dots .$$

Note that for $n = 0, 1, 2, \dots$ we have

$$D^i [(x^2 - 1) P_n^{(\alpha, \beta)}(x)] = (x^2 - 1) D^i P_n^{(\alpha, \beta)}(x) + 2ix D^{i-1} P_n^{(\alpha, \beta)}(x) + i(i - 1) D^{i-2} P_n^{(\alpha, \beta)}(x), \quad i = 2, 3, 4, \dots .$$

Hence we have

$$\sum_{i=0}^{\infty} C_i(x) D^i P_n^{(\alpha+2, \beta+2)}(x) = c_0(n + 2, \alpha, \beta) (1 - x^2) P_n^{(\alpha+2, \beta+2)}(x), \quad n = 0, 1, 2, \dots, \tag{68}$$

where

$$C_i(x) = \begin{cases} 2c_2(x), & i = 0, \\ 4xc_2(x) + 6c_3(x), & i = 1, \\ (x^2 - 1)c_i(x) + 2(i + 1)xc_{i+1}(x) + (i + 1)(i + 2)c_{i+2}(x), & i = 2, 3, 4, \dots . \end{cases}$$

Note that the system of equations (68) has the form (35). So we may apply (36) and use (6) to conclude that for $i = 0, 1, 2, \dots$ we have

$$\begin{aligned} C_i(x) &= (1 - x^2) 2^i \sum_{j=0}^i \frac{\alpha + \beta + 2j + 5}{(\alpha + \beta + j + 5)_{i+1}} c_0(j + 2, \alpha, \beta) P_{i-j}^{(-\alpha-i-3, -\beta-i-3)}(x) P_j^{(\alpha+2, \beta+2)}(x) \\ &= \frac{(\alpha + \beta + 2)^2 (\alpha + \beta + 3)}{(\alpha + 1)(\beta + 1)} (1 - x^2) 2^i \\ &\quad \times \sum_{j=0}^i \frac{\alpha + \beta + 2j + 5}{(\alpha + \beta + j + 5)_{i+1}} \frac{(\alpha + \beta + 4)_{j+1} (\alpha + \beta + 4)_j}{(j + 1)! j!} \\ &\quad \times P_{i-j}^{(-\alpha-i-3, -\beta-i-3)}(x) P_j^{(\alpha+2, \beta+2)}(x). \end{aligned} \tag{69}$$

As before we apply the definition (21) to $P_{i-j}^{(-\alpha-i-3, -\beta-i-3)}(x)$ and the definition (20) to $P_j^{(\alpha+2, \beta+2)}(x)$ to find by changing the order of summations and by using the summation formula (44)

$$\begin{aligned} &\sum_{j=0}^i \frac{\alpha + \beta + 2j + 5}{(\alpha + \beta + j + 5)_{i+1}} \frac{(\alpha + \beta + 4)_{j+1} (\alpha + \beta + 4)_j}{(j + 1)! j!} P_{i-j}^{(-\alpha-i-3, -\beta-i-3)}(x) P_j^{(\alpha+2, \beta+2)}(x) \\ &= (\alpha + \beta + 4) (-1)^i \sum_{\ell=0}^i \frac{(\alpha + 3)_{i-\ell} (-\alpha - \beta - 3)_{i-\ell}}{(i - \ell + 1)! (i - \ell)! \ell! (i - \ell)!} \left(\frac{x - 1}{2}\right)^\ell \\ &\quad \times \sum_{m=0}^{\ell} \frac{(-\ell)_m (\alpha + \beta + 5)_m (\alpha + \beta + 4)_m (\alpha + i - \ell + 3)_m}{(\alpha + 3)_m (i - \ell + 2)_m (i - \ell + 1)_m m!}, \quad i = 0, 1, 2, \dots . \end{aligned}$$

Hence for $i = 0, 1, 2, \dots$ we have

$$\begin{aligned}
 C_i(x) &= \frac{(\alpha + \beta + 2)^2(\alpha + \beta + 3)(\alpha + \beta + 4)}{(\alpha + 1)(\beta + 1)}(1 - x^2)(-2)^i \\
 &\quad \times \sum_{\ell=0}^i \frac{(\alpha + 3)_{i-\ell}(-\alpha - \beta - 3)_{i-\ell}}{(i - \ell + 1)!(i - \ell)!\ell!(i - \ell)!} \\
 &\quad \times {}_4F_3 \left(\begin{matrix} -\ell, \alpha + \beta + 5, \alpha + \beta + 4, \alpha + i - \ell + 3 \\ \alpha + 3, i - \ell + 2, i - \ell + 1 \end{matrix} \middle| 1 \right) \left(\frac{x - 1}{2} \right)^\ell.
 \end{aligned} \tag{70}$$

Now we have by using (39) and (40)

$$\begin{aligned}
 c_i(x) &= \frac{1}{2i!} \sum_{j=0}^{i-2} (-1)^{i-j} j! [(x + 1)^{i-j-1} - (x - 1)^{i-j-1}] C_j(x) \\
 &= \frac{1}{2i!} \sum_{j=0}^{i-2} (-1)^j (i - j - 2)! [(x + 1)^{j+1} - (x - 1)^{j+1}] C_{i-j-2}(x) \\
 &= c_i^{(1)}(x) + c_i^{(2)}(x), \quad i = 2, 3, 4, \dots,
 \end{aligned}$$

where

$$c_i^{(1)}(x) = c_i^{(1)}(\alpha, \beta, x) = \frac{1}{2i!} \sum_{j=0}^{i-2} (-1)^j (i - j - 2)! (x + 1)^{j+1} C_{i-j-2}(x), \quad i = 2, 3, 4, \dots$$

and

$$c_i^{(2)}(x) = c_i^{(2)}(\alpha, \beta, x) = \frac{1}{2i!} \sum_{j=0}^{i-2} (-1)^{j+1} (i - j - 2)! (x - 1)^{j+1} C_{i-j-2}(x), \quad i = 2, 3, 4, \dots$$

Now we will prove (10) and (11). To do this we will first prove (13), which is an easy consequence of the symmetry formula (22). If we write $C_i(x) = C_i(\alpha, \beta, x)$ this symmetry formula gives us

$$C_i(\alpha, \beta, x) = (-1)^i C_i(\beta, \alpha, -x), \quad i = 0, 1, 2, \dots$$

in view of (69). Hence

$$\begin{aligned}
 c_i^{(1)}(\alpha, \beta, x) &= \frac{1}{2i!} \sum_{j=0}^{i-2} (-1)^j (i - j - 2)! (x + 1)^{j+1} C_{i-j-2}(\alpha, \beta, x) \\
 &= \frac{1}{2i!} \sum_{j=0}^{i-2} (-1)^{i+j+1} (i - j - 2)! (-x - 1)^{j+1} C_{i-j-2}(\beta, \alpha, -x) \\
 &= (-1)^i c_i^{(2)}(\beta, \alpha, -x), \quad i = 2, 3, 4, \dots,
 \end{aligned}$$

which proves (13). In order to prove (11) we use (70) and change the order of summations to find for $i = 2, 3, 4, \dots$

$$\begin{aligned}
c_i^{(2)}(x) &= \frac{1}{2i!} \sum_{j=0}^{i-2} (-1)^{j+1} (i-j-2)! (x-1)^{j+1} C_{i-j-2}(x) \\
&= \frac{(\alpha + \beta + 2)^2 (\alpha + \beta + 3) (\alpha + \beta + 4)}{(\alpha + 1)(\beta + 1)} (x^2 - 1) \frac{(-2)^{i-2}}{i!} \\
&\quad \times \sum_{j=0}^{i-2} \sum_{k=0}^{i-j-2} \frac{(\alpha + 3)_{i-j-k-2} (-\alpha - \beta - 3)_{i-j-k-2} (i-j-2)!}{(i-j-k-1)! (i-j-k-2)! k! (i-j-k-2)!} \\
&\quad \times {}_4F_3 \left(\begin{matrix} -k, \alpha + \beta + 5, \alpha + \beta + 4, \alpha + i - j - k + 1 \\ \alpha + 3, i - j - k, i - j - k - 1 \end{matrix} \middle| 1 \right) \left(\frac{x-1}{2} \right)^{j+k+1} \\
&= \frac{(\alpha + \beta + 2)^2 (\alpha + \beta + 3) (\alpha + \beta + 4)}{(\alpha + 1)(\beta + 1)} (x^2 - 1) \frac{(-2)^{i-2}}{i!} \\
&\quad \times \sum_{j=0}^{i-2} \sum_{\ell=j}^{i-2} \frac{(\alpha + 3)_{i-\ell-2} (-\alpha - \beta - 3)_{i-\ell-2} (i-j-2)!}{(i-\ell-1)! (i-\ell-2)! (\ell-j)! (i-\ell-2)!} \\
&\quad \times {}_4F_3 \left(\begin{matrix} -\ell + j, \alpha + \beta + 5, \alpha + \beta + 4, \alpha + i - \ell + 1 \\ \alpha + 3, i - \ell, i - \ell - 1 \end{matrix} \middle| 1 \right) \left(\frac{x-1}{2} \right)^{\ell+1} \\
&= \frac{(\alpha + \beta + 2)^2 (\alpha + \beta + 3) (\alpha + \beta + 4)}{(\alpha + 1)(\beta + 1)} (x^2 - 1) \frac{(-2)^{i-2}}{i!} \\
&\quad \times \sum_{\ell=0}^{i-2} \sum_{j=0}^{\ell} \frac{(\alpha + 3)_{i-\ell-2} (-\alpha - \beta - 3)_{i-\ell-2} (i-j-2)!}{(i-\ell-1)! (i-\ell-2)! (\ell-j)! (i-\ell-2)!} \\
&\quad \times \sum_{m=0}^{\ell-j} \frac{(-\ell + j)_m (\alpha + \beta + 5)_m (\alpha + \beta + 4)_m (\alpha + i - \ell + 1)_m}{(\alpha + 3)_m (i - \ell)_m (i - \ell - 1)_m m!} \left(\frac{x-1}{2} \right)^{\ell+1} \\
&= \frac{(\alpha + \beta + 2)^2 (\alpha + \beta + 3) (\alpha + \beta + 4)}{(\alpha + 1)(\beta + 1)} (x^2 - 1) \frac{(-2)^{i-2}}{i!} \\
&\quad \times \sum_{\ell=0}^{i-2} \sum_{m=0}^{\ell} \frac{(\alpha + 3)_{i-\ell-2} (-\alpha - \beta - 3)_{i-\ell-2}}{(i-\ell-1)! (i-\ell-2)! (i-\ell-2)!} \\
&\quad \times \frac{(\alpha + \beta + 5)_m (\alpha + \beta + 4)_m (\alpha + i - \ell + 1)_m}{(\alpha + 3)_m (i - \ell)_m (i - \ell - 1)_m m!} \left(\frac{x-1}{2} \right)^{\ell+1} \\
&\quad \times \sum_{j=0}^{\ell-m} \frac{(i-j-2)! (-\ell + j)_m}{(\ell-j)!}.
\end{aligned}$$

Now we use the Vandermonde summation formula (41) to obtain

$$\begin{aligned} \sum_{j=0}^{\ell-m} \frac{(i-j-2)!(-\ell+j)_m}{(\ell-j)!} &= \frac{(i-2)!}{\ell!} (-\ell)_m {}_2F_1 \left(\begin{matrix} -\ell+m, 1 \\ -i+2 \end{matrix} \middle| 1 \right) \\ &= \frac{(i-2)!}{\ell!} (-\ell)_m \frac{(-i+1)_{\ell-m}}{(-i+2)_{\ell-m}} = \frac{(i-1)!(i-\ell-2)!}{\ell!(i-\ell-1)!} \frac{(-\ell)_m(i-\ell-1)_m}{(i-\ell)_m}. \end{aligned}$$

Hence

$$\begin{aligned} c_i^{(2)}(x) &= \frac{(\alpha+\beta+2)^2(\alpha+\beta+3)(\alpha+\beta+4)}{(\alpha+1)(\beta+1)i} (x^2-1)(-2)^{i-2} \\ &\quad \times \sum_{\ell=0}^{i-2} \sum_{m=0}^{\ell} \frac{(\alpha+3)_{i-\ell-2}(-\alpha-\beta-3)_{i-\ell-2}}{(i-\ell-1)!(i-\ell-2)!\ell!(i-\ell-1)!} \\ &\quad \times \frac{(-\ell)_m(\alpha+\beta+5)_m(\alpha+\beta+4)_m(\alpha+i-\ell+1)_m}{(\alpha+3)_m(i-\ell)_m(i-\ell)_m m!} \left(\frac{x-1}{2}\right)^{\ell+1} \\ &= \frac{(\alpha+\beta+2)^2(\alpha+\beta+3)(\alpha+\beta+4)}{(\alpha+1)(\beta+1)i} (x^2-1)(-2)^{i-2} \\ &\quad \times \sum_{\ell=0}^{i-2} \frac{(\alpha+3)_{i-\ell-2}(-\alpha-\beta-3)_{i-\ell-2}}{(i-\ell-1)!(i-\ell-2)!\ell!(i-\ell-1)!} \\ &\quad \times {}_4F_3 \left(\begin{matrix} -\ell, \alpha+\beta+5, \alpha+\beta+4, \alpha+i-\ell+1 \\ \alpha+3, i-\ell, i-\ell \end{matrix} \middle| 1 \right) \left(\frac{x-1}{2}\right)^{\ell+1} \end{aligned}$$

for $i = 2, 3, 4, \dots$, which proves (11).

Hence we have proved (9)–(11).

8. The order of the differential equation

For $\alpha > -1$, $\beta > -1$, $M \geq 0$ and $N \geq 0$ the generalized Jacobi polynomials $\{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty}$ satisfy a unique differential equation of the form (2), where the coefficients are given by (4)–(11).

First of all we remark that

$$a_i(\alpha, \beta, x) = \sum_{j=1}^i k_{i,j}^{(a)}(\alpha, \beta)(x+1)^j, \quad i = 1, 2, 3, \dots,$$

where

$$k_{i,1}^{(a)}(\alpha, \beta) = -(\alpha+\beta+2)2^{i-1} \frac{(\beta+3)_{i-1}(-\beta-2)_{i-1}}{(\alpha+1)_{i-1} i! (i-1)!}, \quad i = 1, 2, 3, \dots$$

Since $\alpha > -1$ and $\beta > -1$ we conclude that $k_{i,1}^{(a)}(\alpha, \beta)$ only vanishes if $\beta \in \{0, 1, 2, \dots\}$ and $i \geq \beta + 4$. In the same way we have

$$b_i(\alpha, \beta, x) = \sum_{j=1}^i k_{i,j}^{(b)}(\alpha, \beta)(x - 1)^j, \quad i = 1, 2, 3, \dots,$$

where

$$k_{i,1}^{(b)}(\alpha, \beta) = -(\alpha + \beta + 2)(-2)^{i-1} \frac{(\alpha + 3)_{i-1}(-\alpha - 2)_{i-1}}{(\beta + 1)_{i-1} i! (i - 1)!}, \quad i = 1, 2, 3, \dots$$

Hence, $k_{i,1}^{(b)}(\alpha, \beta)$ only vanishes if $\alpha \in \{0, 1, 2, \dots\}$ and $i \geq \alpha + 4$.

Now we will prove (15). So let $\alpha \in \{0, 1, 2, \dots\}$ and consider $b_i(\alpha, \beta, x)$ given by (8). Suppose that $i \geq 2\alpha + 4$. Then we have

$$(-\alpha - 2)_{i-\ell-1} = 0 \quad \text{for } i - \ell \geq \alpha + 4.$$

Suppose that $i - \ell \leq \alpha + 3$, then we have $\ell \geq i - \alpha - 3 \geq \alpha + 1$. Hence by using (46) and the Vandermonde summation formula (41) we find that

$$\begin{aligned} & {}_3F_2 \left(\begin{matrix} -\ell, \alpha + \beta + 3, \alpha + i - \ell + 2 \\ \beta + i - \ell, i - \ell + 1 \end{matrix} \middle| 1 \right) \\ &= \sum_{n=0}^{\ell} (-1)^n \frac{(-\ell)_n (\alpha + \beta + 3)_n (-\alpha - 1)_n}{(\beta + i - \ell)_n (i - \ell + 1)_n n!} {}_2F_1 \left(\begin{matrix} n - \ell, n + \alpha + \beta + 3 \\ n + \beta + i - \ell \end{matrix} \middle| 1 \right) \\ &= \frac{1}{(\beta + i - \ell)_{\ell}} \sum_{n=0}^{\alpha+1} (-1)^n \frac{(-\ell)_n (\alpha + \beta + 3)_n (-\alpha - 1)_n (i - \ell - \alpha - 3)_{\ell-n}}{(i - \ell + 1)_n n!}. \end{aligned}$$

Since $i - \ell \leq \alpha + 3$ we have $i - \ell - \alpha - 3 \leq 0$. Hence $(i - \ell - \alpha - 3)_{\ell-n} = 0$ for $\ell - n \geq -i + \ell + \alpha + 4$ or $i \geq n + \alpha + 4$. This implies that $b_i(\alpha, \beta, x) = 0$ if $i > n + \alpha + 3$ for all $n \in \{0, 1, 2, \dots, \alpha + 1\}$, hence for $i > \alpha + 1 + \alpha + 3 = 2\alpha + 4$.

In the same way we obtain (14).

Now we will prove (17) and (18). Suppose that $\alpha \in \{0, 1, 2, \dots\}$ and $i = 2\alpha + 4$. Then we have

$$(i - \ell - \alpha - 3)_{\ell-n} = (\alpha + 1 - \ell)_{\ell-n} = 0 \quad \text{for } n \leq \alpha.$$

Hence

$$\begin{aligned} & b_{2\alpha+4}(\alpha, \beta, x) \\ &= (\alpha + \beta + 2)(-2)^{2\alpha+4} \sum_{\ell=\alpha+1}^{2\alpha+3} \frac{(\alpha + 3)_{2\alpha+3-\ell} (-\alpha - 2)_{2\alpha+3-\ell}}{(\beta + 1)_{2\alpha+3-\ell} (2\alpha + 4 - \ell)! (2\alpha + 3 - \ell)! \ell!} \left(\frac{x - 1}{2} \right)^{\ell+1} \\ & \quad \times \frac{(-1)^{\alpha+1}}{(2\alpha + \beta + 4 - \ell)_{\ell}} \frac{(-\ell)_{\alpha+1} (\alpha + \beta + 3)_{\alpha+1} (-\alpha - 1)_{\alpha+1} (\alpha + 1 - \ell)_{\ell-\alpha-1}}{(2\alpha + 5 - \ell)_{\alpha+1} (\alpha + 1)!} \\ &= (\alpha + \beta + 2)(-2)^{2\alpha+4} \frac{(\alpha + \beta + 3)_{\alpha+1}}{(\beta + 1)_{2\alpha+3} (\alpha + 2)!} \sum_{\ell=\alpha+1}^{2\alpha+3} (-1)^{\ell} \frac{(-\alpha - 2)_{2\alpha+3-\ell}}{(2\alpha + 3 - \ell)!} \left(\frac{x - 1}{2} \right)^{\ell+1} \end{aligned}$$

$$\begin{aligned}
 &= - \frac{2^{2\alpha+4}}{(\beta+1)_{\alpha+1}(\alpha+2)!} \sum_{\ell=0}^{\alpha+2} (-1)^\ell \frac{(-\alpha-2)_\ell}{\ell!} \left(\frac{x-1}{2}\right)^{2\alpha+4-\ell} \\
 &= - \frac{2^{2\alpha+4}}{(\beta+1)_{\alpha+1}(\alpha+2)!} \left(\frac{x-1}{2}\right)^{\alpha+2} \left(\frac{x+1}{2}\right)^{\alpha+2} = - \frac{(x^2-1)^{\alpha+2}}{(\beta+1)_{\alpha+1}(\alpha+2)!},
 \end{aligned}$$

which proves (18). The proof of (17) is similar.

In order to prove (16) we first consider $c_i^{(2)}(\alpha, \beta, x)$ given by (11). Let $\alpha, \beta \in \{0, 1, 2, \dots\}$ and suppose that $i \geq 2\alpha + 2\beta + 7$. Then we have

$$(-\alpha - \beta - 3)_{i-\ell-2} = 0 \quad \text{for } i - \ell \geq \alpha + \beta + 6.$$

Suppose that $i - \ell \leq \alpha + \beta + 5$, then we have $\ell \geq i - \alpha - \beta - 5 \geq \alpha + \beta + 2$. Hence by using (45), (46) and the Vandermonde summation formula (41) we find that

$$\begin{aligned}
 & {}_4F_3 \left(\begin{matrix} -\ell, \alpha + \beta + 5, \alpha + \beta + 4, \alpha + i - \ell + 1 \\ \alpha + 3, i - \ell, i - \ell \end{matrix} \middle| 1 \right) \\
 &= \sum_{n=0}^{\ell} (-1)^n \frac{(-\ell)_n (\alpha + \beta + 5)_n (\alpha + \beta + 4)_n (-\alpha - 1)_n}{(\alpha + 3)_n (i - \ell)_n (i - \ell)_n n!} \\
 &\quad \times {}_3F_2 \left(\begin{matrix} n - \ell, n + \alpha + \beta + 5, n + \alpha + \beta + 4 \\ n + i - \ell, n + \alpha + 3 \end{matrix} \middle| 1 \right) \\
 &= \sum_{n=0}^{\alpha+1} (-1)^n \frac{(-\ell)_n (\alpha + \beta + 5)_n (\alpha + \beta + 4)_n (-\alpha - 1)_n}{(\alpha + 3)_n (i - \ell)_n (i - \ell)_n n!} \\
 &\quad \times \sum_{k=0}^{\ell-n} (-1)^k \frac{(n - \ell)_k (n + \alpha + \beta + 5)_k (-\beta - 1)_k}{(n + i - \ell)_k (n + \alpha + 3)_k k!} {}_2F_1 \left(\begin{matrix} n + k - \ell, n + k + \alpha + \beta + 5 \\ n + k + i - \ell \end{matrix} \middle| 1 \right) \\
 &= \sum_{n=0}^{\alpha+1} (-1)^n \frac{(-\ell)_n (\alpha + \beta + 5)_n (\alpha + \beta + 4)_n (-\alpha - 1)_n}{(\alpha + 3)_n (i - \ell)_n (i - \ell)_n n!} \\
 &\quad \times \sum_{k=0}^{\beta+1} (-1)^k \frac{(n - \ell)_k (n + \alpha + \beta + 5)_k (-\beta - 1)_k}{(n + i - \ell)_k (n + \alpha + 3)_k k!} \frac{(i - \ell - \alpha - \beta - 5)_{\ell-n-k}}{(n + k + i - \ell)_{\ell-n-k}} \\
 &= \frac{1}{(i - \ell)_\ell} \sum_{n=0}^{\alpha+1} \sum_{k=0}^{\beta+1} (-1)^{n+k} \frac{(-\ell)_{n+k} (\alpha + \beta + 5)_{n+k} (\alpha + \beta + 4)_n}{(\alpha + 3)_{n+k} (i - \ell)_n n! k!} \\
 &\quad \times (-\alpha - 1)_n (-\beta - 1)_k (i - \ell - \alpha - \beta - 5)_{\ell-n-k}.
 \end{aligned}$$

Since $i - \ell \leq \alpha + \beta + 5$ we have $i - \ell - \alpha - \beta - 5 \leq 0$. Hence $(i - \ell - \alpha - \beta - 5)_{\ell-n-k} = 0$ for $\ell - n - k \geq -i + \ell + \alpha + \beta + 6$ or $i \geq n + k + \alpha + \beta + 6$. This implies that $c_i^{(2)}(\alpha, \beta, x) = 0$ if $i > n + k + \alpha + \beta + 5$ for all $n \in \{0, 1, 2, \dots, \alpha + 1\}$ and $k \in \{0, 1, 2, \dots, \beta + 1\}$, hence for $i > \alpha + 1 + \beta + 1 + \alpha + \beta + 5 = 2\alpha + 2\beta + 7$.

In the same way we find that $c_i^{(1)}(\alpha, \beta, x) = 0$ for $i > 2\alpha + 2\beta + 7$.

Suppose that $\alpha, \beta \in \{0, 1, 2, \dots\}$ and $i = 2\alpha + 2\beta + 7$. Then we have

$$(i - \ell - \alpha - \beta - 5)_{\ell-n-k} = (\alpha + \beta + 2 - \ell)_{\ell-n-k} = 0 \quad \text{for } n + k \leq \alpha + \beta + 1.$$

Hence

$$\begin{aligned} & c_{2\alpha+2\beta+7}^{(2)}(\alpha, \beta, x) \\ &= \frac{(\alpha + \beta + 2)^2(\alpha + \beta + 3)(\alpha + \beta + 4)}{(\alpha + 1)(\beta + 1)(2\alpha + 2\beta + 7)}(x^2 - 1)(-2)^{2\alpha+2\beta+5} \\ & \times \sum_{\ell=\alpha+\beta+2}^{2\alpha+2\beta+5} \frac{(\alpha + 3)_{2\alpha+2\beta+5-\ell}(-\alpha - \beta - 3)_{2\alpha+2\beta+5-\ell}}{(2\alpha + 2\beta + 6 - \ell)!(2\alpha + 2\beta + 5 - \ell)!(2\alpha + 2\beta + 6 - \ell)!\ell!} \left(\frac{x - 1}{2}\right)^{\ell+1} \\ & \times \frac{(-1)^{\alpha+\beta+2}}{(2\alpha + 2\beta + 7 - \ell)_{\ell}} \frac{(-\ell)_{\alpha+\beta+2}(\alpha + \beta + 5)_{\alpha+\beta+2}(\alpha + \beta + 4)_{\alpha+1}}{(\alpha + 3)_{\alpha+\beta+2}(2\alpha + 2\beta + 7 - \ell)_{\alpha+1}(\alpha + 1)!(\beta + 1)!} \\ & \times (-\alpha - 1)_{\alpha+1}(-\beta - 1)_{\beta+1}(\alpha + \beta + 2 - \ell)_{\ell-\alpha-\beta-2} \\ &= \frac{(\alpha + \beta + 2)^2(\alpha + \beta + 3)(\alpha + \beta + 4)}{(\alpha + 1)(\beta + 1)(2\alpha + 2\beta + 7)}(x^2 - 1)(-2)^{2\alpha+2\beta+5} \\ & \times \frac{(\alpha + \beta + 5)_{\alpha+\beta+2}(\alpha + \beta + 4)_{\alpha+1}}{(\alpha + 3)_{\alpha+\beta+2}(2\alpha + 2\beta + 6)!(\alpha + 2)!} \\ & \times \sum_{\ell=\alpha+\beta+2}^{2\alpha+2\beta+5} (-1)^{\ell} \frac{(-\alpha - \beta - 3)_{2\alpha+2\beta+5-\ell}}{(2\alpha + 2\beta + 5 - \ell)!} \left(\frac{x - 1}{2}\right)^{\ell+1} \\ &= \frac{\alpha + \beta + 2}{(\alpha + 1)(\beta + 1)(2\alpha + 2\beta + 7)} \frac{2^{2\alpha+2\beta+5}}{(\alpha + \beta + 1)!(\alpha + \beta + 3)!} (x^2 - 1) \\ & \times \sum_{\ell=0}^{\alpha+\beta+3} (-1)^{\ell} \frac{(-\alpha - \beta - 3)_{\ell}}{\ell!} \left(\frac{x - 1}{2}\right)^{2\alpha+2\beta+6-\ell} \\ &= \frac{\alpha + \beta + 2}{(\alpha + 1)(\beta + 1)(2\alpha + 2\beta + 7)} \frac{2^{2\alpha+2\beta+5}}{(\alpha + \beta + 1)!(\alpha + \beta + 3)!} (x^2 - 1) \\ & \times \left(\frac{x - 1}{2}\right)^{\alpha+\beta+3} \left(\frac{x + 1}{2}\right)^{\alpha+\beta+3} \\ &= \frac{\alpha + \beta + 2}{2(\alpha + 1)(\beta + 1)(2\alpha + 2\beta + 7)} \frac{(x^2 - 1)^{\alpha+\beta+4}}{(\alpha + \beta + 1)!(\alpha + \beta + 3)!}. \end{aligned}$$

Hence, because of the symmetry relation (13) we find that

$$c_{2\alpha+2\beta+7}^{(1)}(\alpha, \beta, x) = -c_{2\alpha+2\beta+7}^{(2)}(\beta, \alpha, -x) = -c_{2\alpha+2\beta+7}^{(2)}(\alpha, \beta, x),$$

which implies that

$$c_{2\alpha+2\beta+7}(\alpha, \beta, x) = c_{2\alpha+2\beta+7}^{(1)}(\alpha, \beta, x) + c_{2\alpha+2\beta+7}^{(2)}(\alpha, \beta, x) = 0.$$

Hence we have proved (16).

In order to prove (19) we first consider $C_i(\alpha, \beta, x) = C_i(x)$ given by (70). We assume that $\alpha, \beta \in \{0, 1, 2, \dots\}$ and $i \geq 2\alpha + 2\beta + 6$. Then we have as before

$$(-\alpha - \beta - 3)_{i-\ell} = 0 \quad \text{for } i - \ell \geq \alpha + \beta + 4.$$

Suppose that $i - \ell \leq \alpha + \beta + 3$, then we have $\ell \geq i - \alpha - \beta - 3 \geq \alpha + \beta + 3$. Hence by using (45), (46) and the Vandermonde summation formula (41) we find that

$$\begin{aligned} & {}_4F_3 \left(\begin{matrix} -\ell, \alpha + \beta + 5, \alpha + \beta + 4, \alpha + i - \ell + 3 \\ \alpha + 3, i - \ell + 1, i - \ell + 2 \end{matrix} \middle| 1 \right) \\ &= \sum_{n=0}^{\ell} (-1)^n \frac{(-\ell)_n (\alpha + \beta + 5)_n (\alpha + \beta + 4)_n (-\alpha - 1)_n}{(\alpha + 3)_n (i - \ell + 1)_n (i - \ell + 2)_n n!} \\ &\quad \times {}_3F_2 \left(\begin{matrix} n - \ell, n + \alpha + \beta + 5, n + \alpha + \beta + 4 \\ n + i - \ell + 1, n + \alpha + 3 \end{matrix} \middle| 1 \right) \\ &= \sum_{n=0}^{\alpha+1} (-1)^n \frac{(-\ell)_n (\alpha + \beta + 5)_n (\alpha + \beta + 4)_n (-\alpha - 1)_n}{(\alpha + 3)_n (i - \ell + 1)_n (i - \ell + 2)_n n!} \\ &\quad \times \sum_{k=0}^{\ell-n} (-1)^k \frac{(n - \ell)_k (n + \alpha + \beta + 5)_k (-\beta - 1)_k}{(n + i - \ell + 1)_k (n + \alpha + 3)_k k!} {}_2F_1 \left(\begin{matrix} n + k - \ell, n + k + \alpha + \beta + 5 \\ n + k + i - \ell + 1 \end{matrix} \middle| 1 \right) \\ &= \sum_{n=0}^{\alpha+1} (-1)^n \frac{(-\ell)_n (\alpha + \beta + 5)_n (\alpha + \beta + 4)_n (-\alpha - 1)_n}{(\alpha + 3)_n (i - \ell + 1)_n (i - \ell + 2)_n n!} \\ &\quad \times \sum_{k=0}^{\beta+1} (-1)^k \frac{(n - \ell)_k (n + \alpha + \beta + 5)_k (-\beta - 1)_k (i - \ell - \alpha - \beta - 4)_{\ell-n-k}}{(n + i - \ell + 1)_k (n + \alpha + 3)_k k! (n + k + i - \ell + 1)_{\ell-n-k}} \\ &= \frac{1}{(i - \ell + 1)_{\ell}} \sum_{n=0}^{\alpha+1} \sum_{k=0}^{\beta+1} (-1)^{n+k} \frac{(-\ell)_{n+k} (\alpha + \beta + 5)_{n+k} (\alpha + \beta + 4)_n}{(\alpha + 3)_{n+k} (i - \ell + 2)_n n! k!} \\ &\quad \times (-\alpha - 1)_n (-\beta - 1)_k (i - \ell - \alpha - \beta - 4)_{\ell-n-k}. \end{aligned}$$

Since $i - \ell \leq \alpha + \beta + 3$ we have $i - \ell - \alpha - \beta - 4 \leq -1$. Hence $(i - \ell - \alpha - \beta - 4)_{\ell-n-k} = 0$ for $\ell - n - k \geq -i + \ell + \alpha + \beta + 5$ or $i \geq n + k + \alpha + \beta + 5$. This implies that $C_i(x) = 0$ if $i > n + k + \alpha + \beta + 4$ for all $n \in \{0, 1, 2, \dots, \alpha + 1\}$ and $k \in \{0, 1, 2, \dots, \beta + 1\}$, hence for $i > \alpha + 1 + \beta + 1 + \alpha + \beta + 4 = 2\alpha + 2\beta + 6$.

Suppose that $\alpha, \beta \in \{0, 1, 2, \dots\}$ and $i = 2\alpha + 2\beta + 6$. Then we have

$$(i - \ell - \alpha - \beta - 4)_{\ell-n-k} = (\alpha + \beta + 2 - \ell)_{\ell-n-k} = 0 \quad \text{for } n + k \leq \alpha + \beta + 1.$$

Hence

$$\begin{aligned}
& C_{2\alpha+2\beta+6}(x) \\
&= \frac{(\alpha + \beta + 2)^2(\alpha + \beta + 3)(\alpha + \beta + 4)}{(\alpha + 1)(\beta + 1)}(1 - x^2)(-2)^{2\alpha+2\beta+6} \\
&\quad \times \sum_{\ell=\alpha+\beta+3}^{2\alpha+2\beta+6} \frac{(\alpha + 3)_{2\alpha+2\beta+6-\ell}(-\alpha - \beta - 3)_{2\alpha+2\beta+6-\ell}}{(2\alpha + 2\beta + 7 - \ell)!(2\alpha + 2\beta + 6 - \ell)!(2\alpha + 2\beta + 6 - \ell)! \ell!} \left(\frac{x - 1}{2}\right)^\ell \\
&\quad \times \frac{(-1)^{\alpha+\beta+2}}{(2\alpha + 2\beta + 7 - \ell)_\ell} \frac{(-\ell)_{\alpha+\beta+2}(\alpha + \beta + 5)_{\alpha+\beta+2}(\alpha + \beta + 4)_{\alpha+1}}{(\alpha + 3)_{\alpha+\beta+2}(2\alpha + 2\beta + 8 - \ell)_{\alpha+1}(\alpha + 1)!(\beta + 1)!} \\
&\quad \times (-\alpha - 1)_{\alpha+1}(-\beta - 1)_{\beta+1}(\alpha + \beta + 2 - \ell)_{\ell-\alpha-\beta-2} \\
&= \frac{(\alpha + \beta + 2)^2(\alpha + \beta + 3)(\alpha + \beta + 4)}{(\alpha + 1)(\beta + 1)}(1 - x^2)(-2)^{2\alpha+2\beta+6} \\
&\quad \times \frac{(\alpha + \beta + 5)_{\alpha+\beta+2}(\alpha + \beta + 4)_{\alpha+1}}{(\alpha + 3)_{\alpha+\beta+2}(2\alpha + 2\beta + 6)!(\alpha + 2)!} \\
&\quad \times \sum_{\ell=\alpha+\beta+3}^{2\alpha+2\beta+6} (-1)^\ell \frac{(-\alpha - \beta - 3)_{2\alpha+2\beta+6-\ell}}{(2\alpha + 2\beta + 6 - \ell)!} \left(\frac{x - 1}{2}\right)^\ell \\
&= \frac{\alpha + \beta + 2}{(\alpha + 1)(\beta + 1)} \frac{2^{2\alpha+2\beta+6}}{(\alpha + \beta + 1)!(\alpha + \beta + 3)!} (1 - x^2) \\
&\quad \times \sum_{\ell=0}^{\alpha+\beta+3} (-1)^\ell \frac{(-\alpha - \beta - 3)_\ell}{\ell!} \left(\frac{x - 1}{2}\right)^{2\alpha+2\beta+6-\ell} \\
&= \frac{\alpha + \beta + 2}{(\alpha + 1)(\beta + 1)} \frac{2^{2\alpha+2\beta+6}}{(\alpha + \beta + 1)!(\alpha + \beta + 3)!} (1 - x^2) \left(\frac{x - 1}{2}\right)^{\alpha+\beta+3} \left(\frac{x + 1}{2}\right)^{\alpha+\beta+3} \\
&= - \frac{\alpha + \beta + 2}{(\alpha + 1)(\beta + 1)} \frac{(x^2 - 1)^{\alpha+\beta+4}}{(\alpha + \beta + 1)!(\alpha + \beta + 3)!}.
\end{aligned}$$

Now we use the fact that

$$C_i(x) = (x^2 - 1)c_i(x) + 2(i + 1)xc_{i+1}(x) + (i + 1)(i + 2)c_{i+2}(x), \quad i = 2, 3, 4, \dots$$

to conclude that

$$C_{2\alpha+2\beta+6}(x) = (x^2 - 1)c_{2\alpha+2\beta+6}(x),$$

which leads to (19).

9. Some remarks

Let $\alpha > -1$ and $\beta > -1$.

The coefficients $\{a_i(x)\}_{i=1}^\infty$ and $\{b_i(x)\}_{i=1}^\infty$ can also be computed in the same way as we computed the coefficients $\{c_i(x)\}_{i=1}^\infty$. Consider the system of equations $S_4 = 0$. First we use (24) to find from (58) that

$$R_n^{(\alpha,\beta)}(x) = \frac{(\alpha + 2)_{n-1}(\alpha + \beta + 2)_n}{2(\beta + 1)_{n-1} n!} (x - 1)P_{n-1}^{(\alpha+2,\beta)}(x), \quad n = 1, 2, 3, \dots$$

Now we use the fact that

$$\frac{(\alpha + 2)_{n-1}(\alpha + \beta + 2)_n}{2(\beta + 1)_{n-1} n!} \neq 0, \quad n = 1, 2, 3, \dots$$

to conclude that

$$\sum_{i=0}^\infty b_i(x)D^i[(x - 1)P_{n-1}^{(\alpha+2,\beta)}(x)] = 0, \quad n = 1, 2, 3, \dots$$

Hence, by shifting n we obtain

$$\sum_{i=1}^\infty b_i(x)D^i[(x - 1)P_n^{(\alpha+2,\beta)}(x)] = -b_0(n + 1, \alpha, \beta)(x - 1)P_n^{(\alpha+2,\beta)}(x), \quad n = 0, 1, 2, \dots$$

Note that for $n = 0, 1, 2, \dots$ we have

$$D^i[(x - 1)P_n^{(\alpha+2,\beta)}(x)] = (x - 1)D^iP_n^{(\alpha+2,\beta)}(x) + iP_{n-1}^{(\alpha+2,\beta)}(x), \quad i = 1, 2, 3, \dots$$

Hence we obtain

$$\sum_{i=0}^\infty B_i(x)D^iP_n^{(\alpha+2,\beta)}(x) = -b_0(n + 1, \alpha, \beta)(x - 1)P_n^{(\alpha+2,\beta)}(x), \quad n = 0, 1, 2, \dots, \tag{71}$$

where

$$B_i(x) = \begin{cases} b_1(x), & i = 0, \\ (x - 1)b_i(x) + (i + 1)b_{i+1}(x), & i = 1, 2, 3, \dots \end{cases}$$

Note that the system of equations (71) has the form (35). So we may apply (36) and use (5) to conclude that for $i = 0, 1, 2, \dots$ we have

$$\begin{aligned} B_i(x) &= -(x - 1)2^i \sum_{j=0}^i \frac{\alpha + \beta + 2j + 3}{(\alpha + \beta + j + 3)_{i+1}} b_0(j + 1, \alpha, \beta)P_{i-j}^{(-\alpha-i-3, -\beta-i-1)}(x)P_j^{(\alpha+2,\beta)}(x) \\ &= -(\alpha + \beta + 2)(x - 1)2^i \\ &\quad \times \sum_{j=0}^i \frac{\alpha + \beta + 2j + 3}{(\alpha + \beta + j + 3)_{i+1}} \frac{(\alpha + 3)_j(\alpha + \beta + 3)_j}{(\beta + 1)_j j!} P_{i-j}^{(-\alpha-i-3, -\beta-i-1)}(x)P_j^{(\alpha+2,\beta)}(x). \end{aligned} \tag{72}$$

As before we can deduce that for $i = 0, 1, 2, \dots$

$$\begin{aligned}
 B_i(x) &= -(\alpha + \beta + 2)(x - 1)(-2)^i \\
 &\times \sum_{\ell=0}^i \frac{(\alpha + 3)_{i-\ell}(-\alpha - 2)_{i-\ell}}{(\beta + 1)_{i-\ell}(i - \ell)!\ell!(i - \ell)!} \\
 &\times {}_3F_2 \left(\begin{matrix} -\ell, \alpha + i - \ell + 3, \alpha + \beta + 3 \\ \beta + i - \ell + 1, i - \ell + 1 \end{matrix} \middle| 1 \right) \left(\frac{x - 1}{2} \right)^\ell.
 \end{aligned}$$

Now we use (37) and (38) with $z = x - 1$ to find that

$$\begin{aligned}
 b_i(x) &= \frac{1}{i!} \sum_{j=0}^{i-1} (-1)^{i-j-1} j! (x - 1)^{i-j-1} B_j(x) \\
 &= \frac{1}{i!} \sum_{j=0}^{i-1} (-1)^j (i - j - 1)! (x - 1)^j B_{i-j-1}(x), \quad i = 1, 2, 3, \dots
 \end{aligned}$$

which leads to (8) after changing the order of summations and using the Vandermonde summation formula (41) as before.

In a similar way the coefficients $\{a_i(x)\}_{i=1}^\infty$ can be computed from the system of equations $S_2 = 0$. In that case we would need (37) and (38) with $z = x + 1$, but it is easier to use the symmetry relation (12) of course.

In [3] Bavinck found the following interesting formula involving Laguerre polynomials:

$$\sum_{k=j}^i k^s L_{i-k}^{(-\alpha-i-1)}(-x) L_{k-j}^{(\alpha+j)}(x) = (-x)^s \delta_{i,j+2s}, \quad i \geq j + 2s, \quad i, j, s \in \{0, 1, 2, \dots\},$$

which holds for all α . In [5] he found an analogue of this formula involving Jacobi polynomials:

$$\begin{aligned}
 2^{i-j} \sum_{k=j}^i \frac{\alpha + \beta + 2k + 1}{(\alpha + \beta + j + k + 1)_{i-j+1}} [k(k + \alpha + \beta + 1)]^s P_{i-k}^{(-\alpha-i-1, -\beta-i-1)}(x) P_{k-j}^{(\alpha+j, \beta+j)}(x) \\
 = (x^2 - 1)^s \delta_{i,j+2s}, \quad i \geq j + 2s, \quad i, j, s \in \{0, 1, 2, \dots\},
 \end{aligned}$$

which holds for $-(\alpha + \beta + 2) \notin \{0, 1, 2, \dots\}$. The case $\alpha + \beta + 1 = 0$ must be understood by continuity. This formula can be applied to (69) and (72). Since we have

$$\frac{(\alpha + 3)_j(\alpha + \beta + 3)_j}{(\beta + 1)_j j!} = \frac{(j + 1)_{\alpha+2}(\beta + j + 1)_{\alpha+2}}{(\beta + 1)_{\alpha+2}(\alpha + 2)!}, \quad \alpha \in \{0, 1, 2, \dots\}$$

and

$$(j + 1)_{\alpha+2}(\beta + j + 1)_{\alpha+2} = \prod_{k=1}^{\alpha+2} [j(j + \alpha + \beta + 3) + k(\alpha + \beta + 3 - k)], \quad \alpha \in \{0, 1, 2, \dots\}$$

this implies that for $\alpha \in \{0, 1, 2, \dots\}$

$$B_i(x) = -(\alpha + \beta + 2)(x - 1) \frac{(x^2 - 1)^{\alpha+2}}{(\beta + 1)_{\alpha+2}(\alpha + 2)!} \delta_{i, 2\alpha+4}, \quad i \geq 2\alpha + 4.$$

Hence, for $\alpha \in \{0, 1, 2, \dots\}$ we have

$$B_i(x) = 0, \quad i > 2\alpha + 4 \quad \text{and} \quad B_{2\alpha+4}(x) = -(x - 1) \frac{(x^2 - 1)^{\alpha+2}}{(\beta + 1)_{\alpha+1}(\alpha + 2)!},$$

which leads to (18) eventually. In a similar way we find for $\alpha, \beta \in \{0, 1, 2, \dots\}$

$$\frac{(\alpha + \beta + 4)_{j+1}(\alpha + \beta + 4)_j}{(j + 1)! j!} = \frac{(j + 2)_{\alpha+\beta+3}(j + 1)_{\alpha+\beta+3}}{(\alpha + \beta + 3)! (\alpha + \beta + 3)!}$$

and

$$(j + 2)_{\alpha+\beta+3}(j + 1)_{\alpha+\beta+3} = \prod_{k=1}^{\alpha+\beta+3} [j(j + \alpha + \beta + 5) + k(\alpha + \beta + 5 - k)],$$

which implies that in view of (69) we have

$$C_i(x) = -\frac{(\alpha + \beta + 2)^2(\alpha + \beta + 3)}{(\alpha + 1)(\beta + 1)} \frac{(x^2 - 1)^{\alpha+\beta+4}}{(\alpha + \beta + 3)! (\alpha + \beta + 3)!} \delta_{i, 2\alpha+2\beta+6}, \quad i \geq 2\alpha + 2\beta + 6.$$

Hence, for $\alpha, \beta \in \{0, 1, 2, \dots\}$ we have

$$C_i(x) = 0, \quad i > 2\alpha + 2\beta + 6 \quad \text{and} \quad C_{2\alpha+2\beta+6}(x) = -\frac{\alpha + \beta + 2}{(\alpha + 1)(\beta + 1)} \frac{(x^2 - 1)^{\alpha+\beta+4}}{(\alpha + \beta + 1)! (\alpha + \beta + 3)!}$$

as before.

If we set $M = 0$ into (2) we get the differential equation

$$N \sum_{i=0}^{\infty} b_i(x) y^{(i)}(x) + (1 - x^2) y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x] y'(x) + n(n + \alpha + \beta + 1) y(x) = 0, \tag{73}$$

satisfied by the polynomials $\{P_n^{\alpha, \beta, 0, N}(x)\}_{n=0}^{\infty}$. From the limit relation (1) it follows that

$$\lim_{\beta \rightarrow \infty} \frac{(-2)^i}{\beta^i} (D^i P_n^{\alpha, \beta, 0, N}) \left(1 - \frac{2x}{\beta}\right) = D^i L_n^{\alpha, N}(x), \quad n = 0, 1, 2, \dots, \quad i = 0, 1, 2, \dots,$$

where $L_n^{\alpha, N}(x)$ denotes the generalized Laguerre polynomial considered in [9]. Note that from (5) we easily find that

$$\lim_{\beta \rightarrow \infty} \frac{b_0(n, \alpha, \beta)}{\beta} = \binom{n + \alpha + 1}{n - 1}, \quad n = 0, 1, 2, \dots$$

Now we use the Vandermonde summation formula (41) to find from (8)

$$\lim_{\beta \rightarrow \infty} \frac{\beta^{i-1}}{(-2)^i} b_i(\alpha, \beta, 1 - 2x/\beta)$$

$$\begin{aligned}
 &= \sum_{\ell=0}^{i-1} \frac{(\alpha+3)_{i-\ell-1}(-\alpha-2)_{i-\ell-1}}{(i-\ell)!(i-\ell-1)!\ell!} {}_2F_1 \left(\begin{matrix} -\ell, \alpha+i-\ell+2 \\ i-\ell+1 \end{matrix} \middle| 1 \right) (-x)^{\ell+1} \\
 &= \sum_{\ell=0}^{i-1} \frac{(\alpha+3)_{i-\ell-1}(-\alpha-2)_{i-\ell-1}(-\alpha-1)_{\ell}}{i!(i-\ell-1)!\ell!} (-x)^{\ell+1} \\
 &= \frac{1}{i!} \sum_{j=1}^i (-1)^{i+j+1} \binom{\alpha+1}{j-1} \binom{\alpha+2}{i-j} (\alpha+3)_{i-j} x^j, \quad i = 1, 2, 3, \dots
 \end{aligned}$$

Hence, if we set $y(x) = P_n^{\alpha, \beta, 0, N}(x)$ into the differential equation (73), change x by $1 - 2x/\beta$, divide by β and take the limit $\beta \rightarrow \infty$ we obtain the differential equation for the polynomials $\{L_n^{\alpha, N}(x)\}_{n=0}^{\infty}$ which was found in [9].

In [10,13] we found all differential equations of the form (3) satisfied by the polynomials $\{P_n^{\alpha, \alpha, M, M}(x)\}_{n=0}^{\infty}$, where $\alpha > -1$ and $M \geq 0$. We emphasize that these differential equations are not of the form (2). The differential equation (2) leads to another one after setting $\beta = \alpha$ and $N = M$.

In [13] we also found differential equations for the polynomials $\{P_n^{\alpha, \pm 1/2, 0, N}\}_{n=0}^{\infty}$, where $\alpha > -1$ and $N \geq 0$. It can be shown that these do coincide with (2) after setting $M = 0$ and $\beta = \pm \frac{1}{2}$. For $\alpha \in \{0, 1, 2, \dots\}$ and $N > 0$ these differential equations have finite order $2\alpha + 4$.

Now we can correct a table conjectured in [8] listing the cases for which the polynomials $\{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty}$ satisfy a finite order differential equation of the form (2) with minimal order:

M, N	α, β	Order
$M = 0, N = 0$	$\alpha > -1, \beta > -1$	2
$N = M > 0$	$\beta = \alpha \in \{0, 1, 2, \dots\}$	$2\alpha + 4$
$M = 0, N > 0$	$\alpha \in \{0, 1, 2, \dots\}, \beta > -1$	$2\alpha + 4$
$M > 0, N = 0$	$\alpha > -1, \beta \in \{0, 1, 2, \dots\}$	$2\beta + 4$
$M > 0, N > 0$	$\alpha, \beta \in \{0, 1, 2, \dots\}$	$2\alpha + 2\beta + 6$

Finally, we thank one of the referees for pointing out the reference to the preprint [19] in which it is explained why the order of the differential equation for these generalized Jacobi polynomials must be of infinite order for other values of the parameters than those mentioned above.

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Further reading

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