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COMPACTNESS FOR OMITTING OF TYPES*

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0. Introduction

In 1970, the most discussed questions in the theory of large cardinals concerned the relationship of measurable, strong compact, and supercompact cardinals. By now the major questions have been solved, the outcome being that strong compactness is not a natural notion for set-theorists while for model-theorists it is measurability and supercompactness which waver, since for them, I suppose, the notion of strong compactness is constant. In the above I am, of course, loosely paraphrasing and refering to the results of Magidor that it is consistent to have the first measurable strongly compact and that it is also consistent to have the first strongly compact cardinal supercompact (and thus larger than the first measurable). In 1970 we did not know this but we could ask what was the significance of these questions for model theory. Strong compactness is defined model-theoretically and measurability has some fairly natural model-theoretic characterizations as well (see e.g. [1, Exercise 4.2.6]). The question, therefore, was to characterize supercompactness which I did (see Theorem 1.1 below), but I did not like the solution at the time. But since the motive was recently repeated in some arguments of Silver (see [5]) and Magidor (see [4]) I looked in this direction once more knowing how it helped them to have things countable and found more interest in it. The result is a kind of compactness for omitting of types.

Section 1 contains a characterization of supercompactness in terms of omitting a type in an infinitary language.

Section 2 introduces the notion of partial algebras and algebraic sets. These turn out to have a close connection with normality of filters but are extremely interesting by themselves. The definition of the integral complements the definition of derivative in our work on modeloids and may eventually be combined with measures or ultrafilters at least. We give two definitions of these notions which show that they are Δ (in ZF).

Section 3 uses the preceding section to define samplings of sets. Samplings provide a unification to different notions "for almost all" explored and effectively used by Barwise, Kueker, Shelah and others (see [6] for references).

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Section 4 relativizes the notions introduced to admissible sets. The relativization is very natural, most notions turn out Σ or better.

Section 5 discusses the paradigm which emerged in Section 1 and uses the preceeding sections to prove an omitting of types result discussed above.

Section 6 uses the result to provide the set-up needed for getting bounds for powers of singular cardinals. The section also contains discussion of the notions with respect to models of set theory.

1. Characterization of supercompactness

Let us recall what a supercompact cardinal is. It is a cardinal κ such that for any $\lambda \geq \kappa$, there is a κ-complete ultrafilter on $P_\kappa(\lambda)(=\{s \subseteq \lambda : |s| < \kappa\})$ which contains $\{s \in P_\kappa(\lambda) : \alpha \in s\}$ for each $\alpha < \lambda$ and is normal, that is if $f : P_\kappa(\lambda) \to \lambda$ is such that $f(s) \in s$ for almost all s (with respect to the ultrafilter), then for some $\alpha < \lambda$ $f(s) = \alpha$ for almost all s.

The characterization turns out to be in terms of realizing a type of a theory and simultaneously omitting another type. Precisely, let T be a theory in the language $L_{\kappa,\kappa}$ ($\kappa \geq \omega$), and let $\Sigma(x, y)$ be its type. We shall deal with types in two variables but all the results generalize to types with x and y standing for less than k variables. We shall assume throughout that the type is closed under conjunctions of less than κ formulas but we do not assume that it is complete. Saying we want realize x and omit y means that we seek a model of

$$(\exists x)[\bigwedge(\exists y)\Sigma \wedge \neg(\exists y)\bigwedge\Sigma] \qquad (*)$$

where $(\exists y)\Sigma = \{(\exists y)\sigma : \sigma \in \Sigma\}$ and $\bigwedge\phi$ is a conjunction of all formulas of ϕ. A model for $(*)$ will contain a c such that $(\exists y)\Sigma(c, y)$, that is c realizes the projection of Σ onto x and the second conjunct in $(*)$ says that the type $\Sigma(c, y)$ (a type in y) is omitted. As we shall see below, $(*)$ is a paradigm for many problems in model theory and we shall refer to it sometimes as the paradigm.

Now given $\Delta \subseteq \Sigma$ we denote by $\alpha_\Delta(x)$ the formula

$$\bigwedge(\exists y)\Delta \wedge \neg(\exists y)\bigwedge\Delta.$$

The paradigm is thus equivalent to $(\exists x)\alpha_\Sigma(x)$.

Before going to the characterization let us recall the notion of closed unbounded subsets of $P_\kappa(X) = \{s \subseteq X : |s| < \kappa\}$ (see [2]). A set $C \subseteq P_\lambda(X)$ is called closed if for any $\subseteq$-chain $\{s_\alpha : \alpha < \mu\} \subseteq C$, $\bigcup\{s_\alpha : \alpha < \mu\} \in C (\mu < \kappa)$. It is unbounded if for any $s \in P_k(X)$, there is $r \in C$ such that $s \subseteq r$. It is known that the closed unbounded subsets on X generate a κ-complete filter which is normal, i.e. if $Y \subseteq X$ and $\{C_y : y \in Y\}$ are members of the filter then

$$\{s : y \in s \to s \in C_y\}$$

is also a member of the filter (see [2] for details; it is assumed that $\kappa > \omega$ is regular and $\kappa < |X|$). We denote this filter by $F_\kappa(X)$.

Theorem 1.1. *The following are equivalent.*

(i) *κ is supercompact;*

(ii) *if T is a theory in $L_{\kappa,\kappa}$ and $\Sigma(x, y)$ is a type such that $\{\Delta \in P_k(\Sigma): T + (\exists x)\alpha_\Delta(x)$ has a model$\}$ is in $P_\kappa(\Sigma)$, then $T + (\exists x)\alpha_\Sigma(x)$ has a model.*

Remark 1.2. (ii), loosely speaking, says that if almost all small paradigms have a solution then the whole paradigm has a solution. This is why we called it "a kind of compactness". If (ii) were phrased "if for all large enough ($|\Delta| > 1$) small ($|\Delta| < \kappa$)Δ's $\subseteq \Sigma$ we have solutions for $(\exists x)(\alpha_\Delta(x)$, then $\cdots$" the equivalence would be also true but it is not the nature of the problem to put it so (see the discussion below).

Proof. In order to prove (i)$\rightarrow$(ii) we assume that κ is supercompact, that we have a theory T in $L_{\kappa,\kappa}$ and a type $\Sigma(x, y)$ of it. Assume that $|\Sigma| = \lambda \geq \kappa$ and let M_Δ be a model of $(\exists x)\alpha_\Delta(x)$ for Δ from

$$A = \{\Delta \in P_k(\Sigma): T + (\exists x)\alpha_\Delta(x) \text{ has a model}\}.$$

Let D be a normal ultrafilter on $P_\kappa(\Sigma)$ as guaranteed by the supercompactness of κ. By [2, Theorem] $A \in D$. It thus makes sense to take the ultraproduct

$$M = \prod_{\Delta \in A} M_\Delta / D.$$

Since D is κ-complete $M \models T$. Let $f(\Delta) \in M_\Delta$ be such that $M_\Delta = \alpha_\Delta(f(\Delta))$ ($\Delta \in A$). Since $\{\Delta \in P_k(\Sigma): \sigma \in \Delta\} \in D$ for every $\sigma \in \Sigma$ we have (again using κ-completeness of D)

$$M \models \bigwedge (\exists y)\Sigma(f/_D, y).$$

Let us assume that we in fact have

$$M \models (\exists y) \bigwedge \Sigma(f/_D, y)$$

and let $g/_D$ be such element. Then for every $\Delta \in A$ there is $\sigma \in \Delta$ such that $M_\Delta \models \neg\sigma(f(\Delta), g(\Delta))$ because $M_\Delta \models \alpha_\Delta(f(\Delta))$; call such σ by $h(\Delta)$. Then $h(\Delta) \in \Delta$ on $A \in D$, so for some $\sigma_0 \in \Sigma$, $h(\Delta) = \sigma_0$ for almost all Δ (mod D). But that means $M \models \neg\sigma_0(f/_D, g/_D)$ thus showing $M \models (\exists x)\alpha_\Delta(x)$.

To prove (ii)$\rightarrow$(i) we let $\lambda \geq \kappa$ and let T be the theory of

$$M = (P_k(\lambda) \cup \lambda, \cdots R \cdots)$$

where $\cdots R \cdots$ is a listing of all elements, all subsets, and all binary relations on the universe $(P_k(\lambda) \cup \lambda)$. Let $\Sigma(x, y)$ be the type

$$\{U(x) \wedge \alpha \in x \wedge y \in x \wedge y \neq \alpha : \alpha < \lambda\}$$

where U is the name of $P_k(\lambda)$ and $\alpha < \lambda$ names itself. Note that the type consists of finite formulas only. Now given $\Delta \subseteq \Sigma$ with $1 < |\Delta| < \kappa$, we see that $T + (\exists x)\alpha_\Delta(x)$ has M as its model by taking for x the set of α's $< \lambda$ which are mentioned in Δ. As the set of these Δ's is closed and unbounded in $P_\kappa(\Sigma)$ we get, by (ii), a model N of $T+(\exists x)\alpha_\Sigma(x)$ which can be considered an $L_{\kappa,\kappa}$ elementary extension of M. Let $a \in N$ be such that $N \models \alpha_\Sigma(a)$ and define D on $P_\kappa(\lambda)$ by

$$D=\{X \subseteq P_\kappa(\lambda) : N \models \mathbf{X}(a)\};$$

here $\mathbf{X}$ is the unary predicate naming X. As is well-known D is a κ complete ultrafilter (since $M \prec N$ in $I_{\kappa,\kappa}$), it contains $\{s \in P_\kappa(\lambda) : \alpha \in s\}$ because

$$a \in \{x : N \models \alpha \in x\}$$

and it is normal: let $f : P_\kappa(\lambda) \to \lambda$ such that $f(s) \in s$ for almost all s. By the definition of D, $N \models \mathbf{f}(a) \in a$ and because the type $\{y \in a \wedge y \neq \alpha : \alpha < \lambda\}$ is omitted in N this means that for some $\alpha < \lambda$, $N \models \mathbf{f}(a) = \alpha$, consequently $\{s : f(s) = \alpha\} \in D$.

Let us now discuss the chances of proving a theorem in $L_{\omega\omega}$ suggested by the characterization. The obvious formulation of such a statement is false as ω is not supercompact. We can see this on a specific example by considering $\Sigma(x, y)$ defined by

$$\Sigma(x, y) = \{m < x \wedge y \neq m \wedge y < x : m < \omega\}$$

Σ is a type of $\mathrm{Th}(\omega, <)$ (m is the mth element above the least element). For every finite $\Delta \subseteq \Sigma$ with $|\Delta| > 1$, $(\exists x)\alpha_\Delta(x)$ has a model, but $(\exists x)\alpha_\Sigma(x)$ does not have one because the linear order would be elementarily equivalent to $(\omega, <)$.

2. Partial X-algebras

In this section, we define certain subsets of X^* which we call X-algebraic and which will turn out to have a close connection with normality of filters. In fact, analyzing normality of certain filters led us naturally to these sets and we found that they were interesting, in their own right, subjects for investigation. Later we learned, thanks to the encyclopedic knowledge of R. Solovay, that we were not the first to use these sets and the functions defined on them. For example, the book of Dubins and Savage, How to Gamble if You Must (McGraw-Hill, 1965, pp. 14–17), is based on this concept (called finitary mappings there). Actually, the concept goes back to Kalmar who investigated these mappings in his paper in Colloquium Mathematicum 5 (1975) 1–5 (with a three lines long title). There are other connections. In coding theory, these sets are called instantaneous codes and in linguistics, prefix-free languages. Hopefully, all these loose ends will be one day cemented together.

The set of all finite sequences of elements of X is denoted by X^*, if $u, v \in X^*$ then uv denotes the concatenation of u and v; 0 denotes the empty sequence.

Definition 2.1. If $\{D_x : x \in X\} \subseteq X^*$ let $D = \int D_x : x \in X = \{xw : w \in D_x\}$. We call D the integration of the sets D_x. The X-algebraic sets are those subsets of X^* which belong to the least collection of subsets of X^* containing $\{0\}$ and closed under integration.

We shall now show that the X-algebraic sets D may be characterized by the following condition (suggested by the referee; our original was simpler but confusing).

$$\text{If } s:\omega\to X \text{ there is a unique finite initial segment of } s \text{ which belongs to } D. \qquad (*)$$

Note that if D satisfies $(*)$ and $u\in D$ then no proper extension of u is in D.

In order to facilitate the proof, we need some definitions.

Definition 2.2. (i) If $D\subseteq X^*$ and $v\in X^*$ let

$$D^v=\{w\in X^*-\{0\}: vw\in D\}$$

and

$$vD=\{vw: w\in D\}.$$

(ii) For $D, E\subseteq X^*$ define $D\leqslant E$ iff for some $v\in X^*$ $vD\subseteq E$.

(iii) Let Al_X denote the set of $D\subseteq X^*$ satisfying $(*)$.

Proposition 2.3. *The relation $\leqslant$ defined above is a well-founded partial order on Al_X.*

Proof. If $D\leqslant E$ because $vD\subseteq E$, and $E\leqslant F$ because $wE\subseteq F$, then $(wv)D\subseteq F$ thus $\leqslant$ is transitive.

If $D\leqslant E$, $E\leqslant D$, and $D\neq E$ we have v, w with $vD\subseteq E$ and $wE\subseteq D$ so $(wv)D\subseteq D$. Since at least one of v and w is non-empty we have a $u\neq 0$ such that $uD\subseteq D$. Because $D\in Al_X$ the sequence $s=uuu\cdots u\cdots$ has a unique initial segment t in D which is impossible: if $t\in D$, then $ut\in D$ and is an initial segment of s as well.

The order is well-founded: let $D_0>D_1>\cdots>D_n>\cdots$ contradict this. Then, for $n>0$, we have $w_n\neq 0$ such that $w_nD_n\subseteq D_{n-1}$. Hence $w_1w_2\cdots w_nD_n\subseteq D_0$. Let t be the initial segment of $w_1w_2\cdots w_n\cdots$ which belongs to D and let n be the least number such that t is a proper initial segment of $w_1w_2\cdots w_n$. Because $w_1\cdots w_nD_n\subseteq D_0$ there is an extension u of $w_1\cdots w_n$ which is in D_0. But then t, $u\in D_0$ and t is a proper initial segment of u; this is impossible.

Proposition 2.3 enables us to define ranking on the sets in Al_X by:

$$r(D)=\text{least ordinal } \alpha \text{ such that if } E<D,\ E\in Al_X, \text{ then } r(E)<\alpha.$$

Theorem 2.5. *A set is X-algebraic iff it satisfies $(*)$.*

Proof. Assume that D is X-algebraic. The representation of D as $\int D_x : x \in X$ is unique so we may prove the implication by induction on the complexity of construction. If $D=\{0\}$ (*) is clear. Assume (*) for D_x, $x \in X$, and let $s:\omega \to X$. Express s as xs' and use (*) to get the initial segment v' of s' which is in D_x. Then $xv' \in D$ and its uniqueness is clear.

We prove the converse by induction on the rank. If $r(D)=0$ then D must be $\{0\}$ as can be easily checked so D is algebraic. If $r(D)>0$ then for every $x \in X$ $D^x(=\{w : xw \in D\})$ satisfies (**) and since $xD^x \subseteq D$ we have $D^x < D$. So, by the inductive hypothesis every D^x is X-algebraic and because

$$D = \int D^x : x \in X$$

so is D.

Examples 2.5. The only algebraic sets of rank 0 and 1 are $\{0\}$ and X. We have, for $n \geqslant 0$

$$\int X^n : x \in X = X^{n+1}$$

so every X^n is X-algebraic and has rank n. If $|X|>1$ then we have sets of rank 2 other than X^2: Take $D=\int D_x : x \in X$ where $D_x = X$ except when $x = x_0$ in which case $D_x = \{0\}$.

Proposition 2.6. *If X is finite then the X-algebraic sets are precisely the sets D satisfying: D is finite and maximal with respect to the property that if v, $w \in D$ then neither is a proper initial segment of the other.*

Proof. For ease in notation, we assume that $X=\{a, b\}$. Assume D_a and D_b are X-algebraic, satisfy the condition and $D=\int D_x : x \in X = aD_a \cup bD_b$. Then D is finite. If v is an initial segment of w, then they start with the same letter, say a, $v = aw'$. Then v' is an initial segment of w' so they can't be both in D_a hence one of v, w is not in D. To show that D is maximal with this property, let $w \in X^*$ and starts with, say b for a change. Then $w = bw'$, w' can be compared with a v in D_b so w is comparable with $bv \in D$.

Conversely, let D be a maximal finite set satisfying the condition and let n be the length of the longest sequence in D. Assume $n>0$. Then D^a and D^b have sequences of lengths $\leqslant n-1$ and this enables us to assume inductively that they are X-algebraic and hence, so is D being $\int D^x : x \in X$.

It is not difficult to see that if X is infinite then for any X-algebraic D $r(D)<|X|^+$ and for any $\alpha<|X|^+$ there is an algebraic d with $r(D)=\alpha$.

Definition 2.7. A partial X-algebra is a function f from an X-algebraic set into X. Its rank, $r(f)$, is the rank of its domain. The functions of rank 0 are called basic functions. The set of partial X-algebras is denoted by P_X.

Remark 2.8. The partial X-algebras are functions which may be defined on arbitrarily long finite sequences but behave in many respects as functions of finitely many arguments. This is best seen in the proof of Theorem 6.1.

The basic functions being $\{(0, x)\}$ are essentially elements of X, the partial X-algebras of rank 1 are simply the functions on X into X, etc.

Given partial X-algebras f_x, $x \in X$ we define

$$\int f_x : x \in X$$

as the function f which at xw is $f_x(w)$ if $w \in \mathrm{dom}(f_x)$ and which is undefined elsewhere.

We shall often use the following operation:

$$f^v(w) = \begin{cases} f(vw) & \text{if } f(vw) \text{ is defined,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Theorem 2.9. (a) *The partial X-algebras form the least set containing tthe basic functions which is closed under the operation of integration.* (b) *Defining $f < g$ iff for some $v \neq 0$ $g^v = f$ we obtain a well-founded partial order on P_X. The rank of f, $r(f)$, is the least α such that if $g < f$ then $r(g) < \alpha$.*

Proof. (a) follows from

$$\mathrm{dom}\left(\int f_x : x \in X\right) = \int \mathrm{dom}\,(f_x) : x \in X.$$

(b) follows from Proposition 2.3 and $f < g$ iff $\mathrm{dom}\,(f) < \mathrm{dom}\,(g)$.

3. Samplings

The notion of sampling generalizes the notion of closed unbounded subsets of $P_\kappa(X)$.

Definition 3.1. A sampling of a set X is a collection S of subsets of X (the elements of S are called samples), such that:

(a) $0 \notin S$ and $X \notin S$;

(b) if $r \subseteq X$ is finite and f is a partial X-algebra then there is an $s \in S$ such that $r \subseteq s$ and s is closed under f.

Example 3.2. If $\omega < \kappa < |X|$ then $S = P_\kappa(X) - \{0\}$ is a sampling because if we take a partial X-algebra f we use the Skolem–Löwenheim argument to find a set of small cardinality containing any given finite set.

Or we can start with an algebra on X and define the samples as subalgebras of size $< \kappa$. In the case when the algebra on X happens to be Jónsson we could define the samples simply as proper subalgebras.

Definition 3.3. Let S be a sampling of X. For every partial X-algebra f we denote by Z_f the set

$$\{s \in S : s \text{ is closed under } f\}.$$

F_S will denote the set of all subsets of S which include Z_f for some partial X-algebra f.

Theorem 3.4. *F_S is a normal filter on S. In fact it is the least normal filter containing the sets $\{s \in S : x \in s\}$ where $x \in X$.*

Proof. Normality means that if C_x with $x \in X = \bigcup S$ are members of the filter then so is the set

$$C = \{s \in S : x \in S \rightarrow s \in C_x\}.$$

We denote the set C by ΔC_x. If $D_x \subseteq C_x$ for each $x \in X$, then $D = \Delta D_x \subseteq C = \Delta C_x$, thus it is enough to consider the case when $C_x = Z_{f_x}$ for some partial X-algebra f_x. Let f be the partial X-algebra $\int f_x : x \in X$. If $s \in Z_f$ then s is closed under every f_x with $x \in s$: if $a_1, \ldots, a_n \in s$ and $f_x(a_1, \ldots, a_n)$ is defined then $f_x(a_1 \cdots a_n) = f(x, a_1 \cdots a_n) \in s$. Therefore $Z_f \subseteq \Delta C_x$ so the diagonal intersection belongs to F_s.

Notice that we still do not know whether F_S is a filter, i.e. normality is not a property of filters only. That F_S is a filter comes from the condition that the samples cover finite sets. As before it is enough to check that if $f_1, \ldots, f_n$ are partial X-algebras then

$$Z_{f_1} \cap \cdots \cap Z_{f_n} \neq 0.$$

Let $x_1, \ldots, x_n$ be in X and define $f_x = f_i$ if $x = x_i$ and for other x's, $f_x = \{(0, x)\}$. Let $s \in S$ be a sample containing $\{x_1, \ldots, x_n\}$ and closed under $f = \int f_x : x \in X$. We have already noticed, that if s is closed under f it is closed under every f_x, with $x \in s$, so the s chosen above is in $Z_{f_1} \cap \cdots \cap Z_{f_n}$.

The filter is non-principal since $\bigcap\{Z_f : f \text{ partial } X\text{-algebra}\} = 0$: if $s \in S$ were in the intersection, $s \neq 0$, from condition (a); let $x \in s$. As there is a $y \in X - s$ and a partial X-algebra mapping X onto y we see that s cannot be closed under all partial X-algebras.

Finally assume we want to construct a normal filter containing $\hat{x} = \{s \in S : x \in s\}$ for every $x \in X$. That means, that every $x \in S$ must be covered by some $s \in S$ which can be phrased that s is closed under the basic function $\{(0, x)\}$. Why should there be samples closed under a given unary function $f : X \rightarrow X$. The reason is if we define $C_x = \widehat{f(x)}$ then the members of the diagonal intersection

$$\{s : x \in S \rightarrow s \in C_x\}$$

are closed under f. In fact $Z_f = \Delta f(x)$. By induction, if we need to put into the filter Z_{f_x} to make it normal we also have to put in Z_f where $f = \int f_x : x \in X$ because

$$Z_f = \{s : x \in s \rightarrow s \in Z_{f_x}\}.$$

In the examples of samplings we gave, the filter is at least ω_1-complete. The reason for that is that condition (b) holds for these samplings in the form: "for every countable $r \subseteq X$ and every partial X-algebra f, there is an $s \in S$ closed under f and containing r". This is immediately seen from the proof of "F_s is closed under finite intersections". The converse of this remark is also true: if F_s is countably complete then the sampling S contains for each countable $r \subseteq X$ its extension $s \in S$.

The next theorem tells us that every sampling has this property.

Proposition 3.5. (i) *If S is a sampling of an infinite set X then F_s is countably complete.*

(ii) *S is a sampling of X iff every countable structure on X, has an elementary substructure in S.*

Proof. Let $r \subseteq X$ be countable, $r = \{x_n : n < \omega\}$. Let $f : X \to X$ be a function such that $f(x_n) = x_{n+1}$. A set $s \in S$ which contains x_0 and is closed under f includes r, so all countable sets are covered and this means F_S is countably complete.

One direction of (ii) is trivial and the other follows from Skolemizing the structure and then taking a countable intersection of sets of the form Z_f, $f : X^n \to X$.

Remark 3.6. The definition of a sampling suggested by this result is simpler than the official definition. However, the official definition is of much more absolute character. In Section 6 we define a set and prove that it is a sampling using the partial X-algebras and that proof reveals even better the finitary properties of these algebras. We have no idea how we could accomplish this via Proposition 3.5(ii).

Remark 3.7. Let S be

$$\{s \subseteq \kappa^+ : \text{order type of } s = \lambda^+\}$$

where $\omega \leqslant \lambda < \kappa$. Then by [1, Proposition 7.3.4(ii)] S is a sampling of κ^+ iff Chang's conjecture holds for the pair (κ^+, κ), (λ^+, λ).

Shelah defines (see [6]) certain filters on subsets of a set A and asks for a better understanding of their interrelations. Although we shall not go into details of it we think that a better understanding may be reached via the notion of samplings. Let us take the filter $\mathbf{E}^1_\kappa(A)$. In our set-up we first define samples, which can be read off the Definition 3.1. The set A is in a set M which contains sets of cardinality less than some fixed cardinal hereditarily. A sample of A is constructed as follows: we form a continuous chain of elementary submodels of some expansion of M, $\langle N_\alpha \mid \alpha < \kappa\rangle$ with the property that $\langle N_\alpha \mid \alpha \leqslant \beta\rangle \in N_{\beta+1}$ (a strictly increasing chain) and take as a sample $A \cap U\{N_\alpha : \alpha < \kappa\}$ (that's how A looks in $U\{N_\alpha : \alpha < \kappa\}$). This way every sample has power κ (we assume $\kappa < |A|$) and this is a sampling of A in our sense since we are allowed to take the expansions of M. Calling S the resulting sampling we find that F_S is not $\mathbf{E}^1_\kappa(A)$, mainly because $\mathbf{E}^1_\kappa(A)$ is κ^+-complete. But if we define F^κ_S by taking as generators

$$\{s \in S : s \text{ is closed under } f \in P\}$$

where $P \subseteq P_A$ is a set of cardinality $\leq \kappa$ we get the same filter. The generators are non-empty since the expansions are allowed to be of size κ. The filter F^κ_S may not be normal but one sees that some remnants of normality remain.

4. Relativization

The results of the last section imply that there are no samplings of countable set. But if a set appears uncountable in some model of set theory it has a sampling in it and it will suffice for the purposes we have in mind. We shall therefore investigate what happens to the notion of partial X-algebras and samplings when we consider them in an admissible set.

In the next proposition we assume that A is an admissible set which contains ω. We also fix an $X \in A$ and assume that it is well-ordered in A. We shall also assume that A satisfies the Σ-choice, i.e. if $R \subseteq A$ is Σ and for some $a \in A$ we have

$$(\forall x \in a)(\exists y) R(x, y),$$

then for some function $f \in A$ defined on a, we have $R(x, f(x))$ for each $x \in a$. These assumptions are satisfied in all L_α's which happen to be admissible as well as in all countable transitive models of ZFC (with or without the power set axiom).

Proposition 4.1. (1) $X^* \in A$. (2) *If* $\{f_x : x \in X\} \in A$ *is a set of partial* X*-algebras, then* $\int f_x : x \in X$ *is in* A.

Proof. The function $\{\langle n, X^n \rangle : n < \omega\}$ is in A and X^* is a Δ_0-subset of its range. This proves (1). (2) is true because $\int f_x : x \in X \subseteq X^*$ and is Δ-definable from $\{f_x : x \in X\}$.

Let us now denote by P^A_X the partial X-algebras which are in A, $P^A_X = A \cap P_X$. By an A-sampling of X we shall mean a set $S \in A$ of subsets of X such that for every $f \in P^A_X$ and every finite $r \subseteq X$ there is an $s \in S$ which is closed under f.

As before we denote by Z_f the set of $s \in S$ which are closed under f and by F^A_S we denote the collection of $Y \subseteq S$ which contain Z_f for some $f \in P^A_X$.

Proposition 4.2. (1) *P_X^A is Σ on A.*

(2) *If S is an A-sampling of X, then F_S^A is a Σ-normal filter on S, which means that if $\{C_x : x \in X\} \subseteq F_S^A$ are such that $\{(xy) : y \in C_x\} (\subseteq X \times S)$ is Σ on A, then the set $\{s \in S : x \in s \rightarrow s \in C_x\}$ is in F_S^A.*

Proof. We define P_X^A by stipulating: "there is a sequence (construction) of length α whose members are functions from a subset of X^{α} into X and the sequence s_β, $\beta < \alpha$ is such that:

(i) s_β is a basic function or

(ii) there is $g : X \rightarrow \beta$ and $s_\beta = \int s_{g(x)} : x \in S$.

It is clear that this is a Σ-definition and that anything in A which satisfies it is a partial X-algebra. Let us show the converse. Given $f \in A \cap P_X$, note that $f^x \in A$ for every $x \in X$; we may assume by inductive assumption on the rank of f that f^x for $x \in X$ satisfies the definition. We define a relation R on A by:

$R(x, s)$ iff s is a sequence satisfying (i) and (ii) and its last element is f^x.

It is clear that R is Σ on A and that $(\forall x \in X)(\exists s) R(x, s)$. Now we use the Σ-choice to get $\langle s_x \mid x \in X \rangle$ such that s_x is a construction of f^x and using the fact that X is well-ordered we combine these into one sequence and top this sequence with f. The result is a construction for f because the function $g(x)$ which equals the length of s_x is in A.

We denote a Σ-definition of P_X^A by $\pi(\cdot)$ and proceed to prove (2). That F_S^A is a filter is proved as before, using Proposition 4.2 (2). Let $R \subseteq X \times S$ be Σ on A and such that for every $x \in X$

$$\{s \in S : R(x, s)\} \in F_S^A.$$

This set need not be in A but by the definition of F_S^A it includes Z_f for some $f \in P_X^A$. Thus if we consider the predicate $T(x, f)$ defined by

$$\pi(f) \wedge (\forall s \in S)[\text{if } s \text{ is closed under } f, \text{ then } R(x, s)]$$

we see that it is Σ on A and that $(\forall x \in X)(\exists f) T(x, f)$; so using the Σ-choice we get $\langle f_x : x \in x \rangle A$, with $f^x \in P_X^A$ and $\Sigma_{f^x} \subseteq \{s \in S : R(x, s)\}$. Hence $f = \int f_x : x \in X$ is P_X^A and we have, as before

$$Z_f \subseteq \{s \in S : x \in s \rightarrow R(x, s)\}.$$

Remark 4.3. The relativization could have been done in terms of Theorem 2.4. In that case we would have gotten as A-partial X-algebras some functions which really are not partial X-algebras because we would check the algebraicity of the domain of the functions for those $s : \omega \rightarrow X$ which are in A. The net effect would be that A-samplings of X might be larger than the A-samplings we have. But even if we would end up with the same samplings it is more pleasant to have P_X^A Σ on A.

The fact that F_X^A is Σ-normal is true if S is just Π on A and $P^A(X)\in A$ as can be seen by inspecting the proof of (2) under these conditions.

5. The paradigm in $L_{\omega\omega}$

Throughout this section T denotes a theory in a countable language and $\Gamma(x, y)$ is a type. As before we denote by $\alpha_\Delta(x)$ the formula

$$\bigwedge(\exists y)\Delta \wedge \neg(\exists y)\bigwedge\Delta$$

which is a formula of $L_{\omega_1,\omega}$. We call $(\exists x)\alpha_\Gamma(x)$ the paradigm and we say it has a solution if there is a model of $T+(\exists x)\alpha_\Gamma(x)$. As we mentioned before, the problem of finding a solution for $(\exists x)\alpha_\Gamma(x)$ patterns quite a few problems in model theory. Let us give some examples:

Example 5.1. Let G_n be the free group on n generators ($n<\omega$). Whether $G_n\equiv G_m$ for n, $m>1$ is still open. This problem can be phrased as asking whether a certain paradigm has a solution. Consider G_n, $n>1$, and let T be the theory of $(G_n, g_1,\ldots, g_n)$ where $g_1,\ldots, g_n$ are free generators of G_n. Let $\Gamma(x, y)$ be the type containing all formulas

$$w(g_1,\ldots, g_n, x)=e \quad \text{iff} \quad \cdots \wedge y\neq w(g_1\cdots g_n, x)$$

where $w(g_1,\ldots, g_n, x)$ is a word in $g_1\cdots g_n$, x (a term) and $\cdots$ is a condition which states when the word is e in such a way that x becomes a new free generator. Now a solution is a group elementarily equivalent to G_n, it has $n+1$ free generators and they generate the whole group (because of the choice of Γ), that is the group is G_{n+1}.

Example 5.2. We have a countable model of ZF and we want to find an end extension of it. Let T be the theory of the model with names for all elements and let $\Gamma(x, y)$ be the type

$$\{x\notin a\wedge y\in a\wedge y\neq b : bEa,\ a\in M\}.$$

Any solution to $(\exists x)\alpha_\Gamma(x)$ is an end extension of M. This example is less typical because the variables x and y are not related, i.e. we have a separation of variables. These paradigms are generally easier to solve. If we consider $\Gamma'(x, y)$ defined by

$$\{\alpha\in x\wedge y\in x\wedge y\neq\alpha : \alpha \text{ ordinal of } M\}$$

the variables are not separated and any solution to the paradigm in this case is a very strong form of end extension in that x becomes the first ordinal after the ordinals of M and, therefore, a solution may not exist.

Let us now go into stating and proving a result which gives sufficient conditions for the existence of a solution for $T+(\exists x)\alpha_\Gamma(x)$. We should add that the statement below is not in the strongest possible form, but making it stronger would result in making it less readable. Another point, more important, is that the idea of the proof applies in situations outside the framework of the theorem but this point shall be illustrated below.

To make the statement of the theorem less bulky, let us enumerate some of its conditions separately: T is a theory, Γ its type and we have an admissible set A which contains ω and Γ, Γ is well-ordered in A, A satisfies the Σ-choice and $T\subseteq A$ is Σ on A.

Theorem 5.3. *If there is an A-sampling S of Γ such that $T+(\exists x)\alpha_\Delta(x)$ has a solution for each $\Delta\in S$, then $T+(\exists x)\alpha_\Gamma(x)$ has a solution.*

Proof. We adjoin to the language of T a new constant c and define a theory in the expanded language:

$$\phi(c)\in T(c) \quad \text{iff } \{\Delta\in S: T+\alpha_\Delta(c)\vdash\Phi(c)\}\in F_S^A.$$

$\Phi(x)$ is a formula of the language of T, $\vdash$ refers, of course, to the provability relation in $L_{\omega_1,\omega}$ and F_S^A is the filter defined in Section 4. We have $T\subseteq T(c)$ and $T(c)$ is consistent because F_S^A is a filter. Also, if $\sigma_1,\ldots,\sigma_n\in\Delta$, then

$$(\exists y)(\sigma_1(c,y)\wedge\cdots\wedge\sigma_n(c,y))\in T(c)$$

because $\{\Delta\in S:\sigma_1,\ldots,\sigma_n\in\Delta\}\in F_S^A$ and if $\sigma_1,\ldots,\sigma_n\in\Delta$, then

$$\alpha_\Delta(c)\vdash(\exists y)(\sigma_1(c,y)\wedge\cdots\wedge\sigma_n(c,y)).$$

Thus, any model of $T(c)$ satisfies the "realizing part" of the paradigm. We have to show that $T(c)$ has a model omitting the type $\Gamma(c,y)$. For this we use the Omitting of Types Theorem (see [1, Theorem 2.2.9]).

We need a criterion for consistency of a formula $\Psi(c,y)$ with $T(c)$. We claim: $\Psi(c,y)$ is consistent with $T(c)$ iff

$$K=\{\Delta\in S: T+\alpha_\Delta(c)+(\exists y)\Psi(c,y)\ \text{consistent}\}$$

is stationary, that is K has a non-empty intersection with every set in F_S^A. If K is stationary and $\Phi(c)\in T(c)$, then

$$K\cap\{\Delta\in S: T+\alpha_\Delta(c)\vdash\Phi(c)\}\neq 0;$$

if Δ is in the intersection we have a model of: $T+\alpha_\Delta(c)+(\exists y)\Psi(c,y)+\Phi(c)$ so by the compactness theorem $T(c)+(\exists y)\Psi(c,y)$ is consistent. If K is not stationary, then

$$\{\Delta\in S: T+\alpha_\Delta(c)\vdash\neg(\exists y)\Psi(c,y)\}\in F_S^A$$

so $\neg(\exists y)\Psi(c,y)\in T(c)$, i.e. $(\exists y)\Psi(c,y)$ is inconsistent with $T(c)$.

Now, let us assume that the assumptions of the Omitting of Types Theorem are not fulfilled; we shall reach a contradiction from this. Hence we have a formula $\Psi(c, y)$ consistent with $T(c)$ which has the property that for each $\sigma \in \Gamma(c, y)$

$$T(c) \vdash (\forall y)(\Psi(c, y) \to \sigma(c, y)).$$

Therefore

$$C_\sigma = \{\Delta \in S : T + \alpha_\Delta(c) \vdash (\forall y)(\Psi(c, y) \to \sigma(c, y))\} \in F_S^A$$

for every $\sigma \in \Gamma$. The relation $\{(\sigma, \Delta) : \Delta \in C_\sigma\}$ is Σ on A because:

(a) the assignment $\sigma \to (\forall y)(\Psi(c, y) \to \sigma(c, y))$ is recursive;

(b) since T is Σ on A and $\alpha_\Delta(c) \in A$, the provability relation in the definition of C_σ is Σ on A (see [3, p. 47, Example 6]).

Because of our assumptions on the admissible set we can use Proposition 4.2 and we find that

$$C = \{\Delta \in S : \in \Delta \in C_\sigma\} \in F_S^A.$$

Because Ψ is consistent with $T(c)$

$$O \neq C \cap \{\Delta \in S : T + \alpha_\Delta(c) + (\exists y)\Psi(c, y) \text{ consistent}\},$$

let Δ be in the intersection. On one hand we have a model M

$$M \models T + \alpha_\Delta(c) + (\exists y)\Psi(c, y).$$

On the other hand if $\sigma \in \Delta$ then

$$M \models (\forall y)(\Psi(c, y) \to \sigma(c, y)).$$

But this contradicts the definition of $\alpha_\Delta(c)$. The Omitting of Types Theorem is now used to give a solution to $T + (\exists x)\alpha_\Gamma(x)$.

Illustration. The reader may find it useful to go through the proof of Theorem 5.3 in a familiar situation. Keisler's 2-cardinal theorem provides a good example. We have a model $(\kappa^+, \kappa, R, \ldots)$, we adjoin to it the closed unbounded sets of κ^+ and relations for membership and being a closed unbounded set. Take a countable elementary substructure, A adjoin to it names for all its elements, and call the theory of the expansion T. Let

$$\Gamma(x, y) = \{U(y) \wedge y \neq a \wedge b < x : a \in U^A, b \in A\}$$

where $U(\cdot)$ is the name of $\kappa (\subseteq \kappa^+)$.

We may now join the proof of 5.1. $T(c)$ is defined by

$$\phi(c) \in T(c) \quad \text{iff} \quad \{b \in A : A \models \phi(b)\} \in F^A$$

where F^A is the filter of "closed unbounded subsets" of A. The rest of the proof may then be read in these terms; it is easier since we do not have to worry about things being Σ. It may also be found that normality of the closed unbounded subsets need not be invoked leaving a room for improvements on this 2-cardinal result.

6. Applications

We shall show an application of Theorem 5.3 itself as well as an application of the method of its proof.

Let us first of all look in the context of this paper at what is Magidor doing in [4]. There he assumes Chang's conjecture to get a special kind of an ultrafilter. This can be achieved as follows. Let T be the theory of

$$(S \cup \omega_2, \in, \ldots)$$

where S consists of subsets of ω_2 whose order type in ω_1 and $\cdots$ enumerates all elements and subsets of the universe. The length of the structure $2^{2^{\omega_1}} = \kappa$. Let $\Gamma(x, y)$ be the type

$$\{\alpha \in x \in S \wedge y \in x \wedge y \neq \alpha : \alpha < \omega_2\}$$

has cardinality ω_2 and we are in no position to apply Theorem 5.3. But if we collapse κ^+ to ω, that is if we work in a universe W where κ^+ is countable, there we have T and Σ countable and embedded in the admissible set A of sets in V which are of cardinality $<\kappa^+$ hereditarily. If Chang's conjecture holds S is an A-sampling of ω_2, which is essentially Γ, and for every $s \in S$ $(S \cup \omega_2, \in, s, \ldots)$ is a solution for $(\exists x)\alpha_s(x)$. All the other conditions of Theorem 5.3 are satisfied and its application yields an elementary extension M of $(S \cup \omega_2, \in, \ldots)$ which contains an element c such that $c \in S^M$ and $M \models a \in c$ iff $a < \omega_2$. We can now define a V-ultrafilter on S by

$$U \in D \quad \text{iff} \quad M \models U(c).$$

Let $V/_D$ be the ultrapower of V using only the functions $f: S \to V$ which are in V, and let $*$ be the elementary embedding of V into $V/_D$. We want to compute the order type of ω_1^*. If $f, g: S \to \omega_1$ then $f/_D < g/_D$ iff $M \models f(c) < g(c) < \omega_1$. Now the order type of ω_1 in M is the same as that of c (we can express this in M) and that is ω_2, so $\omega_1^* \leq \omega_2$. The other inequality follows from considering for $\alpha < \omega_2$, $f_\alpha(s) = \alpha$th element of s. The set-up of [4] is thus established. The procedure used here does not give a better result but it gives a uniform strategy to follow in other situations.

The application using the method of the proof of Theorem 5.3 concerns a situation when we have a model M of ZFC and an elementary embedding $j: M \to N$. These models need not be standard. Given $X \in M$ there are two sets, in general different which can be associated with X. First of all we have $j(X) \in N$ which satisfies the same properties in N as in M. Secondly we have the set $\mathrm{st}_j(X) = \{j(x) : M \models x \in X\}$ which is merely a subset of N. In the case when this subset is an element of N, meaning that for some $Y \in N$ and for every $a \in N$

$$N \models a \in Y \quad \text{iff} \quad a \in \mathrm{st}_j(X)$$

we say that the standard part of X exists in N and we denote the element of N by $\mathrm{st}_j(X)$ or by $\mathrm{st}(X)$.

For example if $M = V$ and $N = V^\kappa/_D$ where D is a κ-complete ultrafilter over κ $(>\omega)$ then $\mathrm{st}(\kappa) = \kappa$ (assuming N is transitive). If D is non-principal over $\kappa = \omega$ the $\mathrm{st}(\kappa)$ does not exist and, for that matter, no standard of any infinite ordinal exists in N. In the opposite direction we have that κ is supercompact iff for every $\lambda \geqslant \kappa$ there is an elementary embedding j of V into M such that κ is the first ordinal moved by j and the standard part of X, $\mathrm{st}_j(X)$, exists for every set $X \subseteq M$ of cardinality λ.

Let us now consider X, $s \in M$ with $M \models S$ is a sampling of X. The type of the sampling S in M is the set

$$\{\phi(v) : M \models \text{“}\{s \in S : \phi(s)\} \in F_S\text{”}\}.$$

This set is a type of the theory of the model M. If M were a set we could also talk about the type of the sampling in $(M, a)_{a \in M}$; it would then be a type of $\mathrm{Th}((M, a)_{a\in M})$ and the following theorem would be true even with this definition of the type of S.

Theorem 6.1. *Let M be a countable model of* ZF, *let X, $S \in M$ be such that $M \models$ “S is a sampling of X”, and let $\tau(v)$ be the type of S in M. For any type $\Sigma(v)$ of* $\mathrm{Th}(M)$ *which extends $\tau(v)$ there is an elementary extension N of M which contains the standard part of X and*

$$N \models \Sigma(\mathrm{st}(X)).$$

Proof. Let $T(c)$ be a theory in the language of M augmented by a new constant c and names for all elements of M (**a** names a) whose axioms are:

$$T(c) = \{\phi(ca_1 \cdots a_n) : \{s \in S \mid M \models \phi[sa_1 \cdots a_n]\} \in F_S^M\}.$$

Here F_S^M is the filter in M determined by S and the set $\{s \in S : M \models \phi[sa_1 \cdots a_n]\}$ denotes the element of M satisfying the definition.

$T(c)$ is a consistent theory and any model of it can be considered as an elementary extension of M. We now show that T locally omits the type

$$\{y \in c \wedge y \neq a : M \models a \in X\}. \tag{1}$$

Let $\psi(yca_1 \cdots a_n)$ be such that

$$T(c) \vdash (\forall y)[\psi(yca_1 \cdots a_n) \to y \in c \wedge y \neq a]$$

for every $a \in M$ satisfying $M \models a \in X$. Then for every such a

$$C_a = \{s \in S : M \models (\forall y)[\psi(ysa_1 \cdots a_n) \to y \in S \wedge y \neq a]\}$$

represents an element of M such that $M \models C_a \in F_S$. Moreover the function which assigns to $a \in^M X$ the set C_a is in M, therefore, since $M \models F_S$ is normal

$$M \models \{s \in S : a \in s \to s \in C_a\} \in F_S.$$

But if $s \in^M S$ is such that for every $a \in^M s$ we have $s \in^M C_a$ then for every $a \in^M s$ we have

$$M \models \psi(ysa_1 \cdots a_n) \to y \in s \wedge y \neq a$$

and from this it follows that

$$M \models \neg(\exists y)\psi(ysa_1 \cdots a_n).$$

By the definition of $T(c)$ this means that

$$T(c) \vdash \neg(\exists y)\psi(yca_1 \cdots a_n).$$

So there is a model N omitting the type and therefore containing $c = \mathrm{st}(X)$, with $N \models \Sigma(\mathrm{st}(X))$.

There is a partial converse to the result. Let $j: M \to N$ be such that $\mathrm{st}(X)$ exists in N and $N \models \mathrm{st}(X) \subseteq j(X)$. Take a formula $\phi(v, j(a_1), \ldots, j(a_n))$ such that

$$N \models \phi[\mathrm{st}(X), j(a_1), \ldots, j(a_n)].$$

We may also assume that $\phi(v)$ implies the formula $v \subseteq j(X)$. We now show that if M is standard the element S of M satisfying

$$M \models v \in S \leftrightarrow \phi(va_1 \cdots a_n)$$

is a sampling of X. Let $f \in M$ be such that $M \models f$ is X-algebraic. Now in N $j(f)$ is $j(X)$-algebraic, but $j(X^*)$ may be different from $j(X)^*$ (in N; things are simple if N is standard). Fortunately, the nature of partial X-algebras is such that it enables us to prove: if

$$N \models w \in \mathrm{dom}\,(j(f)) \cap (\mathrm{st}(X))^*,$$

then w is really finite.

We show this by induction on the rank of f (in M). If the rank is 0 (of M) then the domain of f has one element and the same is true of $j(f)$. If the rank is >0 then we have:

$$M \models (\forall w \in X^*)[w \in \mathrm{dom}\,(f) \leftrightarrow w' \in \mathrm{dom}\,(j(f^{w_0}))]$$

where w_0 is the first letter in w and $w = w_0 w'$. Hence we have

$$N \models (\forall w \in j(X^*))[w \in \mathrm{dom}\,(j(f)) \leftrightarrow w' \in \mathrm{dom}\,(j(f^{w_0}))].$$

Therefore, if $N \models w \in \mathrm{dom}\,(j(f)) \cap \mathrm{st}(X)^*$, then $w_0 \in \mathrm{st}(X)^*$ and $w' \in \mathrm{dom}\,(j(f^{w_0}))$. So $w_0 = j(x_0)$ where $x_0 \in X$ and then $w' \in \mathrm{dom}\,(j(f^{x_0}))$, hence by the inductive assumption w' is finite and so is w. Having this result it is easy to check that

$$N \models \mathrm{st}(X) \text{ is closed under } j(f).$$

Let $r \subseteq X$ be finite. We have

$$N \models (\exists v)[\phi(v, j(a_1), \ldots, j(a_n)) \wedge j(r) \subseteq V \wedge V \text{ is closed under } j(f)].$$

The sentence (without j) is true in M which means that S contains a sample closed under f and including r, consequently $M \models S$ is a sampling. The type of this sampling in M contains all formulas $\Psi(v)$ which are true of st(X) in N and are implied by ϕ, but we have not been able to get the type of S included in that of st(X).

However, the above implies

Corollary 6.2. *Let M be standard $j: M \to N$ elementary, let* $\mathrm{st}(\omega_2^M)$ *exist in N and let $j(\omega_1)$ and* $\mathrm{st}(\omega_2^M)$ *have the same order type. Then $N \models$ Chang's conjecture for* (ω_2, ω_1), (ω_1, ω).

Proof. Define a sampling of ω_2^M using the formula "v has order type $\omega_1 \wedge v \subseteq \omega_2$". This formula is true of $\mathrm{st}(\omega_2^M)$ in N so this defines a sampling. We explained in Section 3 why the existence of such sampling implies Chang's conjecture.

The types of samplings limit and outline the properties st(X) can be made to satisfy. Let us amplify this a bit. For a set X define Φ_X to be the set of all formulas $\phi(v)$ (of the language of set theory) which satisfy:

> if S is a sampling of X then for some $s \in S$, $\phi(s)$
> is true (in the universe).

Intuitively, Φ_X seems to contain the properties which can not be avoided in non-trivial extensions. To give some examples we show that $\Phi_{\omega_1} \neq \Phi_{\omega_2}$: the property "$v$ is a countable ordinal" is in Φ_{ω_1} because there are structure on ω_1 whose elementary substructures consist entirely of ordinals, so every sampling of ω_1 must contain a countable ordinal. The property does not belong to Φ_{ω_2} because the set of subsets of power ω_1 is a sampling of ω_2.

Added in proof

The main argument in this paper is similar in spirit to the proof of the Completeness Theorem for Stationary Logics. See J. Bairwise, M. Kaufmann and M. Makkai, Stationary Logic, Ann. of Math. Logic 13 (1978) 171–224.

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