Ergodic theorems for random clusters

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Abstract

We prove pointwise ergodic theorems for a class of random measures which occurs in Laplacian growth models, most notably in the anisotropic Hastings–Levitov random cluster models. The proofs are based on the theory of quasi-orthogonal functions and uniform Wiener–Wintner theorems.

Keywords: Ergodic theorems; Loewner evolution; Weak convergence

1. Introduction

1.1. General comments

This paper is concerned with the asymptotic behavior of a class of random measures which arise in Laplacian growth models. The main result is an ergodic theorem for these measures, which has been used by Johansson, Sola and Turner in [3] to establish the existence of a limit cluster in anisotropic Hastings–Levitov models. We refer to their paper for further details and motivation.

1.2. Statement of theorems

Let \((X, \mathcal{F}, \mu)\) be a standard probability measure space, and suppose that \(g_k\) is a sequence of bounded and independent, identically distributed real-valued measurable functions on \((X, \mathcal{F})\). Let \(Z\) be a compact and metrizable space, and suppose that \(\theta_k\) is an ergodic Markov chain on \(Z\), defined on \((X, \mathcal{F}, \mu)\), with stationary measure \(\lambda\) and independent of the family \(g_k\). We assume
that the Markov chain $\theta_k$ has a spectral gap with respect to separable and dense normed subspace $H$ of the space of real-valued continuous functions $f$ on $Z$. This means that for every $k$, we have $(\theta_k)_* \mu = \lambda$ and there are constants $C > 0$ and $0 \leq \tau < 1$, such that for all $f$ in $H$ and $k, l \geq 1$, we have the inequality,

$$\left| \int_X f(\theta_k(x)) f(\theta_l(x)) \, d\mu(x) - \left( \int_Z f \, d\lambda \right)^2 \right| \leq C \|f\|_{H}^2 \tau^{|k-l|},$$

where $\| \cdot \|_H$ denotes the norm in $H$. We say that the Markov chain $\theta_k$ is contractive on $H$ if the above inequality holds. Note that this is always the case when all $\theta_k$ are independent and identically distributed.

Suppose that $M_n$ is a martingale defined with respect to some filtration on $(X, F)$, and assume that for some $p > 1$, we have

$$\sum_{n \geq 1} \frac{\|M_n\|_p^p}{n^{p'}} < \infty.$$ 

In particular, this is true for $p \geq 4$ if $M_n$ is the sum of a sequence of independent and identically distributed random variables in $L^4(X, \mu)$ with mean zero.

Let $t_{k,n} = k/n$, and define the sequences $\tilde{M}_{k,n} = M_k/n + t_{k,n}$ and $Y_{k,n} = (\theta_k, \tilde{M}_{k,n})$ on $(X, F)$.

Suppose that $(\rho_k)$ is a quasi-orthogonal sequence (see Definition 3.1) in $L^2(X)$, and $\beta \in \mathbb{R}$. Define the following measurable array,

$$\tilde{\rho}_{k,n}(x) = \frac{1}{n}(\beta + \rho_k(x)), \quad 1 \leq k \leq n.$$ 

We can now formulate the main theorem in this paper.

**Theorem 1.** There is a conull subset $X'$ of $X$ such that for all $x$ in $X'$,

$$\lim_{n \to \infty} \sum_{k=1}^{n} \tilde{\rho}_{k,n}(x) \delta_{\tilde{M}_{k,n}(x)} = \beta m_{[0,1]},$$

in the weak topology on $\mathcal{M}(\mathbb{R})$, where $m_{[0,1]}$ denotes the normalized Lebesgue measure on the interval $[0, 1]$.

Before we embark on the proof of Theorem 1, we discuss an application of the theorem to contractive Markov chains. Suppose that there is a separable and dense subspace of $H$ on which the Markov chain $\theta_k$ above is contractive with an ergodic and stationary measure $\lambda$, and let $g_k$ be a sequence of non-negative bounded independent and identically distributed random variables on $X$. Then we have the following corollary of Theorem 1.

**Corollary 1.1.** There is a conull subset $X'$ of $X$ such that for all $x$ in $X'$, we have the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} g_k(x) \delta_{Y_{k,n}(x)} = \left( \int_X g_1 \, d\mu \right) \lambda \otimes m_{[0,1]}$$

in the weak topology on $\mathcal{M}(Z \times \mathbb{R})$, where $m_{[0,1]}$ denotes the normalized Lebesgue measure on $[0, 1]$. 
Corollary 1.1 admits a generalization to general ergodic weights \((g_k)\). Let \((X, B_X, \mu_X, T)\) be an ergodic probability measure preserving system. Then we have the following theorem.

**Theorem 2.** Suppose that \(f \in L_0^\infty(X, \mu)\), and \(\beta \in \mathbb{R}\). Then there is a conull set \(X' \subseteq X\) such that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (\beta + f(T^k x)) \delta_{\hat{M}_{k,n}(x)} = \beta m_{[0,1]},
\]
in the weak topology on \(\mathcal{M}(\mathbb{R})\).

The paper is organized as follows. In Section 2 we describe the Hastings–Levitov models and give motivation for Corollary 1.1. In Sections 3 and 4 we give complete proofs of Theorems 1 and 2 respectively.

### 2. Hastings–Levitov models

In this section we describe the main application of Corollary 1.1. More details can be found in the paper [3]. Let \(\Delta\) denote the exterior disk
\[
\Delta = \{z \in \mathbb{C}_\infty \mid |z| > 1\},
\]
where \(\mathbb{C}_\infty\) is the Riemann sphere, and consider the unique conformal mapping
\[
\phi_\delta : \Delta \to \Delta \setminus (1, 1 + \delta),
\]
with \(\phi_\delta(z) = C(\delta)z + O(1), C(\delta) > 0\) at infinity. Let \(\theta_1, \theta_2, \ldots\) be i.i.d. random variables on the unit circle \(\mathbb{T}\) with common law \(\nu\), and let \(\delta_1, \delta_2, \ldots\) be positive independent random variables with laws \(\sigma_1, \sigma_2, \ldots\), and independent of \(\theta_1, \theta_2, \ldots\). Define
\[
\phi_{\delta_n}^{\theta_n}(z) = e^{i \theta_n} \phi_{\delta_n}(e^{-i \theta_n} z).
\]
We set \(\phi_0(z) = z\) and
\[
\phi_n(z) = \phi_n \circ \phi_{\delta_n}^{\theta_n}(z).
\]
This will produce a sequence of random conformal maps \(\phi_n : \Delta \to \mathbb{C} \setminus K_n\) with \(K_n\) compact and \(K_{n-1} \subset K_n\). We will refer to the sets \(K_n\) as random clusters.

Recall that a decreasing Loewner chain is a family of conformal mappings
\[
f_t : \Delta \to \mathbb{C} \setminus K_t,
\]
with \(f_t(\infty) = \infty\) and \(f_t'(\infty) > 0\) and
\[
K_{t_1} \subset K_{t_2} \quad \text{if} \quad t_1 < t_2.
\]
It is well known that decreasing Loewner chains can be parameterized by Schwarz–Herglotz integrals of probability measures on \(\mathbb{T}\). More precisely, every decreasing Loewner chain will satisfy the Loewner–Kufarev equation
\[
\frac{\partial}{\partial t} f_t(z) = z f_t'(z) \int_{\mathbb{T}} \frac{z + \zeta}{z - \zeta} d\mu_t(\zeta),
\]
with initial condition \(f_0(z) = z\), and where \(\mu_t\) is a sequence of probability measures on \(\mathbb{T}\) with certain properties.
We will think of the family \( \{ \mu_t \}_{t \geq 0} \) as a locally bounded measure on \( \mathbb{R} \times \mathbb{T} \) via the correspondence,

\[
\mu(\varphi) = \int_{\mathbb{R}} \left( \int_{\mathbb{T}} \varphi(\xi, t) \, d\mu_t(\xi) \right) \, dt,
\]

where \( \varphi \) is a compactly supported continuous function on \( \mathbb{R} \times \mathbb{T} \).

In this paper we will establish almost sure convergence of sequences \( \mu_N \) of random locally bounded measures on \( \mathbb{T} \times \mathbb{R} \), defined on some common probability measure space, of the form

\[
d\mu_N(\xi, t) = \frac{1}{N} \sum_{k=1}^{N} X_k f(\theta_k) \delta_{T_k,N}(t),
\]

where \( f \) is a continuous function on \( \mathbb{T} \) and

\[
T_{k,N} = \frac{1}{N} \sum_{j=1}^{k} X_j,
\]

where \( X_1, X_2, \ldots \) are positive i.i.d. random variables, independent of \( \theta_1, \theta_2, \ldots \). The motivation for these choices can be found in [3]. It turns out that \( \mu_N \) corresponds to the law of the conformal mapping \( \Phi_N \) where the slit lengths \( \delta_j, j = 1, \ldots, N \) are distributed according to \( X_j/N \). We are interested in the asymptotic behavior of these measures. Note that if the \( X_j \)'s are bounded, the sequence \( \mu_N \) is almost surely tight and bounded. Corollary 1.1 was used by Johansson, Sola and Turner [3] to establish the existence of a limit cluster for the Loewner chain constructed above with \( \mu_t = \nu \).

3. Proof of the main theorem

3.1. Random measures

Let \( E \) be a proper metric space, i.e. closed and bounded subsets are compact. Let \( \mathcal{M}(E) \) denote the real vector space of signed measures on \( E \). A sequence \( \eta_n \) in \( \mathcal{M}(E) \) is said to converge vaguely to a measure \( \eta \) in \( \mathcal{M}(E) \) if

\[
\lim_{n \to \infty} \int_E f \, d\eta_n = \int_E f \, d\eta.
\]

for all compactly supported continuous functions on \( E \). The sequence converges weakly if the same is true for bounded and continuous functions. A subset \( A \) of \( \mathcal{M}(E) \) is called tight if for all \( \varepsilon > 0 \), there is a compact set \( K \) such that \( |\nu|(K^c) < \varepsilon \) for all \( \nu \) in \( A \), where \( | \cdot | \) denotes the total variation. We say that \( A \) is bounded if there is a constant \( C \) such that \( |\nu|(E) < C \) for all \( \nu \) in \( A \). We will use the following theorem by Baez–Duarte [2].

**Theorem 3.** A sequence of signed measures \( \eta_n \) converges weakly to \( \eta \) if \( \eta_n \) converges vaguely to \( \eta \) and the sequence is bounded and tight.

Since the space \( C_c(E) \) of continuous functions with compact support in \( E \) is separable, it suffices to establish the vague limit above only for a countable number of elements in \( C_c(E) \). If, in addition, the sequence is tight and bounded, the same is true for weak convergence, in view of Theorem 3.
Let \((X, \mathcal{F}, \mu)\) be a probability space. A random measure on \(E\) is a measurable map \(\pi\) from \(X\) into \(\mathcal{M}(E)\), equipped with the Borel structure coming from the weak topology. We will consider sequences of random measures, defined on some common probability space \((X, \mathcal{F}, \mu)\), with the property that there is a conull subset of \(X'\) such that the sequence \(\pi_{x_n}\) of \(\mathcal{M}(E)\) is tight and bounded for all \(x \in X'\) (but not necessarily uniformly, i.e. we do not require that the compact set \(K\) above can be chosen independently of \(x\), nor that the sequence is almost everywhere uniformly bounded on \(X\)). Thus, if we are able to construct conull subsets \(X_f\) of \(X\) for all \(f\) in a countable dense subset of \(C_c(E)\), with the property that there is a random measure \(\pi\) on \(E\), independent of \(f\), such that
\[
\lim_{n \to \infty} \int_E f \, d\pi_{x_n} = \int_E f \, d\pi_x,
\]
for all \(x \in X_f\), then by taking \(X'' = X' \cap (\cap_f X_f)\), which is again conull, we conclude that the measures \(\pi_{x_n}\) converges vaguely (and thus weakly, since the sequence \(\pi_{x_n}\) is assumed to be almost surely tight and bounded) to \(\pi_x\) for all \(x\) in the set \(X''\).

### 3.2. Fourier analysis

Let \(S\) denote the Schwarz class on \(\mathbb{R}\) (see e.g. [4]). For \(\phi\) in \(S\) we define the Fourier transform to be the function,
\[
\hat{\phi}(\xi) = \int_{\mathbb{R}} \phi(x) e^{-ix\xi} \, dx, \quad \xi \in \mathbb{R}.
\]
It is a standard fact that \(\hat{\phi}\) is in \(S\) and that \(\phi\) can be reconstructed from \(\hat{\phi}\) by the formula,
\[
\phi(x) = \int_{\mathbb{R}} \hat{\phi}(\xi) e^{i\xi x} \, dm(\xi),
\]
where \(m\) denotes the Plancherel measure on \(\mathbb{R}\), see [4].

### 3.3. Quasi-orthogonal functions

We recall the definition of a quasi-orthogonal sequence in a Hilbert space.

**Definition 3.1.** Let \((X, \mathcal{F}, \mu)\) be a measure space. A sequence \(\rho_k\) in \(L^2(X, \mu)\) is called quasi-orthogonal if there is a constant \(C > 0\), such that for all \(c\) in \(\ell^2(\mathbb{N})\), we have the inequality,
\[
\left| \sum_{k,l} c_k \overline{c_l} \langle \rho_k, \rho_l \rangle \right| \leq C \sum_k |c_k|^2.
\]
In particular, a sequence \(\rho_k\) in \(L^2(X, \mu)\) is quasi-orthogonal if there are constants \(C\) and \(0 \leq \tau < 1\) such that
\[
|\langle \rho_k, \rho_l \rangle| \leq C \tau^{|k-l|}, \quad \forall k, l \geq 1.
\]
The following theorem by Kac, Salem and Zygmund [5] shows that the convergence theory of quasi-orthogonal sequences is not too different to the theory of orthogonal sequences.

**Theorem 4.** Suppose that \(\rho_k\) is a quasi-orthogonal sequence in \(L^2(X, \mu)\). There is a constant \(C\) such that for every triangular array of complex numbers \(a_{nk}\), \(k = 1, \ldots, n\), and \(n \geq 1\), we have
the inequality,
\[
\int_X \sup_{1 \leq m \leq n} \left| \sum_{k=1}^m a_{mk} \rho_k(x) \right|^2 \, d\mu(x) \leq C \sum_{k=1}^n |a_{nk}|^2 \|\rho_k\|_2^2 \log^2(1 + k).
\]

In particular, if \(a_{nk}\) is a triangular array with the property that
\[
\lim_{n \to \infty} \sum_{k=1}^n |a_{nk}|^2 \|\rho_k\|_2^2 \log^2(1 + k) = 0,
\]
then we have the limit
\[
\lim_{n \to \infty} \sum_{k=1}^n a_{nk} \rho_k(x) = 0
\]
almost everywhere on \(X\). Similar results have been attained by Teicher [6] for independent random variables.

3.4. Main ergodic theorem

We first indicate how Corollary 1.1 follows from Theorem 1. Recall the notation introduced in Section 1.2. Let \(Z\) be a compact and metrizable space, and suppose that \(\mathcal{H}\) is a separable and dense normed subspace of \(C(Z)\). Let \(\theta_k\) be a contractive Markov chain on \(\mathcal{H}\) with an ergodic and stationary measure \(\lambda\). For \(n \geq 1\), we define,
\[
Y_{k,n} = (\theta_k, \tilde{M}_{k,n}), \quad 1 \leq k \leq n.
\]
Finally, we assume that \(g_k\) is a sequence of bounded and identically distributed non-negative random variables on \(X\), which are independent of the Markov chain \(\theta_k\).

**Corollary 3.1.** There is a conull subset \(X'\) of \(X\) such that for all \(x\) in \(X'\),
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n g_k(x) \delta_{Y_{k,n}(x)} = \left(\int_X g_1 \, d\mu\right) \lambda \otimes m_{[0,1]}
\]
in the weak topology on \(\mathcal{M}(Z \times \mathbb{R})\).

**Proof of Corollary 3.1.** We can without loss of generality assume that \(\int_X g_1 \, d\mu = 1\). Since the sequence of signed measures on \(Z \times \mathbb{R}\),
\[
v_n = \frac{1}{n} \sum_{k=1}^n g_k(x) \delta_{Y_{k,n}(x)}, \quad n \geq 1,
\]
is bounded and tight, it suffices to prove the convergence of \(v_n(f \otimes \phi)\) to \(\lambda(f) m_{[0,1]}(\phi)\) for \(f\) and \(\phi\) in dense and countable collections of functions in \(\mathcal{H}\) and \(C_0(\mathbb{R})\) respectively. Since \(g_k\) and \(\theta_k\) are independent, the sequence
\[
\tilde{\rho}_{k,n} = g_k f(\theta_k)/n, \quad 1 \leq k \leq n,
\]
is again quasi-orthogonal on \((X, \mathcal{F}, \mu)\) with \(\int_X g_k f(\theta_k) \, d\mu = \lambda(f)\). By Theorem 1, there exists, for every fixed \(f\) in \(\mathcal{H}\), a conull subset \(X_f\) of \(X\) such that for all \(x\) in \(X_f\),
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n g_k(x) f(\theta_k(x)) \delta_{\tilde{M}_{k,n}(x)} = \left(\int_X f \, d\lambda\right) m_{[0,1]},
\]
in the weak topology on $\mathcal{M}(\mathbb{R})$. Since $\mathcal{H}$ is countable and dense in $C(Z)$, the intersection $X'$ of all $X_f$ with $f$ in $\mathcal{H}$ is still conull. □

In order to establish Theorem 1 we need some lemmata. The first lemma reduces the analysis to Riemannian integration theory.

**Lemma 1.** There is a conull subset $X'$ of $X$ such that for all $x$ in $X'$,

$$\lim_{n \to \infty} \sum_{k=1}^{n} \tilde{\rho}_{k,n}(x) \left( \delta_{M_{k,n}(x)} - \delta_{t_{k,n}} \right) = 0,$$

in the weak topology on $\mathcal{M}(\mathbb{R})$.

**Proof.** Since the sequence of measures is clearly bounded and tight, it suffices to prove, for every fixed choice of $\phi$ in a countable and dense collection $C$ of functions in $C_0(\mathbb{R})$, the existence of a conull subset $X_{\phi}$ of $X$ with the property that for all $x$ in $X_{\phi}$, we have the limit,

$$\lim_{n \to \infty} \sum_{k=1}^{n} \tilde{\rho}_{k,n}(x) \left( \phi(M_{k,n}(x)) - \phi(t_{k,n}) \right) = 0.$$

We define $X'$ as the intersection of all $X_{\phi}$ when $\phi$ ranges over all elements in $C$.

We can choose the collection $C$ to be contained in $\mathcal{S}$ so that Fourier transform techniques become available. For a fixed $\phi$ in $C$, we pick $\epsilon > 0$ and choose $r > 0$ such that

$$\int_{|\xi| > r} |\hat{\phi}(\xi)| \, dm(\xi) < \frac{\epsilon}{4}.$$

Define, for $|\xi| < r$, the function

$$R_n(x) = \int_{\mathbb{R}} \left[ \sum_{k=1}^{n} \tilde{\rho}_{k,n}(x) e^{i \xi t_{k,n}} \left( e^{i \xi M_{k,n}(x)} - 1 \right) \right] \hat{\phi}(\xi) \, dm(\xi).$$

Choose $\delta > 0$ such that

$$|e^{i \xi t} - 1| < \inf_{k,n \geq 1} \frac{\epsilon}{4 \|\hat{\phi}\|_{\infty} \|\tilde{\rho}_{k,n}\|_{\infty}}$$

for all $|t| < \delta$ and $|\xi| < r$. This is possible since $\tilde{\rho}_{k,n}$ is uniformly bounded from above in $k$ and $n$. Define the maximal function

$$L_n = \sup_{1 \leq k \leq n} |M_k|, \quad n \geq 1,$$

and the sets

$$A_n(\delta) = \{ x \in X \mid L_n(x) < \delta n \}, \quad n \geq 1.$$

Note that if $x$ is in $A_n(\delta)$, then $|R_n(x)| \leq \epsilon$. Thus, if

$$B_n(\epsilon) = \{ x \in X \mid |R_n(x)| > \epsilon \},$$

we have $A_n(\delta) \cap B_n(\epsilon) = \emptyset$ for all $n \geq 1$. By the Borel–Cantelli Lemma and Doob’s Maximal
Theorem,
\[ \sum_{n \geq 1} \mu(B_n(\varepsilon)) = \sum_{n \geq 1} \mu(B_n(\varepsilon) \cap A_n(\delta)) + \sum_{n \geq 1} \mu(B_n(\varepsilon) \cap A_n(\delta)^c) \]
\[ \leq \sum_{n \geq 1} \mu(A_n(\delta)^c) \leq \frac{1}{\delta^p} \sum_{n \geq 1} \|L_n\|_p^p \]
\[ \leq \frac{C^p}{\delta^p} \sum_{n \geq 1} \|M_n\|_p^p \to < +\infty, \]
for all \( \varepsilon > 0 \), and hence \( R_n(x) \to 0 \) almost everywhere on \( X \). \( \square \)

The existence of the limit of the remaining sequence of random measures can be established using quasi-orthogonal functions as the following lemma show.

**Lemma 2.** There is a conull subset \( X' \) of \( X \) such that for all \( x \) in \( X' \),
\[ \lim_{n \to \infty} \sum_{k=1}^{n} \tilde{\rho}_{k, n} \delta_{t_{k, n}} = \beta \cdot m[0, 1] \]
in the weak topology on \( \mathcal{M}(\mathbb{R}) \).

**Proof.** Let \( \beta_{k, n} = \beta/n \). By the theory of Riemann integrals, we have
\[ \sum_{k=1}^{n} \beta_{k, n} \delta_{t_{k, n}} = \beta \cdot m[0, 1], \]
thus it suffices to prove that there is a conull subset \( X' \) of \( X \) such that for all \( x \) in \( X' \), we have
\[ \lim_{n \to \infty} \sum_{k=1}^{n} \rho_{k, n} \delta_{t_{k, n}} = 0, \]
in the weak topology on \( \mathcal{M}(\mathbb{R}) \). Since the sequence above is bounded and tight, we only need to establish the limit for a countable and dense collection \( C \) of functions in \( C_{\phi}(\mathbb{R}) \). By the countable additivity of measures, it suffices to prove that, for every fixed \( \phi \) in \( C \), there exists a conull subset \( X_\phi \) such that for all \( x \) in \( X_\phi \),
\[ \lim_{n \to \infty} \sum_{k=1}^{n} \rho_{k, n}(x) \phi(t_{k, n}) = 0. \]

We define \( X' \) as the intersection of all \( X_\phi \) for \( \phi \) in \( C \), which again is a conull subset of \( X \). By Theorem 4, with \( n a_{nk} = \phi(t_{k, n}) \), we have the maximal inequality,
\[ \int_{X} \sup_{1 \leq m \leq n} \left| \sum_{k=1}^{m} a_{mk} \rho_k(x) \right|^2 \, d\mu(x) \leq C \sum_{k=1}^{n} a_{nk}^2 \log^2(1 + k) \|\rho_k\|_2^2. \]
Since \( \|\rho_k\|_2 \) is bounded in \( k \), we conclude, by an elementary Borel–Cantelli argument, that
\[ \lim_{n \to \infty} \sum_{k=1}^{n} \rho_{k, n} \phi(t_{k, n}) = 0, \]
almost everywhere on \( X \). \( \square \)
4. Generalization to dependent weights

We now sketch a generalization of Corollary 1.1 to the case of not necessarily independent weights \( g_k \). Let \((X, \mathcal{B}_X, \mu_X, T)\) be an ergodic probability measure preserving system. Recall that the Kronecker factor of \((X, \mathcal{B}_X, \mu_X, T)\) is the smallest sub-\(\sigma\)-algebra \( \mathcal{K} \subseteq \mathcal{B}_X \) with respect to which all eigenfunctions are measurable. Bourgain’s uniform Wiener–Wintner theorem [1] asserts that for all \( f \in L^\infty(X, \mu) \) which are orthogonal to \( \mathcal{K}_X \), there is a conull set \( X' \subseteq X \) such that for all \( x \in X' \),

\[
\lim_{n \to \infty} \sup_{\xi \in \mathbb{R}} \left| \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)e^{2\pi i k \xi} \right| = 0.
\]

This result will be very useful in the proof of the following generalization of Corollary 1.1.

**Theorem 5.** Suppose that \( f \in L^0_1(X, \mu) \), and \( \beta \in \mathbb{R} \). Then there is a conull set \( X' \subseteq X \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (\beta + f(T^k x)) \delta_{\tilde{S}_{k,n}(x)} = \beta \, m_{[0,1]},
\]

in the weak topology on \( \mathcal{M}(\mathbb{R}) \).

**Proof.** The line of proof is very similar to the proof of Corollary 1.1, and is divided into two parts. We first assume that \( f \) is in the orthogonal complement of the Kronecker factor, and prove that for all \( \phi \in \mathcal{S} \), with compactly supported Fourier transform \( \hat{\phi} \), there is a conull set \( X_\phi \subseteq X \) such that for all \( x \in X_\phi \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k x) \phi(t_{k,n}) = 0.
\]

To prove this, we note that,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k x) \phi(t_{k,n}) = \lim_{n \to \infty} \int_{\mathbb{R}} \left[ \frac{1}{n} \sum_{k=1}^{n} f(T^k x) e^{i \xi t_{k,n}} \right] \hat{\phi}(\xi) \, dm(\xi)
\]

\[
= \lim_{n \to \infty} \int_{\mathbb{R}} \left[ \frac{1}{n} \sum_{k=1}^{n} f(T^k x) e^{i \xi k} \right] n \hat{\phi}(n \xi) \, dm(\xi).
\]

By Bourgain’s uniform Wiener–Wintner theorem, the expression in the bracket converges uniformly to 0 in \( \xi \) as \( n \to \infty \) for all \( x \in X'' \). Thus, by Hölder’s inequality,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k x) \phi(t_{k,n}) = 0,
\]

for all \( x \in X'' \), which proves the first claim.

Our second claim concerns functions \( f \) which are \( \mathcal{K} \)-measurable. Suppose that \( \psi \) is an eigenfunction of \((X, \mathcal{B}_X, \mu_X, T)\), with eigenvalue \( \lambda \neq 1 \), then, for all compactly supported \( \phi \in \mathcal{S} \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \psi(T^k x) \phi(t_{k,n}) = \lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=1}^{n} \lambda^k \phi(t_{k,n}) \right) \psi(x) = 0,
\]

which is easy to check, and the details are left to the reader. Thus, for any finite linear combination of eigenfunctions \( f \), there is a conull set \( X' \subseteq X \) such that for all \( x \in X' \),
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k x) \delta_{t_k, n} = 0,
\]
in the weak topology on \( \mathcal{M}(\mathbb{R}) \). We will now boost this to hold for all essentially bounded which are measurable with respect to the Kronecker factor. Let \( f \) be essentially bounded and \( \mathcal{K} \)-measurable. By the definition of \( \mathcal{K} \), there is a sequence of finite linear combinations of eigenfunctions, which we will denote by \( f_N \), such that \( f_N \) converge to \( f \) almost everywhere on \( X \) and in \( L^1 \)-norm. We claim that the random measures,

\[
\nu^x_{n,f} = \frac{1}{n} \sum_{k=1}^{n} f(T^k x) \delta_{t_k, n} = 0,
\]
converge to the zero measure on \([0, 1]\) for all \( x \) in some conull set of \( X \). Since we already know this convergence for the measures \( \nu^x_{n,f_N} \) on some conull set \( X' \subseteq X \) which we can take to be independent of \( N \) and containing the generic points for \( f \), it suffices to prove that for all \( x \in X' \),

\[
\lim_{n \to \infty} |\nu^x_{n,f_N}(\phi) - \nu^x_{n,f}(\phi)| = 0,
\]
for all compactly supported \( \phi \in \mathcal{S} \). To prove this, note that, for all \( x \in X' \),

\[
\lim_{n \to \infty} |\nu^x_{n,f_N}(\phi) - \nu^x_{n,f}(\phi)| \leq \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |f_N(T^k x) - f(T^k x)| \|\phi\|_{\infty}
= \|\phi\|_{\infty} \int_X |f - f_N| \, d\mu \to 0,
\]
which proves the last claim.

Finally, for a general \( f \in L^\infty_0(X) \), we can decompose it into a direct sum of an essentially bounded \( \mathcal{K} \)-measurable function and an essentially bounded function orthogonal to \( \mathcal{K} \). Since the theorem holds in both classes of functions, we are done. \( \square \)

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References