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$\aleph$-spaces and spaces with a σ-hereditarily closure-preserving k-network

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Abstract

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We give a necessary and sufficient condition for a regular space with a σ-hereditarily closure-preserving k-network to be an $\aleph$-space and we give some results concerning product spaces with a σ-hereditarily closure-preserving k-network.

Keywords: σ-hereditarily closure preserving, k-network, $\aleph$-space, S_{ω_1}.

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1. Introduction

In this paper, we prove that a regular space X is an $\aleph$-space if and only if X has a σ-hereditarily closure-preserving k-network and contains no closed copy of S_{ω_1}, and we show that $S \times S_{\omega_1}$ has no σ-hereditarily closure-preserving k-network. As applications of these results, we give several characterizations of $\aleph$-spaces and we prove that certain product spaces with a σ-hereditarily closure-preserving k-network are $\aleph$-spaces.

Recall that a family $\mathcal{F}$ of subsets of X is called a *k-network* if for any compact set K and any open set U which contains K, there exist finitely many $F_1, \ldots, F_k \in \mathcal{F}$ such that $K \subseteq F_1 \cup \cdots \cup F_k \subseteq U$. A family $\{F_\alpha : \alpha \in A\}$ of subsets of X is said to be *hereditarily closure-preserving* if for any choice of $S_\alpha \subseteq F_\alpha$, for $\alpha \in A$, the family $\{S_\alpha : \alpha \in A\}$ is closure-preserving, i.e., for every $A' \subseteq A$, $\mathrm{Cl}(\bigcup\{S_\alpha : \alpha \in A'\}) = \bigcup\{\mathrm{Cl}\, S_\alpha : \alpha \in A'\}$. If a family is the union of countably many hereditarily closure-preserving families, then the family is said to be *σ-hereditarily closure-preserving*.

An $\aleph$-*space* is a regular space which has a σ-locally finite k-network, and a σ-*space* is a space which has a σ-locally finite network.

In the following, we shall study the relationship between $\aleph$-spaces and regular spaces with a σ-hereditarily closure-preserving k-network. The latter class of spaces is important mainly because of Foged's result [2] that Lašnev spaces (i.e., continuous images of metrizable spaces under closed mappings) can be characterized as regular Fréchet spaces with a σ-hereditarily closure-preserving k-network.

Let S be the subspace $\{1/n: n \in \mathbb{N}\} \cup \{0\}$ of $\mathbb{R}$ in the usual topology. For each $\alpha < \omega_1$, let $S^{(\alpha)}$ be a copy of S. We denote by S_{ω_1} the quotient space obtained from the topological union $\bigoplus_{\alpha<\omega_1} S^{(\alpha)}$ by mapping all the nonisolated points into one point. Let $\mathbf{0}$ be the nonisolated point of S_{ω_1}. Note that S_{ω_1} is a Lašnev space; by Foged's result, S_{ω_1} has a σ-hereditarily closure-preserving k-network.

If $\mathcal{F}$ is a family of subsets of X and $x \in X$, then $(\mathcal{F})_x = \{F \in \mathcal{F}: x \in F\}$. For terms which are not defined here, refer to [1].

2. $\aleph$-spaces and the space S_{ω_1}

Lemma 2.1. *If X is a T_1-, σ-space and every compact subset of X is finite, then we can represent X as $X = \bigcup_{n\in\mathbb{N}} X_n$ in such a way that, for every n, X_n is a closed and discrete subset of X.*

Proof. Let $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ be a σ-locally finite network of X which is closed under finite intersections. Let $X_n = \{x \in X: \{x\} \in \mathcal{F}_n\}$. Then X_n is a closed and discrete subset of X. We show that $X = \bigcup_{n\in\mathbb{N}} X_n$. Let $x \in X$, and enumerate $(\mathcal{F})_x$ as $\{F'_k: k \in \mathbb{N}\}$. Let $F_k = \bigcap_{n\leqslant k} F'_n$ for each k. There must exist $k \in \mathbb{N}$ such that $F_k = \{x\}$. (Otherwise we can find a sequence of distinct points $\{x_{k_n}: n \in \mathbb{N}\}$ of X such that $x_{k_n} \in F_{k_n}$ and $x \neq x_{k_n}$ for each n. Then $\{x_{k_n}: n \in \mathbb{N}\} \cup \{x\}$ is an infinite compact subset of X.) Let $n \in \mathbb{N}$ be such that $F_k \in \mathcal{F}_n$. Then $x \in X_n$; it follows that $X = \bigcup_{n\in\mathbb{N}} X_n$. □

Lemma 2.2. *If $\mathcal{F}$ is a hereditarily closure-preserving family in a regular space X, then $\{\mathrm{Cl}\, F: F \in \mathcal{F}\}$ is also hereditarily closure-preserving in X* [7, Lemma 1].

Lemma 2.3. *Let $\mathcal{F}$ be a hereditarily closure-preserving family of closed subsets of a Hausdorff space X and let C be a compact subset of X. Then there exists a finite subset A of C such that only finitely many members of $\mathcal{F}$ meet $C\setminus A$* [10, Theorem 2.1].

Note that, for a hereditarily closure-preserving closed family $\mathcal{F}$ of X, the set $D = \{x \in X: |(\mathcal{F})_x| \geqslant \omega\}$ is closed and, by Lemma 2.3, every compact subset of D is finite.

Lemma 2.4. *Let X be a Hausdorff space which contains no closed copy of S_{ω_1}. If $\mathscr{F}$ is a hereditarily closure-preserving family of closed subsets of X, then for every $x \in X$ there are at most countably many $F \in \mathscr{F}$ such that the set $F \setminus \{x\}$ contains a sequence converging to x.*

Proof. Assume that there exists a point $z \in X$, an uncountable subfamily $\mathscr{H}$ of $\mathscr{F}$, and for every $H \in \mathscr{H}$, a subset $S_H = \{y_{H,n} : n \in \mathbb{N}\}$ of $H \setminus \{z\}$ such that $y_{H,n} \to z$. By the remark made after Lemma 2.3, we can assume that, for every $H \in \mathscr{H}$, $\mathscr{F}$ is point finite at every $y_{H,n}$, $n \in \mathbb{N}$. It follows that, for every $H \in \mathscr{H}$, the family $\{F \in \mathscr{F}: F \cap S_H \neq \emptyset\}$ is countable. As a consequence, the family $\{S_H : H \in \mathscr{H}\}$ is star-countable, and hence σ-disjoint. Since $\mathscr{H}$ is uncountable, there exists a subfamily $\mathscr{J}$ or $\mathscr{H}$ such that, $|\mathscr{J}| = \omega_1$ and for all H, $H' \in \mathscr{J}$, if $H \neq H'$, then $S_H \cap S_{H'} = \emptyset$. Now it easily follows, since $\mathscr{J}$ is hereditarily closure-preserving, that the subspace $\{z\} \cup \bigcup_{H \in \mathscr{J}} S_H$ is closed in X and homeomorphic with S_{ω_1}, a contradiction. □

Lemma 2.5. *Let X be a regular space with a σ-hereditarily closure-preserving k-network such that X contains no closed copy of S_{ω_1}, and let D be a closed and discrete subset of X. Then there exists a σ-discrete family $\mathscr{H}$ consisting of closed subsets of X such that for every compact set $K \subseteq X$ and for every $d \in K \cap D$, the family $\{H \in \mathscr{H}: d \in \operatorname{Int}_K(K \cap H)\}$ is a network of d in X.*

Proof. By Lemma 2.3, X has a closed k-network $\mathscr{F} = \bigcup_{n \in \mathbb{N}} \mathscr{F}_n$ such that, for every $n \in \mathbb{N}$, $\mathscr{F}_n$ is hereditarily closure-preserving and $\mathscr{F}_n \subseteq \mathscr{F}_{n+1}$. For each $x \in X$, let $\mathscr{F}(x) = \{F \in \mathscr{F}$: there exists a compact set $K \subseteq F$ such that $x \in \operatorname{Cl}(K \setminus \{x\})\}$. Since X is a σ-space (see [11]), every compact subset of X is metrizable, and it follows from Lemma 2.4 that, for each $x \in X$, the family $\mathscr{F}(x)$ is countable; hence we can write $\{\bigcup \mathscr{F}': \mathscr{F}' \subseteq \mathscr{F}(x)$ and $\mathscr{F}'$ is finite$\} \cup \{\{x\}\} = \{F_k(x): k \in \mathbb{N}\}$.

For all $d \in D$ and $n \in \mathbb{N}$, let $G_n(d)$ be a closed neighborhood of d such that $G_n(d) \subseteq X \setminus \bigcup\{F \in \mathscr{F}_n : d \notin F\}$, and let $S_n(d) = \bigcup\{F \in \mathscr{F}_n : F \cap D = \{d\}\}$. Note that, for each $n \in \mathbb{N}$, since $\mathscr{F}_n$ is hereditarily closure-preserving, the family $\{G_n(d) \cap S_n(d): d \in D\}$ is closed and discrete. It follows that, for all $n \in \mathbb{N}$ and $k \in \mathbb{N}$, the family $\mathscr{H}_{n,k} = \{F_k(d) \cap G_n(d) \cap S_n(d): d \in D\}$ is closed and discrete. To show that the σ-discrete and closed family $\mathscr{H} = \bigcup_{n,k \in \mathbb{N}} \mathscr{H}_{n,k}$ satisfies the condition of the lemma, let $K \subseteq X$ be compact, let $d \in K \cap D$ and let O be a neighborhood of d in X. Let V be a closed neighborhood of d such that $V \subseteq O \setminus (D \setminus \{d\})$. Since $\mathscr{F}$ is a k-network of X, there exists a finite subfamily $\mathscr{F}'$ of $\mathscr{F}$ such that $K \cap V \subseteq \bigcup \mathscr{F}' \subseteq O \setminus (D \setminus \{d\})$. Let $n \in \mathbb{N}$ be such that $\mathscr{F}' \subseteq \mathscr{F}_n$ and let $k \in \mathbb{N}$ be such that $F_k(d) = \bigcup(\mathscr{F}' \cap \mathscr{F}(d)) \cup \{d\}$. Then $d \in F_k(d) \cap G_n(d) \cap S_n(d) \subseteq \bigcup \mathscr{F}' \subseteq O$. We complete the proof by showing that $d \in \operatorname{Int}_K(K \cap F_k(d) \cap G_n(d) \cap S_n(d))$. Since $d \in \operatorname{Int}_K(K \cap V) \subseteq \operatorname{Int}_K(K \cap \bigcup \mathscr{F}')$ and since $\mathscr{F}'$ is a finite family of closed subsets of X covering $K \cap V$, we have that $d \in \operatorname{Int}_K(K \cap \bigcup(\mathscr{F}')_d)$. Since $(\bigcup \mathscr{F}') \cap (D \setminus \{d\}) = \emptyset$, we have that $\bigcup(\mathscr{F}')_d \subseteq S_n(d)$. From the foregoing it follows that $d \in \text{Int}_K(K \cap S_n(d))$. Since $d \in \text{Int}_X G_n(d)$, we have that $d \in \text{Int}_K(K \cap G_n(d))$. It remains to prove that $d \in \text{Int}_K(K \cap F_k(d))$. For every $F \in \mathscr{F}'$, let $E_F = X \setminus \text{Cl}(F \cap K \setminus \{d\})$, and let $E = \bigcap\{E_F: F \in \mathscr{F}' \text{ and } d \in E_F\}$. Then E is a neighborhood of d and we have that $\{F \in \mathscr{F}': F \cap K \cap E \neq \emptyset\} \subseteq \mathscr{F}' \cap \mathscr{F}(d)$. It follows that $K \cap \bigcup \mathscr{F}' \cap E \subseteq \bigcup(\mathscr{F}' \cap \mathscr{F}(d)) \cup \{d\} = F_k(d)$. Since $d \in \text{Int}_K(K \cap \bigcup \mathscr{F}')$ and $d \in \text{Int}_X E$, it follows from the foregoing that $d \in \text{Int}_K(K \cap F_k(d))$. We have shown that $d \in \text{Int}_K(K \cap S_n(d))$, $d \in \text{Int}_K(K \cap G_n(d))$ and $d \in \text{Int}_K(K \cap F_k(d))$; it follows that $d \in \text{Int}_K(K \cap F_k(d) \cap G_n(d) \cap S_n(d))$. □

Theorem 2.6. *Let X be a regular space which has a σ-hereditarily closure-preserving k-network. Then X is an $\aleph$-space if and only if X contains no closed copy of S_{ω_1}.*

Proof. *Necessity* is immediate since every subspace of an $\aleph$-space is an $\aleph$-space and S_{ω_1} is not an $\aleph$-space [5, Example 9.2].

Sufficiency. By Lemma 2.2, we may assume that X has a closed k-network $\mathscr{F} = \bigcup_{n \in \mathbb{N}} \mathscr{F}_n$, where $\mathscr{F}_n$ is hereditarily closure-preserving and $\mathscr{F}_n \subseteq \mathscr{F}_{n+1}$ for each $n \in \mathbb{N}$. Assume that X contains no closed copy of S_{ω_1}. Let $D_n = \{x \in X: |(\mathscr{F}_n)_x| \geqslant \omega\}$. By Lemma 2.1 and the remark made after Lemma 2.3, we can represent each D_n as $D_n = \bigcup_{k \in \mathbb{N}} D_{n,k}$ in such a way that, for every $k \in \mathbb{N}$, $D_{n,k}$ is closed and discrete in X and $D_{n,k} \subseteq D_{n,k+1}$. For all n and k, let $\mathscr{H}_{n,k}$ be a σ-discrete family satisfying the condition of Lemma 2.5 (with $\mathscr{H} = \mathscr{H}_{n,k}$ and $D = D_{n,k}$).

For every $n \in \mathbb{N}$, since $\mathscr{F}_n$ is closure-preserving, closed and point-finite in $X \setminus D_n$, $\mathscr{F}_n$ is locally finite in $X \setminus D_n$. For all $n \in \mathbb{N}$ and $k \in \mathbb{N}$, let $S_{n,k} = \bigcup\{F \in \mathscr{F}_k : F \cap D_n = \emptyset\}$ and $\mathscr{F}_{n,k} = \{F \cap S_{n,k}: F \in \mathscr{F}_n\}$; note that $\mathscr{F}_{n,k}$ is locally finite in X.

We show that the σ-locally finite family $\mathscr{L} = \bigcup_{n,k \in \mathbb{N}} \mathscr{H}_{n,k} \cup \bigcup_{n,k \in \mathbb{N}} \mathscr{F}_{n,k}$ is a k-network for X. Let $K \subseteq O \subseteq X$ where K is compact and O is open. Then there exists a finite subfamily $\mathscr{F}'$ of $\mathscr{F}$ such that $K \subseteq \bigcup \mathscr{F}' \subseteq O$. Let $n \in \mathbb{N}$ be such that $\mathscr{F}' \subseteq \mathscr{F}_n$. The set $K \cap D_n$ is finite, by Lemma 2.3, and hence there exists $k \in \mathbb{N}$ such that $K \cap D_n \subseteq D_{n,k}$. By Lemma 2.5, there exists, for every $d \in K \cap D_n$, a set $H_d \in \mathscr{H}_{n,k}$ such that $d \in \text{Int}_K(K \cap H_d)$ and $H_d \subseteq O$. Let $K' = K \setminus \bigcup\{\text{Int}_K(K \cap H_d): d \in K \cap D_n\}$. Then K' is a compact set and $K' \subseteq X \setminus D_n$; as a consequence, there exists a finite subfamily $\mathscr{F}''$ of $\mathscr{F}$ such that $K' \subseteq \bigcup \mathscr{F}'' \subseteq X \setminus D_n$. Let $l \in \mathbb{N}$ be such that $\mathscr{F}'' \subseteq \mathscr{F}_l$. Then $\bigcup \mathscr{F}'' \subseteq S_{n,l}$ and hence $K' \subseteq S_{n,l}$. Let $\mathscr{J} = \{H_d: d \in K \cap D_n\} \cup \{F \cap S_{n,l}: F \in \mathscr{F}'\}$. Then $\mathscr{J}$ is a finite subfamily of $\mathscr{L}$ and it follows from the foregoing that $K \subseteq \bigcup \mathscr{J} \subseteq O$. □

Gruenhage, Michael and Tanaka have shown that S_{ω_1} is not a quotient s-image of a metrizable space and S_{ω_1} has no point-countable closed k-network [5, Example 9.2, Theorem 6.1; 9, Theorem 3.2]. Moreover, it is easy to check that $\chi(S_{\omega_1}) > \omega_1$. Consequently, Theorem 2.6 has the following corollaries:

Corollary 2.7. *A regular space X is an $\aleph$-space if and only if X has a σ-hereditarily closure-preserving k-network and a point-countable closed k-network.*

Corollary 2.8. *If X is a regular quotient s-image of a metrizable space, then X is an $\aleph$-space if and only if X has a σ-hereditarily closure-preserving k-network.*

In particular, if a Lašnev space X is a quotient s-image of a metrizable space, then X is an $\aleph$-space, and hence (see [4, 8]), X is a closed s-image of a metrizable space.

Corollary 2.9. *If X is a regular space with a σ-hereditarily closure-preserving k-network and $\chi(X) \leq \omega_1$, then X is an $\aleph$-space.*

Remark. For Lašnev spaces, the result of Corollary 2.9 is due to Gao [3]. Gao and Hattori [4] and Lin [8] have shown that a *regular space X is an $\aleph$-space if and only if X has a σ-closure-preserving and point-countable closed k-network.* This result suggests the question whether in Corollary 2.7 above the condition "σ-hereditarily closure-preserving" can be changed into "σ-closure-preserving". However, that cannot be done, as shown by the space X appearing in [5, example 9.8]. The results given in [5] show that the space X is not an $\aleph$-space, while it is easy to show that X has a σ-closure-preserving base and X has a point-countable k-network consisting of compact sets. Moreover, X is a k-space but not a Fréchet space. The following problem is still open.

Problem. Is a stratifiable Fréchet space an $\aleph$-space if the space has a point-countable closed k-network?

3. Product spaces with a σ-hereditarily closure-preserving k-network

Theorem 3.1. *$S \times S_{\omega_1}$ has no σ-hereditarily closure-preserving k-network.*

Proof. By Lemma 2.2, if a regular space has a σ-hereditarily closure-preserving k-network, then the space has a σ-hereditarily closure-preserving closed k-network. By Lemma 2.3, if $\mathscr{F}$ is a σ-hereditarily closure-preserving closed k-network for a regular space X, then, for every compact set $K \subseteq X$, the family $\{F \in \mathscr{F}: |F \cap K| = \omega\}$ is countable. Hence to prove Theorem 3.1, it suffices to show that for every closed k-network $\mathscr{F}$ of $S \times S_{\omega_1}$, the family $\{F \in \mathscr{F}: |(S \times \{\mathbf{0}\}) \cap F| = \omega\}$ is uncountable.

Let $\mathscr{F}$ be a closed k-network of $S \times S_{\omega_1}$. By transfinite induction, choose $F_\alpha \in \mathscr{F}$ and $z_\alpha \in (S \times (S^{(\alpha)} \setminus \{\mathbf{0}\})) \cap F_\alpha$, for each $\alpha < \omega_1$, so that $|(S \times \{\mathbf{0}\}) \cap F_\alpha| = \omega$ and $F_\alpha \neq F_{\alpha'}$ for every $\alpha' < \alpha$, as follows:

Since $S \times S^{(0)}$ is compact, some finite subfamily $\mathscr{F}'$ of $\mathscr{F}$ covers $S \times S^{(0)}$. It is easy to see that there exists $F_0 \in \mathscr{F}'$ and $z_0 \in (S \times (S^{(0)} \setminus \{\mathbf{0}\})) \cap F_0$ such that $|(S \times \{\mathbf{0}\}) \cap F_0| = \omega$.

Let $\alpha < \omega_1$ and assume that, for every $\beta < \alpha$, we have chosen $F_\beta \in \mathscr{F}$ and $z_\beta \in (S \times (S^{(\beta)} \setminus \{0\})) \cap F_\beta$ such that $|(S \times \{0\}) \cap F_\beta| = \omega$. Note that the set $Z = \{z_\beta : \beta < \alpha\}$ is closed and $Z \cap (S \times S^{(\alpha)}) = \emptyset$; hence there exists a finite $\mathscr{F}' \subseteq \mathscr{F}$ such that $S \times S^{(\alpha)} \subseteq \bigcup \mathscr{F}' \subseteq S \times S_{\omega_1} \setminus Z$. There exists $F_\alpha \in \mathscr{F}'$ and $z_\alpha \in (S \times (S^{(\alpha)} \setminus \{0\})) \cap F_\alpha$ such that $|(S \times \{0\}) \cap F_\alpha| = \omega$. Since $F_\alpha \cap Z = \emptyset$, we have that $F_\alpha \neq F_\beta$ for each $\beta < \alpha$. This completes the inductive construction. □

Remark. Consider the following statement:

> There exists a closed k-network $\mathscr{F}$ of $S \times S_{\omega_1}$ such that, for every compact and infinite subset K of $S \times S_{\omega_1}$, the family $\{F \in \mathscr{F}: K \subseteq F\}$ is countable. (*)

The above proof shows that the statement obtained from (*) by replacing "$K \subseteq F$" with "$|K \cap F| = \omega$" is not valid. Statement (*), however, turns out to be independent of ZFC: it is quite easy to see that CH⇒(*) and (*)⇒($\mathfrak{s} = \omega_1$) (refer to [13] for the definition and properties of the cardinal $\mathfrak{s}$).

The following result, which follows directly from Theorems 2.6 and 3.1, is a generalization of [7, Theorem 1]:

Corollary. *If X is a regular space and $S \times X$ has a σ-hereditarily closure-preserving k-network, then X is an $\aleph$-space.*

Theorem 3.2. *If X and Y are regular spaces such that both of them have a σ-hereditarily closure-preserving k-network, then $X \times Y$ has a σ-hereditarily closure-preserving k-network if and only if either both X and Y are $\aleph$-spaces or in one of them every compact subset is finite.*

Proof. *Necessity.* Assume that $X \times Y$ has a σ-hereditarily closure-preserving k-network and Y is not an $\aleph$-space. We show that every compact subset of X is finite. By Theorem 2.6, Y contains a closed copy of S_{ω_1}. It follows from Theorem 3.1 that X cannot contain a copy of S. Since X is a σ-space, every compact subset of X is metrizable. Every infinite compact metrizable space contains a copy of S. As a consequence, every compact subset of X is finite.

Sufficiency. If both X and Y are $\aleph$-spaces, then $X \times Y$ is an $\aleph$-space, and hence it has a σ-hereditarily closure-preserving k-network. Assume that every compact subset of X is finite. Then, by Lemma 2.1, we can represent X as $X = \bigcup_{n \in \mathbb{N}} X_n$ in such a way that, for each $n \in \mathbb{N}$, X_n is closed and discrete. Note that $\{\{x\}: x \in X\}$ is a k-network for X. Let $\mathscr{H} = \bigcup_{k \in \mathbb{N}} \mathscr{H}_k$ be a k-network for Y such that each $\mathscr{H}_k$ is hereditarily closure-preserving. Let $\mathscr{F}_{n,k} = \{\{x\} \times H: x \in X_n \text{ and } H \in \mathscr{H}_k\}$. Then $\mathscr{F} = \bigcup_{n,k \in \mathbb{N}} \mathscr{F}_{n,k}$ is a σ-hereditarily closure-preserving k-network for $X \times Y$. □

As seen in the above proof, every σ-space without infinite compact subsets is an $\aleph$-space; hence we have the following consequence of Theorem 3.2.

Corollary 3.3. *Let X and Y be regular spaces. If $X \times Y$ has a σ-hereditarily closure-preserving k-network, then either X or Y is an $\aleph$-space.*

Corollary 3.4. *Let X be a regular space such that X^2 has a σ-hereditarily closure-preserving k-network. Then X is an $\aleph$-space.*

Since every infinite product of two-point spaces is an infinite compact space, Theorem 3.2 has the following consequence.

Corollary 3.5. *For each $n \in \mathbb{N}$, let X_n be a regular space which has at least two points. If $\prod_{n \in \mathbb{N}} X_n$ has a σ-hereditarily closure-preserving k-network, then it is an $\aleph$-space.*

Harley and Tanaka [6, Theorem 1; 12, Theorem of Remark 4.1] showed that the property of "being a Fréchet space" is poorly behaved among products of Lašnev spaces, i.e., *if X and Y are Lašnev spaces, then $X \times Y$ is a Fréchet space if and only if either both X and Y are metrizable or one of them is discrete.* Since every Lašnev space is a regular k-space, the next result, which follows from Theorem 3.2, shows that the property of "having a σ-hereditarily closure-preserving k-network" is also poorly behaved among products of Lašnev spaces.

Corollary 3.6. *If X and Y are regular k-spaces and $X \times Y$ has a σ-hereditarily closure-preserving k-network, then either both X and Y are $\aleph$-spaces or one of them is discrete.*

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