Theoretical Computer Science 411 (2010) 631–659

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Definable transductions and weighted logics for texts

Christian Mathissen *

Institut für Informatik, Universität Leipzig, 04009 Leipzig, Germany

ARTICLE INFO

Article history:
Received 24 April 2009
Received in revised form 5 September 2009
Accepted 29 September 2009
Communicated by Z. Esik

Keywords:
Weighted automata
Weighted logics
Monadic second-order logic
MSO-definable transductions
Parenthesizing automata
Branching automata
Recognizable series
Texts

ABSTRACT

A text is a word together with an additional linear order on it. We study quantitative models for texts, i.e. text series which assign to texts elements of a semiring. We introduce an algebraic notion of recognizability following Reutenauer and Bozapalidis as well as weighted automata for texts combining an automaton model of Lodaya and Weil with a model of Ésik and Németh. After that we show that both formalisms describe the text series definable in a certain fragment of weighted logics as introduced by Droste and Gastin. In order to do so, we study certain definable transductions and show that they are compatible with weighted logics.

1. Introduction

Texts as introduced by Rozenberg and Ehrenfeucht [21] extend the model of words by an additional linear order. This additional order gives a word a tree-like hierarchy and enriches its structure. The theory of texts originates in the theory of 2-structures (cf. [20]) and it turns out that texts represent an important subclass of 2-structures, namely T-structures [22]. Moreover, Ehrenfeucht and Rozenberg proposed texts as a well-suited model for natural texts that may carry in its tree-like structure grammatical information [22, p. 264]. A number of authors [23,38,39] have investigated classes of text languages such as the families of context-free, equational or recognizable text languages and developed a language theory. In particular, the fundamental result of Büchi and Elgot [9,24] on the coincidence of recognizable and definable languages has been extended to texts [39].

In this paper we consider quantitative aspects of texts and study text series, i.e. functions form the domain of texts into a semiring. Extending the result of Hoogeboom and ten Pas mentioned above as well as results of Lodaya and Weil [46] and Ésik and Németh [30] we show that three different formalisms are expressively equivalent and yield the same classes of text series. More precisely, we show that the class of series which are algebraically recognizable, the class of series that are definable in a certain fragment of Droste and Gastin's weighted logics and the class of series that describe the behavior of a certain class of weighted automata coincide. Let us discuss the first two formalisms in more detail.

We consider a weighted algebraic recognizability concept for general algebras following a line of research initiated by Reutenauer [58] and continued by Bozapalidis et al. [8,7,6]. It generalizes weighted automata on words and trees as well as the notion of recognizable languages as defined by Mezei and Wright in the 1960s [55]. The algebraic notion of recognizability has attracted much attention. One reason for this is the universality of this concept, since it is at hand whenever we define an algebraic structure (external structure) on the class of objects, i.e. whenever we define operations

* Tel.: +49 341 97 32285; fax: +49 341 97 39254.
E-mail address: mathissen@informatik.uni-leipzig.de.

0304-3975/$ – see front matter
doi:10.1016/j.tcs.2009.09.040

on the class. Even more importantly, algebraic recognizability is closely connected to combinatorial regularity and logical definability in many important cases, such as words, trees and traces; see [67] for a survey. In this paper, we define algebraic recognizability for series, i.e. functions from a general algebra into a semiring, and, furthermore, generalize the notion of a syntactic algebra to the weighted setting.

Recently, Droste and Gastin [13] introduced weighted logics over words and showed a Büchi-type characterization for weighted automata over words. They enrich the language of monadic second-order logic with values from a semiring in order to add quantitative expressiveness. The semantics of a formula in their framework is a formal power series. In [13] they showed a Büchi–Elgot-type characterization for weighted automata over words. Since they define their logic for arbitrary commutative semirings, the framework is very flexible. This way, one may now, for example, express *how often* a certain property holds, *how much* execution time a process needs, or *how* reliable it is. The result of Droste and Gastin was extended to infinite words, (infinite) trees, nested words, pictures, traces and message sequence charts [16,18,53,50,54,4,17,57]. Let us also note that a restriction of Łukasiewicz's multi-valued logic coincides with Droste and Gastin's weighted logics [61].

In order to show our main result we will establish a general translation technique for weighted logics. Therefore, we study a certain subclass of Courcelle's definable transductions [10] and show that they preserve definability with respect to weighted logics. We then refine the transductions from texts to terms and vice versa given by Hoogeboom and ten Pas such that they are compatible with weighted logics. This approach admits the advantage that decidability results for the emptiness and equivalence problem, then come almost for free as a corollary.

We now summarize the content of each section of this paper and its structure.

Section 2. Here we start by recalling the definition of a recognizable subset of a Σ-algebra, which is due to Mezei and Wright [55]. This notion subsumes recognizable subsets of a free monoid and is of outstanding importance in theoretical computer science as it leads to many decidability and minimization results. Further, we develop the concept of a recognizable series from a Σ-algebra into a semiring and show that this definition generalizes the notion of a recognizable subset of Mezei and Wright. Moreover, we will see that for words and trees recognizable series form the same class of series as the behaviors of weighted automata. The notion of a recognizable series always depends on a set of operations defined on the class under consideration. Next, we turn to syntactic congruences and syntactic algebras of series. The concepts we introduce are in a line with definitions of Reutenauer [58] and Bozapalidis et al. [8,7,6] for series over words and trees. Adapting an idea of Reutenauer, we show that whenever the underlying semiring is a ring or a locally finite semiring then a series is recognizable if and only if its syntactic algebra is a finitely generated semimodule. Moreover, we give two counterexamples for the case where the underlying semiring does not meet this assumption. Let us point out that the concepts introduced in this section were independently considered by Fülöp and Steinby [31].

Section 3. As described before, Droste and Gastin [13] considered so-called weighted logics and showed a Büchi–Elgot-type characterization for weighted automata over words. In [14, Open Problem 1] they ask what happens when considering weighted logics for general relational structures and which results of classical model theory can be developed also for weighted logics. This section can be seen as a first contribution to this question. We investigate weighted logics and different fragments thereof for arbitrary relational structures. In particular, we consider the fragment sRMSO, which was proposed in [14]. It can be defined whenever the structures under consideration are equipped with a linear order, since in this case classical monadic second-order logic can be embedded into its weighted counterpart. In fact, we demonstrate with a counterexample that this is not possible if the structures admit non-trivial automorphisms. Furthermore, we investigate how sRMSO and other fragments of weighted logics behave with respect to translations of formulae in the framework of Courcelle's definable transductions [10]. We prove a transfer theorem and show that under certain conditions these transductions are compatible with weighted logics. In the subsequent part of the paper we will use this theorem to easily transfer results on weighted logics between different classes of combinatorial structures.

Section 4. In this section we draw our attention to texts. Hoogeboom and ten Pas [39] defined operations on the class of texts and showed that the languages which are recognizable with respect to these operations coincide with the languages definable in MSO. They achieved this by encoding texts as trees and applying the corresponding characterization for tree languages which was given by Doner, Thatcher and Wright [64,12]. Applying the results of the two previous sections and using the transductions given by Hoogeboom and ten Pas, we first extend the result of [39] to a weighted setting where we need the assumption that the underlying semiring is locally finite or a ring. This assumption already indicates that our extension is not a straightforward adaption of the unweighted case. We will overcome the restrictions on the semiring by introducing a new automaton model. We call these automata weighted branching and parenthesizing automata. These automata form a joint extension of the model of branching automata of Lodaya and Weil [46] and the model of parenthesizing automata of Ésik and Németh [30]. We will show that the behaviors of these automata are precisely the recognizable series from texts into a semiring and that these automata are precisely as expressive as the fragment sRMSO of weighted logics. Let us point out that this result now holds for any commutative semiring.

An extended abstract of this paper appeared as [49]. This paper differs from it in the following way. First, full proofs are included. Second, in Section 2 we give two examples for series which are recognizable but whose syntactic algebra is not finitely generated. Third, in [49] recognizable text series were characterized using a fragment of weighted logics called RMSO. Here we use the fragment sRMSO which has the advantage of being decidable. Fourth, and most important, we introduce a new model of weighted automata, which turns out to describe precisely the recognizable text series. These automata permit us to drop the restrictions on the semiring we had to use in the main result of [49]. This main result now holds for any commutative semiring.

2. Recognizable series over general algebras

Let us start by fixing some notations. We assume in the following that Δ is an alphabet[1] and $\Sigma = (\Sigma, \mathrm{rk})$ is a ranked alphabet interpreted as a functional signature where $\mathrm{rk}(f) \in \mathbb{N}$ denotes the rank of f for all $f \in \Sigma$. We let $\Sigma^{(k)} = \{f \in \Sigma \mid \mathrm{rk}(f) = k\}$ for any $k \in \mathbb{N}$. A Σ-algebra $(\mathcal{C}, (f^{\mathcal{C}})_{f\in\Sigma})$ consists of a set $\mathcal{C}$ together with an *interpretation* $f^{\mathcal{C}} : \mathcal{C}^{\mathrm{rk}(f)} \to \mathcal{C}$ for any $f \in \Sigma$. If the operations are clear from the context, we simply denote $(\mathcal{C}, (f^{\mathcal{C}})_{f\in\Sigma})$ by $\mathcal{C}$. Interpretations of function symbols f of rank 0 are called *constants* and will be identified with an element $f^{\mathcal{C}} \in \mathcal{C}$. The set Δ^* of words together with concatenation $\cdot$ as a binary operation and the empty word ε as a constant forms an algebra. Alternatively, we may equip the set of words Δ^* with unary operations $\cdot a$ for any $a \in \Delta$, interpreted as the right-concatenation of letters, i.e. consider the algebra $(\Delta^*, (\cdot a)_{a\in\Delta})$. The free Σ-algebra over Δ is denoted $T_\Sigma(\Delta)$ and comprises all Σ-terms or equivalently all Σ-trees over Δ.

In the rest of this section, let $\mathcal{C}$ be a Σ-algebra.

Let us assume that Δ generates $\mathcal{C}$. We denote by $\eta_{\mathcal{C}} : T_\Sigma(\Delta) \to \mathcal{C}$ the unique epimorphism extending the identity $\mathrm{id}(\Delta)$ on Δ, called the *natural epimorphism of* $\mathcal{C}$. Let $X = \{x_1, x_2, \ldots\}$ and let $\tau \in T_\Sigma(\Delta \cup X)$. Let $x_{i_1}, \ldots, x_{i_k}$ with $i_1 < \ldots < i_k$ those elements of X that appear in τ. Since $T_\Sigma(\Delta \cup \{x_{i_1}, \ldots, x_{i_k}\})$ is the free algebra in the class of all Σ-algebras, any function $\alpha : \Delta \cup \{x_{i_1}, \ldots, x_{i_k}\} \to \mathcal{C}$ extends to a homomorphism α, in particular the function $\alpha_{s_1,\ldots,s_k}$ mapping x_{i_j} to s_j for all $1 \le j \le k$ and mapping a to itself for all $a \in \Delta$. Now, τ defines a k-ary operation (*polynomial function*) $\tau^{\mathcal{C}}$ on $\mathcal{C}$ given by $\tau^{\mathcal{C}}(s_1, \ldots, s_k) = \alpha_{s_1,\ldots,s_k}(\tau)$. Again we sometimes omit the superscript $\mathcal{C}$. We conclude that for any $\Sigma' \subseteq T_\Sigma(\Delta \cup X)$ we can turn $\mathcal{C}$ into a Σ'-algebra, again denoted by $\mathcal{C}$. In the following we will in particular consider the set $\mathrm{Ctx}(\Sigma, \Delta) \subseteq T_\Sigma(\Delta \cup \{x_1\})$ of *contexts*, which is the set of trees where x_1 appears exactly once.

2.1. Recognizable languages and recognizable series

The notion of a recognizable subset[2] of $\mathcal{C}$ is due to Mezei and Wright [55]. It generalizes the notion of a regular language, i.e. of a regular subset of a finitely generated free monoid, to arbitrary algebras.

Definition 2.1 (*Mezei & Wright [55]*). A language $L \subseteq \mathcal{C}$ is *(Σ-)recognizable* if there is a finite Σ-algebra $\mathscr{A}$ and a homomorphism $\varphi : \mathcal{C} \to \mathscr{A}$ such that $\varphi^{-1}(\varphi(L)) = L$.

We say that a language $L \subseteq \Delta^*$ is *regular* if there is a finite automaton $\mathcal{A}$ such that L is the language accepted by $\mathcal{A}$. By the well known Myhill–Nerode theorem, $L \subseteq \Delta^*$ is regular iff it is $\{\cdot\}$-recognizable (Myhill's characterization) iff it is $\{(\cdot a)_{a\in\Delta}\}$-recognizable (Nerode's characterization).

Similarly for $T_\Sigma(\Delta)$, a *tree automaton* is a tuple $\mathcal{A} = (Q, \delta, F)$, where Q is a finite set of states, $F \subseteq Q$ and $\delta = (\delta_f)_{f\in\Sigma\cup\Delta}$ is a family of sets $\delta_f \subseteq Q^{\mathrm{rk}(f)+1}$. Let $\delta_* \subseteq Q \times T_\Sigma(\Delta)$ be the smallest set such that $\delta_a \times \{a\} \subseteq \delta_*$ for any $a \in \Delta \cup \Sigma^{(0)}$, and for any $n \in \mathbb{N}$ and $f \in \Sigma^{(n)}$, if $(q_1, t_1), \ldots, (q_n, t_n) \in \delta_*$ and $(q_1, \ldots, q_n, q) \in \delta_f$, then $(q, f(t_1, \ldots, t_n)) \in \delta_*$. Let $\mathscr{L}(\mathcal{A}) = \{t \in T_{\Sigma(\Delta)} \mid \exists q \in F : (q, t) \in \delta_*\}$ denote the *language accepted by* $\mathcal{A}$. A language $L \subseteq T_\Sigma(\Delta)$ is *regular* if there is a tree automaton $\mathcal{A}$ such that $\mathscr{L}(\mathcal{A}) = L$. Again, a language $L \subseteq T_\Sigma(\Delta)$ is regular iff it is Σ-recognizable (see e.g. [32]). Moreover, recognizable languages of finitely generated algebras can be characterized by tree automata:

Proposition 2.2 (*cf. [11]*). *Let $\mathcal{C}$ be finitely generated by Δ. A language $L \subseteq \mathcal{C}$ is recognizable iff $\eta_{\mathcal{C}}^{-1}(L) \subseteq T_\Sigma(\Delta)$ is a regular tree language.*

Similarly to Definition 2.1, we introduce a concept of recognizability for (formal $\mathcal{C}$-)series, i.e. for functions from $\mathcal{C}$ to a semiring $\mathbb{K}$. For this, we follow ideas of Reutenauer [58] as well as Bozapalidis et al. [8,7,6], who studied series over words and trees, respectively. After that we motivate the definition by showing that it generalizes several concepts, namely the concept of regular word and tree series on the one hand and the notion of recognizable languages of arbitrary Σ-algebras, as defined above, on the other. Series over general algebras were also considered by Kuich [42], although with an emphasis on equationally defined series.

A *semiring* $\mathbb{K}$ is an algebraic structure $(\mathbb{K}, +, \cdot, 0, 1)$ such that $(\mathbb{K}, +, 0)$ is a commutative monoid, $(\mathbb{K}, \cdot, 1)$ is a monoid[3], multiplication distributes over addition (from both sides) and 0 is absorbing, i.e. $0 \cdot k = k \cdot 0 = 0$ for all $k \in \mathbb{K}$. If $\mathbb{K}$ admits additive inverses, i.e. if $(\mathbb{K}, +, 0)$ is a group, then $\mathbb{K}$ is a *ring*. So, for example, the set of integers with the usual operations form a ring $(\mathbb{Z}, +, \cdot, 0, 1)$. If multiplication is commutative, then $\mathbb{K}$ is a *commutative semiring*. For example, the smallest subsemiring of any semiring $\mathbb{K}$ is commutative. In particular, the semiring of natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$ forms a commutative semiring. If addition is idempotent, i.e. $k + k = k$ for all $k \in \mathbb{K}$, then $\mathbb{K}$ is an *idempotent semiring*. Important examples are given by the semiring of formal languages $(\mathscr{P}(\Delta^*), \cup, \cdot, \emptyset, \{\varepsilon\})$, where $\mathscr{P}(\Delta^*)$ denotes the powerset of Δ^* and $\cdot$ the concatenation of languages, the tropical semiring $(\mathbb{Z} \cup \{\infty\}, \min, +, \infty, 0)$ and the arctic or max-plus

1 We use the term alphabet to indicate that a set is finite.

2 We also refer to a subset $X \subseteq \mathcal{C}$ as a $\mathcal{C}$-*language* or simply *language* if $\mathcal{C}$ is clear from the context.

3 As usual $\cdot$ binds more strongly than $+$.

semiring ($\mathbb{Z} \cup \{-\infty\}$, max, $+$, $-\infty$, 0). The latter have been used to model real-time systems or discrete event systems. They are commutative and idempotent. Idempotent semirings possess the property that any finitely generated submonoid of $(\mathbb{K}, +, 0)$ is finite. Semirings with this latter property are called *additively locally finite*. For example, any field of characteristic $\neq 0$ is additively locally finite. Another important example for a commutative, idempotent, and thus additively locally finite semiring is the probabilistic semiring ([0, 1], max, $\cdot$, 0, 1). A semiring is *zero-sum free* if $k + k' = 0$ implies $k = 0$ or $k' = 0$. Moreover, we call a semiring *locally finite* if any finitely generated subsemiring is finite. For example, any Boolean algebra, the min-max semiring ($\mathbb{R}_+ \cup \{\infty\}$, max, min, 0, ∞) and the fuzzy semiring ([0, 1], max, min, 0, 1) are each locally finite. We denote by $\mathbb{B}$ the 2-valued Boolean algebra ($\{0, 1\}, \vee, \wedge, 0, 1$) and refer to it as the Boolean semiring.

In all of the following, let $\mathbb{K}$ *be a commutative semiring such that* $0 \neq 1$.

A $\mathbb{K}$-*semimodule* M is a commutative monoid $(M, +, 0)$ together with a scalar multiplication $. : \mathbb{K} \times M \to M$ such that for all $k, l \in \mathbb{K}$ and $m, n \in M$ we have

$$k.(m + n) = k.m + k.n, \qquad (k + l).m = k.m + l.m, \qquad (k \cdot l).m = k.(l.m),$$
$$1.m = m, \qquad 0.m = 0.$$

Note that we use the same symbols $+$ and 0 in both M and $\mathbb{K}$. Furthermore, observe that from these axioms we get $k.0 = k.(0.0) = (k \cdot 0).0 = 0.0 = 0$ for all $k \in \mathbb{K}$. We interpret $\mathbb{K}$-semimodules as algebras over the signature $(+, (k.)_{k\in\mathbb{K}})$. If M is generated by some set X, we say that X *spans* M. If $\mathbb{K}$ is a ring, then M is called a $\mathbb{K}$-*module*. A module having only finitely generated submodules is called *Noetherian*.

Example 2.3. 1. Any commutative monoid can naturally be interpreted as an $\mathbb{N}$-semimodule (and vice versa: every $\mathbb{N}$-semimodule can be obtained this way), every commutative semiring is a semimodule over itself and every commutative ring is a module over itself. In the latter case, the submodules are the ideals as considered in classical algebra and the finitely generated submodules are precisely the finitely generated ideals. A commutative ring is *Noetherian* if it is a Noetherian module over itself.

2. For any set Q and any $\mathbb{K}$-semimodule M the set M^Q of all mappings from Q to M forms again a semimodule by letting $(f+g)(q) = f(q)+g(q)$ and $(k.f)(q) = k.f(q)$ for any $f, g \in M^Q$, $k \in \mathbb{K}$ and $q \in Q$. The unit element is the zero mapping $0 : q \mapsto 0$. In particular $\mathbb{K}^Q$ and $\mathbb{K}^{\mathcal{C}}$ form $\mathbb{K}$-semimodules. They can be turned into semirings using the pointwise product $\odot$. In the following, we denote $\mathbb{K}^{\mathcal{C}}$ by $\mathbb{K}\langle\langle\mathcal{C}\rangle\rangle$. For $S \in \mathbb{K}\langle\langle\mathcal{C}\rangle\rangle$ and $s \in \mathcal{C}$ we write (S, s) for the value of S at s. Elements of $\mathbb{K}\langle\langle\mathcal{C}\rangle\rangle$ are called *(formal* $\mathcal{C}$*-)series*, and elements of $\mathbb{K}\langle\langle\Delta^*\rangle\rangle$ are called *formal power series*.
3. The *support* of a series $S : \mathcal{C} \to \mathbb{K}$ is the set $\text{supp}(S) = \{s \in \mathcal{C} \mid (S, s) \neq 0\}$. The set $\mathbb{K}\langle\mathcal{C}\rangle$ of series with finite support is a subsemimodule of $\mathbb{K}\langle\langle\mathcal{C}\rangle\rangle$. It is the free $\mathbb{K}$-semimodule over $\mathcal{C}$. Hence, any function $S : \mathcal{C} \to \mathbb{K}$ extends linearly to $\mathbb{K}\langle\mathcal{C}\rangle$. We will not distinguish between S and its linear extension. A series $P \in \mathbb{K}\langle\mathcal{C}\rangle$ is called a *polynomial*.

Let M, M' be two $\mathbb{K}$-semimodules. Let $n \in \mathbb{N}_+$. A mapping $\mu : M^n \to M'$ is *multilinear* if $\mu(m_1, \dots, k.m_i + m_i', \dots, m_n) = k.\mu(m_1, \dots, m_i, \dots, m_n) + \mu(m_1, \dots, m_i', \dots, m_n)$ for all $1 \leq i \leq n$, $m_i', m_1, \dots, m_n \in M$ and $k \in \mathbb{K}$. A multilinear mapping $\mu : M \to M'$ is also called *linear* and is nothing but a semimodule homomorphism. A linear mapping $\mu : M \to \mathbb{K}$ is called a *linear form*. Alexandrakis and Bozapalidis [7] introduced the notion of a $\mathbb{K}$-Σ-algebra. A $\mathbb{K}$-Σ*-algebra* $\mathscr{A} = (\mathscr{A}, (f)_{f\in\Sigma})$ consists of a $\mathbb{K}$-semimodule $\mathscr{A}$ together with multilinear operations f (of rank $\text{rk}(f)$). We interpret $\mathbb{K}$-Σ-algebras as algebras over the signature $(+, (k\cdot)_{k\in\mathbb{K}}, (f)_{f\in\Sigma})$. Thus, any $\mathbb{K}$-Σ-algebra is also a Σ-algebra. A homomorphism of $\mathbb{K}$-Σ-algebras is called a $\mathbb{K}$-Σ*-homomorphism* and a congruence is called a $\mathbb{K}$-Σ*-congruence*. A $\mathbb{K}$-Σ-algebra $\mathscr{A}$ is said to have *finite rank* if $\mathscr{A}$ is a finitely generated $\mathbb{K}$-semimodule, i.e. if there is a finite set spanning $\mathscr{A}$.

Example 2.4. 1. Let Q be a finite set. The $\mathbb{K}$-semimodule $\mathbb{K}^{Q\times Q}$ can be turned into a $\mathbb{K}$-$\{\cdot\}$-algebra (where $\cdot$ is binary) by means of the usual matrix multiplication, i.e. by letting $(m \cdot m')_{i,j} = \sum_{k\in Q} m_{i,k} \cdot m'_{k,j}$ for any $m, m' \in \mathbb{K}^{Q\times Q}$ and $i, j \in Q$. In particular, $\cdot$ is associative, i.e. $(\mathbb{K}^{Q\times Q}, \cdot, E)$ is a monoid where E denotes the usual unit matrix. In fact, together with the componentwise addition $\mathbb{K}^{Q\times Q}$ forms a semiring.

2. We equip the $\mathbb{K}$-semimodule $\mathbb{K}\langle\mathcal{C}\rangle$ with multilinear operations in order to make it a $\mathbb{K}$-Σ-algebra. We define for all $n \in \mathbb{N}$, $f \in \Sigma^{(n)}$, $P_1, \dots, P_n \in \mathbb{K}\langle\mathcal{C}\rangle$ and $s \in \mathcal{C}$:

$$(f(P_1, \dots, P_n), s) = \sum_{\substack{s_1,\dots,s_n\in\mathcal{C} \\ f(s_1,\dots,s_n)=s}} (P_1, s_1) \cdot \ldots \cdot (P_n, s_n).$$

Note that, as the P_i are polynomials, the sum is in fact finite. It is not hard to see that this definition indeed gives a multilinear operation for each $f \in \Sigma$. Hence, $\mathbb{K}\langle\mathcal{C}\rangle$ is a $\mathbb{K}$-Σ-algebra and thus a Σ-algebra. Identifying $s \in \mathcal{C}$ with the polynomial that maps s to 1 and any other element of $\mathcal{C}$ to 0, $\mathcal{C}$ becomes a subalgebra of $\mathbb{K}\langle\mathcal{C}\rangle$.
3. Let R be a commutative ring that is finitely generated over Δ. Let $(M, \cdot, 1)$ be the free commutative monoid over Δ. Then $\mathbb{K}\langle M\rangle$ is a $\mathbb{K}$-$\{\cdot\}$-algebra. It is again a semiring; in particular, $\mathbb{N}\langle M\rangle$ is the free commutative semiring over Δ. If $\mathbb{K}$ is a ring, then $\mathbb{K}\langle M\rangle$ is a ring and $\mathbb{Z}\langle M\rangle$ is the free commutative ring over Δ[4]. Clearly, $\mathbb{Z}$ is Noetherian, and thus by Hilbert's Basis

[4] $\mathbb{Z}\langle M\rangle$ is the ring of polynomials over commuting variables Δ as considered in classical algebra.

Theorem $\mathbb{Z}\langle M\rangle$ is Noetherian. As $\mathbb{Z}\langle M\rangle$ is free over Δ, the identity on Δ extends to an epimorphism from $\mathbb{Z}\langle M\rangle$ onto R. We conclude that R is Noetherian, too, since the preimage of any ideal I is an ideal of $\mathbb{Z}\langle M\rangle$ which is finitely generated by some set G, the image of which generates I.

4. It is not hard to see that $\mathbb{K}\langle T_\Sigma(\Delta)\rangle$ is the free $\mathbb{K}$-Σ-algebra over Δ. Hence, for any $\mathbb{K}$-Σ-algebra $\mathscr{A}$, any mapping $\mu : \Delta \to \mathscr{A}$ extends uniquely to a $\mathbb{K}$-Σ-homomorphism $\mu_{\mathscr{A}} : \mathbb{K}\langle T_\Sigma(\Delta)\rangle \to \mathscr{A}$.

Next, we define the notion of a recognizable series and show that it generalizes Definition 2.1.

Definition 2.5. A series $S : \mathcal{C} \to \mathbb{K}$ is *(Σ-)recognizable* if there is a $\mathbb{K}$-Σ-algebra $\mathscr{A}$ of finite rank, a Σ-homomorphism $\varphi : \mathcal{C} \to \mathscr{A}$ and a linear form $\gamma : \mathscr{A} \to \mathbb{K}$ such that $\gamma \circ \varphi = S$. We call the pair (φ, γ) a *representation* of S.

Remark 2.6. We note that as for Definition 2.1 the definition is independent of the set of constants. Moreover, if $\varphi(s) = \varphi(s')$, then $\gamma \circ \varphi(s) = \gamma \circ \varphi(s')$. Hence, $\ker(\varphi) \subseteq \ker(S)$, where for any function f the kernel of f is the equivalence relation $\ker(f) = \{(x, y) \mid f(x) = f(y)\}$.

First, we show that Definition 2.5 generalizes Definition 2.1. For a language $L \subseteq \mathcal{C}$, let $\mathbb{1}_L : \mathcal{C} \to \mathbb{K}$ denote the characteristic series of L whose value is given for all $s \in \mathcal{C}$ by $(\mathbb{1}_L, s) = 1$ if $s \in L$ and 0 otherwise. As mentioned before, we identify $s \in \mathcal{C}$ with $\mathbb{1}_{\{s\}}$.

Proposition 2.7. *Let $L \subseteq \mathcal{C}$ be recognizable. Then $\mathbb{1}_L$ is recognizable.*

Proof. Let $L \subseteq \mathcal{C}$ be recognizable. There is thus a finite Σ-algebra $\mathscr{A}$ and a Σ-homomorphism $\varphi : \mathcal{C} \to \mathscr{A}$ such that $\varphi^{-1}(\varphi(L)) = L$. Since $\mathscr{A}$ is a subalgebra of $\mathbb{K}\langle\mathscr{A}\rangle$ (cf. Example 2.4(2)), we may interpret φ as a homomorphism from $\mathcal{C}$ to $\mathbb{K}\langle\mathscr{A}\rangle$. Define $\gamma : \mathscr{A} \to \mathbb{K}$ by letting $\gamma(m) = 1$ if $m \in \varphi(L)$ and $\gamma(m) = 0$ otherwise. Since $\mathbb{K}\langle\mathscr{A}\rangle$ is the free semimodule over $\mathscr{A}$, there is a unique extension of γ to a linear form $\gamma : \mathbb{K}\langle\mathscr{A}\rangle \to \mathbb{K}$. Clearly, $\gamma \circ \varphi(s) = (\mathbb{1}_{\varphi(L)}, \varphi(s)) = (\mathbb{1}_L, s)$ for any $s \in \mathcal{C}$. □

The following proposition is well known for words [60,63] and trees [47, Inverse Image Theorem].

Proposition 2.8. *Let $\mathcal{C}$ be finitely generated by Δ. Let $\mathbb{K}$ be a locally finite semiring or let $\mathbb{K}$ be a ring and let $S : \mathcal{C} \to \mathbb{K}$ be recognizable such that $S(\mathcal{C}) \subseteq \mathbb{K}$ is finite. Moreover, let $A \subseteq \mathbb{K}$. Then $S^{-1}(A)$ is recognizable.*

Proof (*Similar to [47, Inverse Image Theorem] and [2, Theorem 2.7]*). Let a representation of S be given by $(\varphi : \mathcal{C} \to \mathscr{A}, \gamma)$. Assume that $\mathscr{A}$ is spanned by $m_1, \ldots, m_n$. Now, let $f \in \Sigma \cup \Delta$ with $\mathrm{rk}(f) = k$ and let $i_1, \ldots, i_k \in [n]$. Then choose some $\delta_f(i_1, \ldots, i_k)_j \in \mathbb{K}$ such that $f(m_{i_1}, \ldots, m_{i_k}) = \sum_{1\le j\le n} \delta_f(i_1, \ldots, i_k)_j.m_j$. Let L be the semiring, respectively ring, generated by the finite set $\{\delta_f(i_1, \ldots, i_k)_j \mid f \in \Sigma \cup \Delta, j, i_1, \ldots, i_k \in [n]\} \cup \{\gamma(m_1), \ldots, \gamma(m_n)\}$. For any $\tau \in \mathrm{Ctx}(\Sigma, \Delta)$, let $\varphi(\tau) \in \mathrm{Ctx}(\Sigma, \varphi(\Delta))$ be given by replacing a in τ by $\varphi(a)$ for all $a \in \Delta$. Consider the mapping $\mu : \mathrm{Ctx}(\Sigma, \Delta) \to L^n$ given by $\mu(\tau)_i = \gamma(\varphi(\tau)(m_i))$ $(1 \le i \le n)$ and its linear extension $\mu : L\langle\mathrm{Ctx}(\Sigma, \Delta)\rangle \to L^n$. If $\mathbb{K}$ is locally finite, L is again finite. If $\mathbb{K}$ is a ring, L is Noetherian (cf. Example 2.4(3)). Hence, the L-semimodule L^n is either finite or a Noetherian L-module by [45, Proposition X.1.4]). There are thus $l \in \mathbb{N}$ and $\tau_1, \ldots, \tau_l \in \mathrm{Ctx}(\Sigma, \Delta)$ such that $\mu(\tau_1), \ldots, \mu(\tau_l)$ span the L-subsemimodule $\mu(L\langle\mathrm{Ctx}(\Sigma, \Delta)\rangle)$. Without loss of generality, we assume that τ_1 is just the distinguished variable x_1.

Consider $X = \{((S, \tau_i(s)))_{1\le i\le l} \mid s \in \mathcal{C}\} \subseteq L^l$, which is finite by assumption. We will define Σ-operations on X. For $f \in \Sigma$ with $\mathrm{rk}(f) = k$ and $s_1, \ldots, s_k \in \mathcal{C}$, let $f((S, \tau_i(s_1))_{1\le i\le l}, \ldots, (S, \tau_i(s_k))_{1\le i\le l}) = ((S, \tau_i(f(s_1, \ldots, s_k))))_{1\le i\le l})$. These operations are well defined. Indeed, let $s'_1, \ldots, s'_k \in \mathcal{C}$ with $(S, \tau_i(s_1))_{1\le i\le l} = (S, \tau_i(s'_1))_{1\le i\le l}, \ldots, (S, \tau_i(s_k))_{1\le i\le l} = (S, \tau_i(s'_k))_{1\le i\le l}$. We need to show that $((S, \tau_i(f(s_1, \ldots, s_k))))_{1\le i\le l}) = ((S, \tau_i(f(s'_1, \ldots, s'_k))))_{1\le i\le l})$. Using the facts that the operations on $\mathscr{A}$ are multilinear, that $\mu(\tau_1), \ldots, \mu(\tau_l)$ span $\mu(L\langle\mathrm{Ctx}(\Sigma, \Delta)\rangle)$ and that Δ generates $\mathcal{C}$, we get that for $1 \le i \le l$ there are $\tau \in \mathrm{Ctx}(\Sigma, \Delta)$ as well as $\lambda_1, \ldots, \lambda_n, \lambda'_1, \ldots, \lambda'_l \in L$ such that

$$\begin{aligned}(S, \tau_i(f(s_1, \ldots, s_k))) &= \gamma(\varphi(\tau)(\varphi(s_1))) = \sum_{1\le j\le n} \gamma(\varphi(\tau)(\lambda_j.m_j)) = \sum_{1\le j\le n} \lambda_j \cdot \mu(\tau)_j \\ &= \sum_{1\le j\le n} \lambda_j \cdot \sum_{1\le j'\le l} \lambda'_{j'} \cdot \mu(\tau_{j'})_j = \sum_{1\le j'\le l} \lambda'_{j'} \cdot \sum_{1\le j\le n} \lambda_j \cdot \mu(\tau_{j'})_j = \sum_{1\le j'\le l} \lambda'_{j'} \cdot (S, \tau_{j'}(s_1)).\end{aligned}$$

Similarly, we get $(S, \tau_i(f(s'_1, \ldots, s_k))) = \sum_{1\le j'\le l} \lambda'_{j'} \cdot (S, \tau_{j'}(s'_1))$ and hence $(S, \tau_i(f(s_1, \ldots, s_k))) = (S, \tau_i(f(s'_1, \ldots, s_k)))$. Applying this argument successively k-times gives $(S, \tau_i(f(s_1, \ldots, s_k))) = (S, \tau_i(f(s'_1, \ldots, s'_k)))$. Since this holds for all $1 \le i \le l$, we conclude that the operations are well defined.

Now, let $h : \mathcal{C} \to X$ be given by $h(s) = ((S, \tau_i(s))_{1\le i\le l})$. Clearly, this is a homomorphism. Since $S^{-1}(A) = h^{-1}(\{(k_i)_{1\le i\le l}) \mid k_1 \in A\})$ we conclude that $S^{-1}(A)$ is recognizable. □

Remark 2.9. Note that in particular $L \subseteq \mathcal{C}$ is recognizable iff $\mathbb{1}_L : \mathcal{C} \to \mathbb{B}$ is recognizable. Moreover, $\mathbb{1}_\emptyset$ and $\mathbb{1}_\mathcal{C}$ are recognizable for any semiring $\mathbb{K}$.

We now show that the proposed notion of recognizable series coincides with the well-known notion of the behavior of a weighted automaton over words and the behavior of a weighted tree automaton (over trees in $T_\Sigma(\Delta)$). For an overview on weighted automata over words the reader is referred to the survey articles [59,43,2,41,15]. For an overview on weighted automata over trees the reader may consult [29,15]. A *weighted automaton* is a tuple $\mathcal{A} = (Q, \lambda, \mu, \gamma)$ such that Q is a finite set of states, $\lambda, \gamma : Q \to \mathbb{K}$ and $\mu : \Delta \to \mathbb{K}^{Q \times Q}$. Since Δ^* is the free monoid over Δ, there is a unique monoid homomorphism $\mu : \Delta^* \to \mathbb{K}^{Q\times Q}$ extending μ. Now, the *behavior* $\| \mathcal{A} \|: \Delta^* \to \mathbb{K}$ of $\mathcal{A}$ is defined by $(\| \mathcal{A} \|, w) = \sum_{q,q' \in Q} \lambda(q) \cdot \mu(w)_{q,q'} \cdot \gamma(q')$ for all $w \in \Delta^*$. We say that a formal power series $S : \Delta^* \to \mathbb{K}$ is *regular* if it is the behavior of a weighted automaton.

Proposition 2.10. *A formal power series $S : \Delta^* \to \mathbb{K}$ is regular iff it is $\{\cdot\}$-recognizable iff it is $\{(\cdot a)_{a\in\Delta}\}$-recognizable.*

Proof. Let $S = \|\mathcal{A}\|$ for a weighted automaton $\mathcal{A} = (Q, \lambda, \mu, \gamma)$. Define the linear form $\kappa(m) = \sum_{q,q'\in Q} \lambda(q) \cdot m_{q,q'} \cdot \gamma(q')$. Then $\kappa \circ \mu = S$ and thus S is $\{\cdot\}$-recognizable.

Let S be $\{\cdot\}$-recognizable. Since the operation $\cdot a$ is the polynomial function of the context $\cdot(x_1, a)$, any $\{\cdot\}$-homomorphism is an $\{(\cdot a)_{a\in\Delta}\}$-homomorphism. Thus S is $\{(\cdot a)_{a\in\Delta}\}$-recognizable.

We now show that S is regular if S is $\{(\cdot a)_{a\in\Delta}\}$-recognizable. A similar idea was used in [2, Lemma 1.2]. Let S be $\{(\cdot a)_{a\in\Delta}\}$-recognizable. There is thus a $\mathbb{K}$-$\{(\cdot a)_{a\in\Delta}\}$-algebra $\mathscr{A}$ of finite rank spanned by $m_1, \ldots, m_n \in \mathscr{A}$, a $\{(\cdot a)_{a\in\Delta}\}$-homomorphism $\varphi : \Delta^* \to \mathscr{A}$ and a linear form $\gamma : \mathscr{A} \to \mathbb{K}$ such that $\gamma \circ \varphi = S$. For all $1 \le i, j \le n$ and $a \in \Delta$ choose $\mu(a)_{i,j} \in \mathbb{K}$ such that $m_i \cdot a = \sum_{j=1}^{n} \mu(a)_{i,j}.m_j$. Now extend μ to a monoid homomorphism $\mu : \Delta^* \to \mathbb{K}^{n\times n}$. Moreover, choose $\lambda(i) \in \mathbb{K}$ $(1 \le i \le n)$ such that $\varphi(\varepsilon) = \sum_{i=1}^{n} \lambda(i).m_i$. This defines $\lambda : [n] \to \mathbb{K}$. Let $\kappa : [n] \to \mathbb{K}$ be given by $\kappa(i) = \gamma(m_i)$ $(1 \le i \le n)$. We show by induction on $w \in \Delta^*$ that $\varphi(w) = \sum_{1\le i,j\le n} \lambda(i) \cdot \mu(w)_{i,j}.m_j$. For $w = \varepsilon$ this is clear by definition. Now, for the induction step suppose that the claim holds for u and let $w = u \cdot a$ for some $a \in \Delta$. Then $\varphi(ua) = \varphi(u) \cdot a = (\sum_{1\le i,k\le n} \lambda(i) \cdot \mu(u)_{i,k}.m_k) \cdot a = \sum_{1\le i,k\le n} \lambda(i) \cdot \mu(u)_{i,k}.(m_k \cdot a) = \sum_{1\le i,k,j\le n} \lambda(i) \cdot \mu(u)_{i,k} \cdot \mu(a)_{k,j}.m_j = \sum_{1\le i,j\le n} \lambda(i) \cdot \mu(ua)_{i,j}.m_j$. We conclude that $\gamma(\varphi(w)) = \sum_{1\le i,j\le n} \lambda(i) \cdot \mu(w)_{i,j} \cdot \kappa(j)$. Hence, S is regular. □

A *weighted tree automaton* $\mathcal{A}$ is a tuple (Q, δ, κ), where Q is a finite set of states, $\kappa : Q \to \mathbb{K}$ and $\delta = (\delta_f)_{f\in\Sigma\cup\Delta}$ is a family of mappings $\delta_f : Q^{\mathrm{rk}(f)} \to \mathbb{K}^Q$. Let $f \in \Sigma$ with $\mathrm{rk}(f) = k$. We extend δ_f to $\delta_f : \underbrace{\mathbb{K}^Q \times \ldots \times \mathbb{K}^Q}_{k} \to \mathbb{K}^Q$ by letting, for all $v_1, \ldots, v_k \in \mathbb{K}^Q$ and $q \in Q$,

$$\delta_f(v_1, \ldots, v_k)_q = \sum_{q_1,\ldots,q_k\in Q} \delta_f(q_1, \ldots, q_k)_q \cdot (v_1)_{q_1} \cdot \ldots \cdot (v_k)_{q_k}.$$

Note that the δ_f are multilinear. Hence, they turn $\mathbb{K}^Q$ into a $\mathbb{K}$-Σ-algebra and in particular into a Σ-algebra. As $T_\Sigma(\Delta)$ is the free Σ-algebra over Δ, there is a unique homomorphism $\delta : T_\Sigma(\Delta) \to \mathbb{K}^Q$ extending $\delta : \Delta \to \mathbb{K}^Q : a \mapsto \delta_a$. Now, the *behavior* $\|\mathcal{A}\|: T_\Sigma(\Delta) \to \mathbb{K}$ of $\mathcal{A}$ is defined by $(\|\mathcal{A}\|, t) = \sum_{q\in Q} \delta(t)_q \cdot \kappa(q)$. We say that a formal tree series is *regular* if it is the behavior of a weighted tree automaton. The following proposition was established by Borchardt for fields [5].

Proposition 2.11. *Let $S : T_\Sigma(\Delta) \to \mathbb{K}$. Then S is regular iff it is recognizable.*

Proof. (*Only if*). Immediately clear from the definition of a weighted tree automaton.

(*If*). Let $(\varphi : T_\Sigma(\Delta) \to \mathscr{A}, \gamma)$ be a representation of S such that $\mathscr{A}$ is a $\mathbb{K}$-Σ-algebra of finite rank spanned by $m_1, \ldots, m_n \in \mathscr{A}$. We set $Q = [n]$. Let $f \in \Sigma \cup \Delta$ with $\mathrm{rk}(f) = k$ and let $i_1, \ldots, i_k \in Q$. Then choose $\delta_f(i_1, \ldots, i_k)_j \in \mathbb{K}$ such that $f(m_{i_1}, \ldots, m_{i_k}) = \sum_{1\le j\le n} \delta_f(i_1, \ldots, i_k)_j.m_j$. This defines $\delta_f : Q^k \to \mathbb{K}^Q$. We define $\kappa : Q \to \mathbb{K}$ by setting $\kappa(i) = \gamma(m_i)$ for all $i \in Q$. Let $\mathcal{A} = (Q, \delta, \kappa)$. By induction we show that $\varphi(t) = \sum_{j\in Q} \delta(t)_j.m_j$. For $a \in \Delta \cup \Sigma^{(0)}$ this is the definition. Now, for the induction step let $k \in \mathbb{N}$ and $f \in \Sigma^{(k)}$. Suppose that the claim holds for $t_1, \ldots, t_k$. Then

$$\begin{aligned}
\varphi(f(t_1, \ldots, t_k)) &= f(\varphi(t_1), \ldots, \varphi(t_k)) = f\left(\sum_{j\in Q} \delta(t_1)_j.m_j, \ldots, \sum_{j\in Q} \delta(t_k)_j.m_j\right) \\
&= \sum_{i_1,\ldots,i_k\in Q} \delta(t_1)_{i_1} \cdots \delta(t_k)_{i_k}.f(m_{i_1}, \ldots, m_{i_k}) \\
&= \sum_{i_1,\ldots,i_k\in Q} \delta(t_1)_{i_1} \cdots \delta(t_k)_{i_k}. \sum_{j\in Q} \delta_f(i_1, \ldots, i_k)_j.m_j \\
&= \sum_{j\in Q} \sum_{i_1,\ldots,i_k\in Q} \delta_f(i_1, \ldots, i_k)_j \cdot \delta(t_1)_{i_1} \cdots \delta(t_k)_{i_k}.m_j = \sum_{j\in Q} \delta(t)_j.m_j.
\end{aligned}$$

Hence, $\|\mathcal{A}\| = \gamma \circ \varphi = S$. □

Let us remark that weighted nested word automata and weighted hedge automata have also been characterized in the spirit of the last proposition [52].

In the following lemma we collect some immediate observations concerning closure properties of the class of recognizable series for future reference. Let $\mathcal{C}'$ be a Σ-algebra and $\psi : \mathcal{C} \to \mathcal{C}'$ a function. Moreover, let $S \in \mathbb{K}\langle\langle \mathcal{C}' \rangle\rangle$ (cf. Example 2.3(2)). Then $\psi^{-1}(S)$ denotes the composition $S \circ \psi$.

Lemma 2.12. *Let $\mathcal{C}, \mathcal{C}'$ be Σ-algebras, let $S, T : \mathcal{C} \to \mathbb{K}$ and let $k \in \mathbb{K}$. Let $\psi : \mathcal{C}' \to \mathcal{C}$ be a homomorphism and let $\Sigma' \subseteq T_\Sigma(\Delta \cup X)$.*

1. *If S and T are Σ-recognizable, then so are $k.S$ and $S + T$.*
2. *If S is Σ-recognizable, then so is $\psi^{-1}(S)$.*
3. *S is Σ'-recognizable if it is Σ-recognizable.*
4. *If $\Sigma \subseteq \Sigma'$, then S is Σ'-recognizable iff it is Σ-recognizable.*
5. *If $\mathcal{C}'$ is a subalgebra of $\mathcal{C}$ and S is Σ-recognizable, then $S_{|\mathcal{C}'}$ is Σ-recognizable.*

2.2. Syntactic congruences and syntactic algebras

In this subsection, let us assume that the finite set Δ generates $\mathcal{C}$. As the kernel of a homomorphism is a congruence and conversely any congruence gives rise to a homomorphism, namely the natural homomorphism, we immediately get that $L \subseteq \mathcal{C}$ is recognizable iff L is a union of congruence classes of a finite index congruence. We can now associate to a language L a canonical homomorphism, respectively congruence, namely the coarsest congruence such that L is the union of equivalence classes. This congruence $\sim_L$ is called the *syntactic congruence*. It can also be given explicitly by defining $s \sim_L s'$ if $\tau^{\mathcal{C}}(s) \in L \Leftrightarrow \tau^{\mathcal{C}}(s') \in L$ for any $\tau \in \mathrm{Ctx}(\Sigma, \Delta)$. The syntactic congruence of some regular language $L \subseteq \Delta^*$, where Δ^* is equipped with right-concatenation, describes the minimal deterministic automaton of L.

A similar notion of a syntactic ideal was also considered by Reutenauer for formal power series over rings [58]. Bozapalidis et al. considered the syntactic ideal of a tree series over fields [8,7,6]. We note that the syntactic ideal is not a generalization of the syntactic congruence for languages. In particular, over the Boolean semiring it does not correspond to a deterministic automaton. However, this is the case for the syntactic congruence of a regular language. Here, we generalize the notion of a syntactic ideal to the notion of a syntactic congruence and to the notion of a syntactic algebra for series $S : \mathcal{C} \to \mathbb{K}$. We note that our definition was independently used by Fülöp and Steinby [31] for series over fields. Let $S : \mathcal{C} \to \mathbb{K}$ and let $\sim_S = \{(P_1, P_2) \in \mathbb{K}\langle\mathcal{C}\rangle \times \mathbb{K}\langle\mathcal{C}\rangle \mid (S, \tau(P_1)) = (S, \tau(P_2)) \text{ for all } \tau \in \mathrm{Ctx}(\Sigma, \Delta)\}$[5]. It is not hard to see that this is a $\mathbb{K}$-Σ-congruence which we call the *syntactic congruence* of S. Let $\sim$ be any congruence contained in $\ker(S)$ and let $P_1 \sim P_2$. Then $\tau(P_1) \sim \tau(P_2)$ for any $\tau \in \mathrm{Ctx}(\Sigma, \Delta)$ as $\sim$ is a congruence. Therefore, we have $(S, \tau(P_1)) = (S, \tau(P_2))$ for all $\tau \in \mathrm{Ctx}(\Sigma, \Delta)$. This shows that $\sim \subseteq \sim_S$ and, hence, that $\sim_S$ is the coarsest congruence fully contained in $\ker(S)$. We define $\mathscr{A}_S = \mathbb{K}\langle\mathcal{C}\rangle / \sim_S$, the *syntactic* $\mathbb{K}$-Σ-*algebra* of S. Note that this definition is independent of the choice of Δ as a generating set of $\mathcal{C}$. Moreover, note that $\mathscr{A}_S$ is of finite rank iff there is a finite set $G \subseteq \mathcal{C}$ such that the congruence classes of elements in G span the syntactic algebra (as a semimodule).

Proposition 2.13. *A series $S : \mathcal{C} \to \mathbb{K}$ is recognizable if $\mathscr{A}_S$ is of finite rank.*

Proof. Let φ be the natural $\mathbb{K}$-Σ-epimorphism $\varphi : \mathbb{K}\langle\mathcal{C}\rangle \to \mathscr{A}_S$. Define $\gamma : \mathscr{A}_S \to \mathbb{K}$ by letting $\gamma([P]_{\sim_S}) = (S, P)$. This is well defined as $\ker(\varphi) \subseteq \ker(S)$. Moreover, γ is a linear form as $S : \mathbb{K}\langle\mathcal{C}\rangle \to \mathbb{K}$ is linear by definition. Clearly, $\gamma \circ \varphi = S$. □

Theorem 2.14. *Let $\mathbb{K}$ be a commutative ring or a commutative and locally finite semiring and let $\mathcal{C}$ be finitely generated by Δ. A series $S : \mathcal{C} \to \mathbb{K}$ is recognizable iff $\mathscr{A}_S$ is of finite rank.*

Proof (*Similar to the Proof of Theorem II.1.2 in [58]*). By Proposition 2.13 we only have to show the only-if part. Let $(\varphi : \mathcal{C} \to \mathscr{A}, \gamma)$ be a representation for S and hence $\ker(\varphi) \subseteq \ker(S)$. Assume that $\mathscr{A}$ is spanned by $m_1, \ldots, m_n$. We extend φ linearly to $\varphi : \mathbb{K}\langle\mathcal{C}\rangle \to \mathscr{A}$. Now, let $f \in \Sigma \cup \Delta$ with $\mathrm{rk}(f) = k$ and let $i_1, \ldots, i_k \in [n]$. Then choose some $\delta_f(i_1, \ldots, i_k)_j \in \mathbb{K}$ such that $f(m_{i_1}, \ldots, m_{i_k}) = \sum_{1 \le j \le n} \delta_f(i_1, \ldots, i_k)_j . m_j$. Let L be the semiring, respectively ring, generated by the finite set $\{\delta_f(i_1, \ldots, i_k)_j \mid f \in \Sigma \cup \Delta, j, i_1, \ldots, i_k \in [n]\}$. As in the proof of Proposition 2.8, we conclude that the L-subsemimodule $L < m_1, \ldots, m_n >$ of $\mathscr{A}$ spanned by $m_1, \ldots, m_n$ using only coefficients from L is either finite or a Noetherian L-module. Thus, $\varphi(L\langle\mathcal{C}\rangle) \subseteq L < m_1, \ldots, m_n >$ is spanned by a finite set and consequently $\varphi(\mathbb{K}\langle\mathcal{C}\rangle) = \mathbb{K}\langle\mathcal{C}\rangle / \ker(\varphi)$ is a $\mathbb{K}$-Σ-algebra of finite rank. Since $\ker(\varphi) \subseteq \ker(S)$, we have $\ker(\varphi) \subseteq \sim_S$. This shows that $\mathscr{A}_S = \mathbb{K}\langle\mathcal{C}\rangle / \sim_S$ is of finite rank, too. □

Wang [66, Subsection 4] asked the question whether the last theorem holds for arbitrary commutative semirings in the case where $\mathcal{C}$ is a finitely generated free monoid. Next, we want to point out that the answer is negative. Let $S : \Delta^* \to \mathbb{K}$ be a formal power series. The *Hankel matrix* H_S of S is an infinitary matrix in $\mathbb{K}^{\Delta^* \times \Delta^*}$ given by $(H_S)_{u,v} = (S, uv)$. We may regard the set of words as the free monoid or we may equip it with right-concatenation. In the latter case, for two words w, w' we have $w \sim_S w'$ iff $(S, wu) = (S, w'u)$ for all $u \in \Delta^*$. Thus the syntactic algebra is isomorphic (as a $\mathbb{K}$-semimodule) to the subsemimodule of $\mathbb{K}\langle\langle\Delta^*\rangle\rangle$ spanned by the rows of the Hankel matrix. Hence, this submodule is of finite rank iff the syntactic algebra with respect to right-concatenation is of finite rank. The syntactic congruence in the monoid case, however, is a refinement of the syntactic congruence with respect to right-concatenation, and thus if the syntactic algebra in the monoid case is of finite rank, then so is the subsemimodule spanned by the rows of the Hankel matrix. Exercise II.3.1 in [59] shows that this does not need to be of finite rank even if S is recognizable. Therefore for both cases the theorem does not hold in general. In the following counterexamples we show this fact explicitly.

5 Here we switch from S to its linear extension $S : \mathbb{K}\langle\mathcal{C}\rangle \to \mathbb{K}$ (cf. Example 2.3).

Example 2.15 (*Counterexample, cf. [59, Exercise II.3.1]*). We consider the semiring of the natural numbers $\mathbb{N}$ and the regular formal power series $S : \{a\}^* \to \mathbb{N}$ given by $(S, a^n) = n$. Let $n \in \mathbb{N}$ and let $P = \sum_{1\le i\le m} k_i.a^i$ be a polynomial such that $a^n \sim_S P$ (where $\sim_S$ is defined with respect to the monoid structure of $\{a\}^*$). Then by definition $k + n + l = (S, a^k a^n a^l) = \sum_{1\le i\le m} k_i \cdot (S, a^k a^i a^l) = (k + l) \cdot \sum_{1\le i\le m} k_i + \sum_{1\le i\le m} k_i \cdot i$ for all $k, l \in \mathbb{N}$.For $k = l = 0$ we get $n = \sum_{1\le i\le m} k_i \cdot i$. For $k + l = 1$ we get $n + 1 = \sum_{1\le i\le m} k_i + \sum_{1\le i\le m} k_i \cdot i$. Thus $\sum_{1\le i\le m} k_i = 1$. We conclude that $a^n = P$. Hence, $a^n \sim_S P$ iff $P = a^n$. This shows that there is no finite set $F \subset \{a\}^*$ such that $\{[a^n]_{\sim_S} \mid a^n \in F\}$ spans the semimodule $\mathscr{A}_S$.

Example 2.16 (*Counterexample*). Now we consider the arctic semiring $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$ and the alphabet $\{a\}$. Consider the weighted automaton $\mathcal{A} = (Q, \lambda, \mu, \gamma)$ with $Q = \{1, 2\}$, where λ, μ and γ are given by

$$\lambda(1) = 0, \lambda(2) = -\infty, \qquad \mu(a)_{1,1} = \mu(a)_{2,2} = -\infty, \qquad \mu(a)_{1,2} = \mu(a)_{2,1} = 1, \qquad \gamma(1) = 0, \gamma(2) = -\infty.$$

The behavior of the automaton $\mathcal{A} = (Q, \lambda, \mu, \gamma)$ is

$$a^n \mapsto \begin{cases} n & \text{if } n \text{ is even} \\ -\infty & \text{otherwise.} \end{cases}$$

Consider the regular series S given by the pointwise sum of $\|\mathcal{A}\|$ and the characteristic function (with respect to the semiring $\mathbb{R}_{\max}$) of the regular language of words a^n such that n is odd. Thus S maps a^n to n if n is even and to 0 otherwise. By our results of the previous subsection, S is recognizable. Suppose for contradiction that the semimodule $\mathscr{A}_S$ is spanned by the congruence classes of the elements of some finite set $F \subset \{a\}^*$, i.e. $\mathscr{A}_S$ is spanned by $[a]_{\sim_S}, [a^2]_{\sim_S} \ldots, [a^n]_{\sim_S}$ for some $n \ge 1$. Hence, $[a^{2n}]_{\sim_S} = \sum_{1\le i\le n} k_i.[a^i]_{\sim_S}$ for some $k_i \in \mathbb{R}_{\max}$ (where the sum and product are taken in $\mathbb{R}_{\max}$). That is to say $(S, a^{2n+l}) = \max_{1\le i\le n} k_i + (S, a^{i+l})$ for all $l \in \mathbb{N}$. For $l = 1$ we obtain $0 = (S, a^{2n+1}) = \max_{1\le i\le n} k_i + (S, a^{i+1})$ and thus $k_i \le 0$ for all $1 \le i \le n$. But then for $l = 0$ we get $2n = (S, a^{2n}) \le \max_{1\le i\le n}(S, a^i) \le n$, a contradiction.

Lemma 2.17. *Let $\mathbb{K}$ be a commutative ring or a commutative and locally finite semiring and let $\mathcal{C}$ be finitely generated by Δ. Let $\mathcal{C}'$ be a Σ-algebra and let $\psi : \mathcal{C} \to \mathcal{C}'$ be an epimorphism. Then $S : \mathcal{C}' \to \mathbb{K}$ is recognizable iff $\psi^{-1}(S)$ is recognizable.*

Proof. (*Only if*). This is Lemma 2.12(2).

(*If*). Let $\psi^{-1}(S) : \mathcal{C} \to \mathbb{K}$ be recognizable. Hence, $\mathscr{A}_{\psi^{-1}(S)}$ is of finite rank by Theorem 2.14. We show that this implies that $\mathscr{A}_S$ is of finite rank, too. We may extend ψ linearly to a $\mathbb{K}$-Σ-homomorphism $\psi : \mathbb{K}\langle\mathcal{C}\rangle \to \mathbb{K}\langle\mathcal{C}'\rangle$. We have

$$\begin{aligned} P_1 \sim_{\psi^{-1}(S)} P_2 &\iff (\psi^{-1}(S), \tau(P_1)) = (\psi^{-1}(S), \tau(P_2)) \quad \text{for all } \tau \in \mathrm{Ctx}(\Sigma, \Delta) \\ &\iff (S, \psi(\tau(P_1))) = (S, \psi(\tau(P_2))) \quad \text{for all } \tau \in \mathrm{Ctx}(\Sigma, \Delta) \\ &\iff (S, \tau(\psi(P_1))) = (S, \tau(\psi(P_2))) \quad \text{for all } \tau \in \mathrm{Ctx}(\Sigma, \psi(\Delta)) \\ &\iff \psi(P_1) \sim_S \psi(P_2). \end{aligned}$$

There is, hence, an epimorphism from $\mathscr{A}_{\psi^{-1}(S)}$ to $\mathscr{A}_S$. Thus, we conclude that $\mathscr{A}_S$ is of finite rank, too, and therefore S must be recognizable by Proposition 2.13. □

Corollary 2.18. *Let $\mathbb{K}$ be a commutative ring or a commutative and locally finite semiring and let $\mathcal{C}$ be finitely generated by Δ. A series $S : \mathcal{C} \to \mathbb{K}$ is recognizable iff $\eta_{\mathcal{C}}^{-1}(S)$ is recognizable.*

If $\mathbb{K}$ is a field, a $\mathbb{K}$-semimodule is a vector space as considered in classical linear algebra and a $\mathbb{K}$-semimodule of finite rank is a finite-dimensional vector space. Now, if we have a representation $(\varphi : \mathcal{C} \to \mathscr{A}, \gamma)$ of some recognizable series $S : \mathcal{C} \to \mathbb{K}$, we may extend φ linearly to $\varphi : \mathbb{K}\langle\mathcal{C}\rangle \to \mathscr{A}$. Then $\varphi(\mathbb{K}\langle\mathcal{C}\rangle)$ is a $\mathbb{K}$-Σ-subalgebra of $\mathscr{A}$ and thus again a finite-dimensional vector space. Moreover, since $\ker(\varphi) \subseteq \ker(S)$ we have $\ker(\varphi) \subseteq \sim_S$, and thus there is an epimorphism from $\varphi(\mathbb{K}\langle\mathcal{C}\rangle)$ onto $\mathscr{A}_S$. We conclude that $\mathscr{A}_S$ has the smallest dimension among all representations. The syntactic algebra of some regular series $S : \Delta^* \to \mathbb{K}$ with respect to right-concatenation, which is isomorphic to the semimodule spanned by the columns of the Hankel matrix, can be used to construct a minimal automaton for S [2]. Moreover, we note that syntactic algebras were used to develop learning algorithms for recognizable formal power series and recognizable tree series over fields [1,33].

3. Relational structures and weighted logics

The connection between automata and logic, first considered by Büchi and Elgot [9,24], is of outstanding importance in theoretical computer science and led to many applications. However, a characterization of weighted automata in terms of logic was lacking for a long time, until in 2005 Droste and Gastin [13] considered so-called weighted logics. In this section we will consider weighted logics for arbitrary relational structures and investigate how different fragments behave with respect to translations of formulae. Let us start by recalling classical monadic second-order logic.

3.1. Classical monadic second-order logic

A *relational signature* (σ, ρ) consists of a set σ of *relation symbols*, each element of which is equipped with an *arity* through $\rho : \sigma \to \mathbb{N}_+$. We only write σ, if ρ is clear from the context. A σ*-structure* $s = (V(s), (R^s)_{R\in\sigma})$ consists of a set $V(s)$, its *domain*, together with a relation R^s of arity $\rho(R)$ for every relation symbol $R \in \sigma$. We also write R for R^s if s is clear. An *isomorphism* is a bijection $\varphi : V(s) \to V(s')$ between two structures s and s' such that for all $R \in \sigma$ and for all $v_1, \ldots, v_{\rho(R)} \in V(s)$ we have $(v_1, \ldots, v_{\rho(R)}) \in R^s$ iff $(\varphi(v_1), \ldots, \varphi(v_{\rho(R)})) \in R^{s'}$. In this case s and s' are said to be *isomorphic*. An *automorphism* φ of s is an isomorphism $\varphi : V(s) \to V(s)$.

In this section let $\mathcal{C}$ be a class of σ-structures. Subsequently, we only consider relational structures with finite domains. Moreover, we will distinguish relational structures only up to isomorphisms.

Example 3.1. 1. We identify a word $a_1 \ldots a_n \in \Delta^*$ with the relational structure $([n], (\mathrm{Lab}_a)_{a\in\Delta}, \leq)$, where $\mathrm{Lab}_a = \{i \in [n] \mid a_i = a\}$ $(a \in \Delta)$ and $\leq$ is the usual order on $[n]$.

2. Let $m \in \mathbb{N}$ be minimal such that $\mathrm{rk}(f) \leq m$ for all $f \in \Sigma$. We identify a tree $t \in T_\Sigma(\Delta)$ with a relational structure $(V(t), (\mathrm{Lab}_f)_{f\in\Sigma\cup\Delta}, (E_i)_{1\leq i\leq m})$. The domain of a tree is a finite, non-empty, prefix-closed subset of $(\mathbb{N}_+)^*$. Intuitively, a tree has unary relations Lab_f for the labeling and binary relations E_i such that $(x, y) \in E_i$ expresses that y is the i-th child of x. More precisely, the definition is given inductively as follows:
   (a) If $t = a$ for some $a \in \Sigma^{(0)} \cup \Delta$, then $V(t) = \{\varepsilon\}$, $\mathrm{Lab}^t_a = \{\varepsilon\}$ and $\mathrm{Lab}^t_f = E^t_i = \emptyset$ for all $f \in \Sigma \cup \Delta \setminus \{a\}$ and for all $1 \leq i \leq m$.
   (b) If $n \in \mathbb{N}_+, f \in \Sigma^{(n)}$ and $t = f(t_1, \ldots, t_n)$ for some $t_1, \ldots, t_n \in T_\Sigma(\Delta)$, then $V(t) = \{\varepsilon\} \cup \bigcup_{1\leq i\leq n} i \cdot V(t_i)$. Moreover, $\mathrm{Lab}^t_f = \{\varepsilon\} \cup \bigcup_{1\leq i\leq n} i \cdot \mathrm{Lab}^{t_i}_f$ and $\mathrm{Lab}^t_g = \bigcup_{1\leq i\leq n} i \cdot \mathrm{Lab}^{t_i}_g$ for all $g \in \Delta \cup \Sigma \setminus \{f\}$. Furthermore, $\mathrm{E}_i = \{(u, ui) \mid ui \in V(t)\}$ for all $1 \leq i \leq m$.

   We denote the corresponding signature by (E_i). A vertex $w \in V(t)$ which is a proper prefix of some $v \in V(t)$ is called an *inner vertex* or *inner node*. Any other vertex is called a *leaf*. The vertex ε is called the *root*.

Now we review classical monadic second-order logic for relational structures over the signature σ. The set $\mathrm{MSO}(\sigma)$ (if the signature is known from the context, then we sometimes drop σ) of *monadic second-order formulae* over σ is given by the following grammar.

$$\varphi ::= x = y \mid R(x_1, \ldots, x_{\rho(R)}) \mid x \in X \mid \varphi \vee \varphi \mid \neg\varphi \mid \exists x.\varphi \mid \exists X.\varphi$$

where R ranges over σ, where x, y, x_j are first-order variables and where X is a second-order variable. As usual we abbreviate $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$, $\varphi \to \psi = \neg\varphi \vee \psi$, $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$, $\forall x.\varphi = \neg(\exists x.\neg\varphi)$ and $\forall X.\varphi = \neg(\exists X.\neg\varphi)$ for any $\varphi, \psi \in \mathrm{MSO}(\sigma)$.

Let $\varphi \in \mathrm{MSO}(\sigma)$ and let $\mathrm{Free}(\varphi)$ denote the set of variables that occur free in φ. Let $\mathcal{V}$ be a finite set of first-order and second-order variables and let $s \in \mathcal{C}$. A $(\mathcal{V}, s)$*-assignment* γ is a mapping from $\mathcal{V}$ to $V(s) \cup \mathscr{P}(V(s))$ such that first-order variables are mapped to elements of $V(s)$ and second-order variables are mapped to subsets of $V(s)$. For $v \in V(s)$ and $T \subseteq V(s)$ we denote by $\gamma[x \to v]$ the $(\mathcal{V} \cup \{x\}, s)$-assignment which equals γ on $\mathcal{V} \setminus \{x\}$ and assumes v at x; by $\gamma[X \to T]$ we denote the $(\mathcal{V} \cup \{X\}, s)$-assignment which equals γ on $\mathcal{V} \setminus \{X\}$ and assumes T at X. Now, let $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$ and γ be a $(\mathcal{V}, s)$-assignment. Then we write $(s, \gamma) \models \varphi$ if φ holds true in s under the assignment γ. A closed formula, that is one without free variables, is called a *sentence*.

For a formula φ we write $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_m)$ to denote the fact that $\mathrm{Free}(\varphi) \subseteq \{x_1, \ldots, x_n, X_1, \ldots, X_m\}$. In this case we write $s \models \varphi[v_1, \ldots, v_n, T_1, \ldots, T_m]$ whenever we have $(s, \gamma) \models \varphi$ if $\gamma(x_i) = v_i$ and $\gamma(X_i) = T_i$. This is justified by the fact that $(s, \gamma) \models \varphi$ only depends on the restriction $\gamma_{|\mathrm{Free}(\varphi)}$ of γ to $\mathrm{Free}(\varphi)$. In particular, we simply write $s \models \varphi$ for sentences φ. A formula $\varphi(x_1, \ldots, x_k) \in \mathrm{MSO}(\sigma)$ which has only free first-order variables together with the ordered list of first-order variables $x_1, \ldots, x_k$ defines a relation φ^s on the domain of a σ-structure s as follows: $\varphi^s = \{(v_1, \ldots, v_k) \in V(s)^k \mid s \models \varphi[v_1, \ldots, v_k]\}$.

Now, let $\varphi \in \mathrm{MSO}(\sigma)$ and $\mathcal{V} \supseteq \mathrm{Free}(\varphi)$ be a finite set of variables, then $\mathscr{L}_\mathcal{V}(\varphi) = \{(s, \gamma) \mid (s, \gamma) \models \varphi\}$. Moreover, $\mathscr{L}(\varphi) = \mathscr{L}_{\mathrm{Free}(\varphi)}(\varphi)$. Two formulae $\varphi, \varphi' \in \mathrm{MSO}(\sigma)$ are said to be *equivalent* if $\mathscr{L}(\varphi) = \mathscr{L}(\varphi')$. Let $Z \subseteq \mathrm{MSO}(\sigma)$. A language $L \subseteq \mathcal{C}$ is *Z-definable (relatively to $\mathcal{C}$)* iff $L = \mathscr{L}(\varphi)$ for a sentence $\varphi \in Z$. Formulae containing no quantification at all are called *propositional*. First-order formulae, i.e. formulae in $\mathrm{MSO}(\sigma)$ containing only quantification over first-order variables, are collected in $\mathrm{FO}(\sigma)$. The class $\mathrm{EMSO}(\sigma)$ consists of all formulae φ of the form $\exists X_1. \ldots \exists X_m.\psi$, where $\psi \in \mathrm{FO}(\sigma)$.

Theorem 3.2 (*Büchi, Elgot [9,24]*). *Let Δ be an alphabet. Then $L \subseteq \Delta^*$ is regular iff it is* MSO*-definable iff it is* EMSO*-definable.*

Theorem 3.3 (*Thatcher & Wright, Doner [64,12]*). *Let Δ be an alphabet and let Σ be a ranked alphabet. Then $L \subseteq T_\Sigma(\Delta)$ is regular iff it is* MSO*-definable iff it is* EMSO*-definable.*

3.2. Weighted logics

Let us now come to weighted logics. We now define weighted monadic second-order logic, introduced in [13] for words, for relational structures over the signature σ. The set $\mathrm{MSO}(\mathbb{K},\sigma)$ (again we drop σ if it is known from the context) of *weighted* MSO *formulae* over $\mathbb{K}$ and σ is given by the following grammar:

$$\begin{aligned}\varphi ::= {} & k \mid x = y \mid R(x_1,\ldots,x_{\rho(R)}) \mid x \in X \mid \neg(x = y) \mid \neg R(x_1,\ldots,x_{\rho(R)}) \mid \neg(x \in X)\\ & \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x.\varphi \mid \exists X.\varphi \mid \forall x.\varphi \mid \forall X.\varphi\end{aligned}$$

where $k \in \mathbb{K}$, where R ranges over σ, where x, y, x_j are first-order variables and where X is a second-order variable. Note that we allow negation only for *atomic formulae* i.e. for the formulae $x = y$, $R(x_1,\ldots,x_{\rho(R)})$ and $x \in X$. This is because in general semirings we do not have a natural complement, and for this reason it will not be clear how to define the semantics of negation for values other than 0 and 1 (cf. [13]).

Definition 3.4. Let $\varphi \in \mathrm{MSO}(\mathbb{K},\sigma)$ and $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$. Moreover, let $s \in \mathcal{C}$ and let γ be a $(\mathcal{V}, s)$-assignment. The weighted semantics $[\![\varphi]\!]_{\mathcal{V}}$ of φ is a function assigning to each such pair (s,γ) an element of $\mathbb{K}$. It is given inductively as follows. For $k \in \mathbb{K}$ we put $[\![k]\!]_{\mathcal{V}}(s,\gamma) = k$. For all other atomic formulae and their negations φ the semantics $[\![\varphi]\!]_{\mathcal{V}}$ is given by the characteristic function $\mathbb{1}_{\mathscr{L}_{\mathcal{V}}(\varphi)}$. Moreover, we define

$$\begin{aligned}
([\![\varphi \vee \psi]\!]_{\mathcal{V}}, (s,\gamma)) &= ([\![\varphi]\!]_{\mathcal{V}}, (s,\gamma)) + ([\![\psi]\!]_{\mathcal{V}}, (s,\gamma)),\\
([\![\varphi \wedge \psi]\!]_{\mathcal{V}}, (s,\gamma)) &= ([\![\varphi]\!]_{\mathcal{V}}(s,\gamma)) \cdot ([\![\psi]\!]_{\mathcal{V}}, (s,\gamma)),\\
([\![\exists x.\varphi]\!]_{\mathcal{V}}, (s,\gamma)) &= \sum_{v \in V(s)} ([\![\varphi]\!]_{\mathcal{V}\cup\{x\}}, (s, \gamma[x \to v])),\\
([\![\exists X.\varphi]\!]_{\mathcal{V}}, (s,\gamma)) &= \sum_{T \subseteq V(s)} ([\![\varphi]\!]_{\mathcal{V}\cup\{X\}}, (s, \gamma[X \to T])),\\
([\![\forall x.\varphi]\!]_{\mathcal{V}}, (s,\gamma)) &= \prod_{v \in V(s)} ([\![\varphi]\!]_{\mathcal{V}\cup\{x\}}, (s, \gamma[x \to v])),\\
([\![\forall X.\varphi]\!]_{\mathcal{V}}, (s,\gamma)) &= \prod_{T \subseteq V(s)} ([\![\varphi]\!]_{\mathcal{V}\cup\{X\}}, (s, \gamma[X \to T])).
\end{aligned}$$

Recall that, by our general assumption, $V(s)$ is finite; hence the sums and products occurring above are defined. In the following we write $[\![\varphi]\!]$ for $[\![\varphi]\!]_{\mathrm{Free}(\varphi)}$. Observe that in the case where φ is a sentence, $[\![\varphi]\!]$ is a series from $\mathcal{C}$ to $\mathbb{K}$.

Remark 3.5. 1. A formula $\varphi \in \mathrm{MSO}(\mathbb{K},\sigma)$ which does not contain a subformula $k \in \mathbb{K}$ can also be interpreted as an element of $\mathrm{MSO}(\sigma)$. Conversely, any unweighted formula which only contains negations of atomic formulae is also a weighted formula. Note, moreover, that using the abbreviations introduced for classical MSO we can pull negation through to atomic formulae without altering the *unweighted* semantics.

2. Let $\mathbb{K}$ be the Boolean semiring $\mathbb{B}$. Then it is easy to see that weighted logics and classical monadic second-order logic coincide. In this case k is either 0 (false) or 1 (true). Sometimes it will be convenient to use these constants also in classical MSO.
3. As for classical MSO, one can easily see by induction that weighted formulae assign to isomorphic structures the same values. Hence, also for weighted MSO it is justified to consider relational structures only up to isomorphism.
4. Let $\varphi \in \mathrm{MSO}(\mathbb{K},\sigma)$ contain no subformula k ($k \in \mathbb{K}$). It can easily be shown by induction that, if $([\![\varphi]\!], (s,\gamma)) \neq 0$, then $(s,\gamma) \models \varphi$. Moreover, the converse also holds iff the least subsemiring of $\mathbb{K}$ is zero-sum free. This is in particular the case for idempotent semirings $\mathbb{K}$, where additionally we have $([\![\varphi]\!], (s,\gamma)) = 1$ iff $(s,\gamma) \models \varphi$.

Example 3.6. 1. Let $\mathbb{K} = \mathbb{N}$ be the semiring of the natural numbers. Let Δ be a finite alphabet, let $a \in \Delta$ and let $w = a_1 \ldots a_n \in \Delta^*$. Then $([\![\exists x.\mathrm{Lab}_a(x)]\!], w)$ counts the number of a's in w.

2. Again, let $\mathbb{K} = \mathbb{N}$, let Δ be a finite alphabet and let $w = a_1 \ldots a_n \in \Delta^*$. Consider the formula $\varphi = \forall x.\exists y.1$. Then $([\![\exists x.1]\!], w) = n$ and $([\![\forall y.\exists x.1]\!], w) = n^n$. The latter is not regular as it grows too fast (cf. Example 3.4 in [13]).

The following lemma shows that the definition of the semantics is consistent in the sense that for the value only the assignment of the free variables matter. It can easily be shown by induction.

Lemma 3.7 (*cf. [13]*). *Let $s \in \mathcal{C}$, let $\varphi \in \mathrm{MSO}(\mathbb{K},\sigma)$ and let $\mathcal{V} \supseteq \mathrm{Free}(\varphi)$ be a finite set. Moreover, let γ be a $(\mathcal{V}, s)$-assignment. Then*

$$([\![\varphi]\!]_{\mathcal{V}}, (s,\gamma)) = ([\![\varphi]\!], (s, \gamma_{|\mathrm{Free}(\varphi)})).$$

Let $Z \subseteq \mathrm{MSO}(\mathbb{K},\sigma)$. A series $S : \mathcal{C} \to \mathbb{K}$ is *Z-definable* if $S = [\![\varphi]\!]$ for a sentence $\varphi \in Z$. We let $\mathrm{FO}(\mathbb{K},\sigma)$ consist of all weighted formulae without second-order quantifiers.

As pointed out earlier, a formula $\varphi(x,y) \in \mathrm{MSO}(\sigma)$ defines a binary relation $R = \varphi^s$ for any $s \in \mathcal{C}$. In this way we can derive a new relational structure $s(\varphi)$ from a given $s \in \mathcal{C}$. Even though we can then easily translate a formula $\psi \in \mathrm{MSO}(\sigma \cup R)$ into a formula $\psi' \in \mathrm{MSO}(\sigma)$ such that $s \models \psi'$ iff $s(\varphi) \models \psi$, it is not clear if we can translate weighted formulae such that the values are preserved. The following lemma indicates why this can be problematic.

Lemma 3.8. *Let $\mathcal{C}$ be the class of graphs where the vertices are labeled with a or b. Then there is no sentence $\varphi \in \mathrm{MSO}(\mathbb{N})$ such that $[\![\varphi]\!] = \mathbb{1}_{\mathscr{L}(\exists x.\mathrm{Lab}_a(x))}$.*

Proof. We show that $\mathbb{1}_{\mathscr{L}(\exists x.\mathrm{Lab}_a(x))}$ cannot be defined relatively to the subclass $\mathcal{C}'$ of $\mathcal{C}$ consisting of all graphs having no edge at all. Assume for contradiction that there is a sentence $\varphi \in \mathrm{MSO}(\mathbb{N})$ such that, for all $G \in \mathcal{C}'$, $([\![\varphi]\!], G) = 1$ if G has a vertex labeled a and $([\![\varphi]\!], G) = 0$ otherwise. We may assume that for any variable occurring in φ there is a unique quantifier binding this variable. Moreover, since our semiring is $\mathbb{N}$, we may assume that φ does not contain constant formulae other than 0 and 1 and hence is a formula in classical MSO. By removing all existential quantifiers, in the following we transfer φ into a sentence φ', which hence contains only universal quantifiers and has the property that $G \models \varphi$ iff $G \models \varphi'$ for all $G \in \mathcal{C}'$. Recall that since $\mathbb{N}$ is zero-sum free we then have $G \models \varphi'$ iff $([\![\varphi]\!], G) \neq 0$. First, we extend our logic by adding the new atomic formulae $\mathrm{Sing}_{\mathcal{V}}(F)$, where $\mathcal{V}$ is a finite set of variables and $F \subseteq \{a, b\} \times \{0, 1\}^{\mathcal{V}}$. The free variables of $\mathrm{Sing}_{\mathcal{V}}(F)$ are the elements of $\mathcal{V}$. Let $G \in \mathcal{C}'$ and let γ be a $(\mathcal{V}, G)$-assignment. We let $(G, \gamma) \models \mathrm{Sing}_{\mathcal{V}}(F)$ iff there is exactly one vertex v of G with the following property: There is an element $(c, f) \in F$ such that v is labeled with c, $\gamma(x) = v$ iff $f(x) = 1$ and $v \in \gamma(X)$ iff $f(X) = 1$ for all first-order variables $x \in \mathcal{V}$ and all second-order variables $X \in \mathcal{V}$. As for the other atomic formulae, we define its weighted semantics by letting the value be 1 if the formula holds true and 0 otherwise.

Now, we replace all subformulae in φ of the form $\exists x.\psi$ by $\exists X.\mathrm{Sing}_{\{X\}}(\{(a, 1), (b, 1)\}) \wedge \psi[x/X]$, where $\psi[x/X]$ is obtained by replacing any subformulae $\vartheta(x)$ of ψ, which is an atomic formula or a negation of an atomic formula and contains x, by $\forall y.\neg(y \in X) \vee (y \in X \wedge \vartheta(y))$. Hence, the formula we obtain does not contain existential first-order quantification. Observe that so far we have not changed the weighted semantics.

Next, we successively remove existential second-order quantification proceeding from the innermost quantifiers to the outermost. Let $\widetilde{\varphi}$ be the formula we obtained so far. Let $\exists X.\widetilde{\psi}$ be a subformula which does not contain another existential quantifier. Let $\mathcal{V} = \mathrm{Free}(\widetilde{\psi}) \setminus \{X\}$. Moreover, let $\mathcal{V}_1$ consist of all first-order variables of $\mathcal{V}$ and $\mathcal{V}_2$ consist of all second-order variables of $\mathcal{V}$. Let $F \subseteq \{a, b\} \times \{0, 1\}^{\mathcal{V}}$. We define the formula $\widetilde{\psi}[X/F]$ by replacing in $\widetilde{\psi}$ all subformulae of the form $y \in X$ and $\mathrm{Sing}_{\mathcal{V}'}(F')$ as follows. A formula $y \in X$ is replaced by

$$\bigvee_{(c,f)\in F} \Big(\mathrm{Lab}_c(y) \ \wedge \bigwedge_{z\in\mathcal{V}_1, f(z)=1} y = z \ \wedge \bigwedge_{z\in\mathcal{V}_1, f(z)=0} \neg(y = z) \ \wedge \bigwedge_{Z\in\mathcal{V}_2, f(Z)=1} y \in Z \ \wedge \bigwedge_{Z\in\mathcal{V}_2, f(Z)=0} \neg(y \in Z)\Big).$$

If $y \in X$ occurs negated, then we pull negation down to atomic formulae. A formula $\mathrm{Sing}_{\mathcal{V}'}(F')$ is replaced only if $X \in \mathcal{V}'$. In this case it will be replaced by $\mathrm{Sing}_{\mathcal{V}'\setminus\{X\}\cup\mathcal{V}}(F'')$, where we obtain F'' from F' by proceeding over all $(c', f') \in F'$ as follows. If $f'(X) = 1$, then replace (c', f') by all pairs (c', f''), where f'' is an extension of $f'_{|\mathcal{V}'\setminus\{X\}}$ such that $(c', f''_{|\mathcal{V}}) \in F$. In the case where $f'(X) = 0$, replace (c', f') by all pairs (c', f''), where f'' is an extension of $f'_{|\mathcal{V}'\setminus\{X\}}$ such that $(c', f''_{|\mathcal{V}}) \notin F$. Now, let $G \in \mathcal{C}'$ and γ be a $(\mathcal{V}, G)$-assignment. Let T_F be the set of all vertices of (G, γ) which are labeled by some $f \in F$. Observe that $(G, \gamma) \models \widetilde{\psi}[X/F]$ iff $(G, \gamma[X \to T_F]) \models \widetilde{\psi}$. Now, we replace $\exists X.\widetilde{\psi}$ in $\widetilde{\varphi}$ by

$$\bigvee_{F\subseteq\{a,b\}\times\{0,1\}^{\mathcal{V}}} \widetilde{\psi}[X/F].$$

We will argue that this manipulation preserves the unweighted semantics of $\widetilde{\varphi}$. This is clear, if we could replace $\exists X.\widetilde{\psi}$ with 0 and would still preserve the weighted semantics of $\widetilde{\varphi}$. Otherwise there is some $G \in \mathcal{C}'$, some assignment $\gamma : \mathcal{V} \to V(G) \cup \mathscr{P}(V(G))$ and some $T \subseteq V(G)$ such that $([\![\widetilde{\psi}]\!], (G, \gamma[X \to T])) \neq 0$, and this is not ruled out by multiplication with 0 when calculating $([\![\widetilde{\varphi}]\!], G)$. Consider the subformula $\exists X.\psi$ of φ which corresponds to $\exists X.\widetilde{\psi}$. As we proceed from the innermost quantifiers to the outermost, we conclude that $([\![\psi]\!], (G, \gamma[X \to T])) \neq 0$, and this is not ruled out by multiplication with 0 when calculating $([\![\varphi]\!], G)$. Consider an automorphism Φ of the graph (G, γ). Clearly, this is also an isomorphism $\Phi : (G, \gamma[X \to T]) \to (G, \gamma[X \to \Phi(T)])$, and hence $([\![\psi]\!], (G, \gamma[X \to T])) = ([\![\psi]\!], (G, \gamma[X \to \Phi(T)]))$. We conclude that T must be invariant under all automorphisms of (G, γ), since otherwise $([\![\varphi]\!], (G, \gamma)) \geq ([\![\psi]\!], (G, \gamma[X \to T]) + ([\![\psi]\!], (G, \gamma[X \to \Phi(T)])) \geq 2$, using that our semiring is $\mathbb{N}$. Now observe that the subsets of $V(G)$ which are invariant under all automorphisms of (G, γ) are precisely the sets $(T_F)_{F\subseteq\{a,b\}\times\{0,1\}^{\mathcal{V}}}$ as defined above. We conclude that our manipulation preserves the unweighted semantics.

After replacing all existential quantifiers we end up with a formula φ' which has the same unweighted semantics as φ and contains only universal quantifiers. Let $G \in C'$ such that G has vertices labeled with b and at least two vertices labeled with a. Hence, $G \models \varphi'$. Consider the subgraph G' of G which consists of the vertices labeled with b only. Observe that for all formulae $\mathrm{Sing}_{\mathcal{V}}(F)$ in φ' and all $(\mathcal{V}, G')$-assignments γ, we have, if $(G, \gamma) \models \mathrm{Sing}_{\mathcal{V}}(F)$, then $(G', \gamma) \models \mathrm{Sing}_{\mathcal{V}}(F)$. Moreover, observe that no formula $\mathrm{Sing}_{\mathcal{V}}(F)$ occurs negated. Using this and the fact that φ' does only contain universal quantifiers it is easy to see that $G' \models \varphi'$. But since $\mathbb{N}$ is zero-sum free we have $G' \models \varphi'$ iff $([\![\varphi]\!], G') \neq 0$ iff there is a vertex in G' labeled a. Contradiction! □

The last lemma motivates us to only consider classes of structures $\mathcal{C}$ such that for all formulae $\varphi \in \mathrm{MSO}(\sigma)$ there are formulae $\varphi^+, \varphi^- \in \mathrm{MSO}(\mathbb{K}, \sigma)$ such that $[\![\varphi^+]\!] = \mathbb{1}_{\mathscr{L}(\varphi)}$ and $[\![\varphi^-]\!] = \mathbb{1}_{\mathscr{L}(\neg\varphi)}$. We therefore require that each $s \in \mathcal{C}$ is equipped with a linear order.

3.3. Classes of ordered structures

From now on for the rest of this section we assume that there is a binary relation symbol $\leq \in \sigma$ such that $\leq^s$ is a linear order for all $s \in \mathcal{C}$. Note that the assumption that a class of relational structures is equipped with a linear order is very natural and has been proven useful and important in many different situations, for example in descriptive complexity, where it is common to consider so-called "built-in relations" and in particular a "built-in linear order". In order to stress how natural the assumption of a linear order is, Immerman, a developer of descriptive complexity, said [40]:

> "An unordered graph makes sense mathematically, but you can't store such an object in a computer as far as I know."

Following the ideas of [14] we show how to define for any classical (unweighted) MSO formula φ formulae φ^+ and φ^- such that $[\![\varphi^+]\!] = \mathbb{1}_{\mathcal{L}(\varphi)}$ and $[\![\varphi^-]\!] = \mathbb{1}_{\mathcal{L}(\neg\varphi)}$. In the end this will lead to a syntactically defined fragment sRMSO$(\mathbb{K}, \sigma) \subseteq$ MSO$(\mathbb{K}, \sigma)$ which over words is equally expressive as weighted automata. We will give the definition of φ^+ and φ^- inductively.

1. If φ is of the form $x = y$, $R_i(x_1, \ldots, x_{\rho(i)})$, $x \in X$, then $\varphi^+ = \varphi$ and $\varphi^- = \neg\varphi$.
2. If $\varphi = \neg\psi$, then $\varphi^+ = \psi^-$ and $\varphi^- = \psi^+$.
3. If $\varphi = \psi \vee \psi'$, then $\varphi^+ = \psi^+ \vee (\psi^- \wedge \psi'^+)$ and $\varphi^- = \psi^- \wedge \psi'^-$.

The problem that arises is that by definition of the semantics $\vee$ gets translated by means of $+$. Hence, in order to find φ^+ in the case $\varphi = \psi \vee \psi'$, we only evaluate ψ' if ψ evaluates to 0, otherwise we might end up with a sum greater than one. A similar problem occurs for $\exists x.$ and $\exists X$. Therefore we define as follows:

4. If $\varphi = \exists x.\psi(x)$, then $\varphi^+ = \exists x.(\psi(x)^+ \wedge \forall y.\, y < x \overset{+}{\rightarrow} \psi(y)^-)$ and $\varphi^- = \forall x.\psi(x)^-$,

where $y < x \overset{+}{\rightarrow} \psi(y)^-$ is an abbreviation for $x \leq y \vee (y < x \wedge \psi(y)^-)$. In order to deal with set quantification, we have to define a linear order on the subsets of the domain or, since we have a linear order, equivalently on words (of fixed length) over the alphabet $\{0, 1\}$. We take the strict lexicographic order $<$, which is given by the following formula.

$$X < Y = \exists y.\, y \in Y \wedge \neg(y \in X) \wedge \forall z.\, z < y \rightarrow (z \in X \leftrightarrow z \in Y).$$

Observe that we get $[\![X < Y]\!] = \mathbb{1}_{\mathcal{L}(X<Y)}$. Now we proceed:

5. If $\varphi = \exists X.\psi(X)$, then $\varphi^+ = \exists X.(\psi(X)^+ \wedge \forall Y.Y < X \overset{+}{\rightarrow} \psi(Y)^-)$ and $\varphi^- = \forall X.\psi(X)^-$,

where $Y < X \overset{+}{\rightarrow} \psi(Y)^-$ is an abbreviation for $(Y < X)^- \vee (Y < X \wedge \psi(Y)^-)$.

All formulae φ^+ or φ^- given in this way for some $\varphi \in$ MSO(σ) are called *syntactically unambiguous*. Observe that, if φ is syntactically unambiguous, then $[\![\varphi]\!]_{\mathcal{V}} = \mathbb{1}_{\mathcal{L}_{\mathcal{V}}(\varphi)}$ for any finite set of variables $\mathcal{V} \supseteq$ Free(φ). For $\varphi \in$ MSO(σ) and any $\psi \in$ MSO$(\mathbb{K}, \sigma)$ we let $\varphi \overset{+}{\rightarrow} \psi = \varphi^- \vee (\varphi^+ \wedge \psi)$. Note that $[\![\varphi \overset{+}{\rightarrow} \psi^+]\!] = \mathbb{1}_{\mathcal{L}(\varphi\rightarrow\psi)}$ for all $\psi \in$ MSO(σ).

We define aUMSO$(\mathbb{K}, \sigma)$, the collection of *almost unambiguous* formulae, to be the smallest subset of MSO$(\mathbb{K}, \sigma)$ containing all constants k ($k \in \mathbb{K}$) and all syntactically unambiguous formulae which is closed under conjunction and disjunction. Clearly, from (2) in the definition above we get that we can write any syntactically unambiguous formula φ both as ψ^+ as well as ψ'^- for some appropriate $\psi, \psi' \in$ MSO(σ) and hence from (3) we obtain that if φ and φ' are syntactically unambiguous then so is $\varphi \wedge \varphi'$. Now, using the distributivity, observe that for any $\psi \in$ aUMSO$(\mathbb{K}, \sigma)$ there is a formula ψ' of the form $\psi' = \bigvee_{i=1}^n k_i \wedge \psi_i$ for some $k_i \in \mathbb{K}$ and syntactically unambiguous ψ_i such that $[\![\psi]\!] = [\![\psi']\!]$ (cf. [14]). We are now ready to define the fragment sRMSO$(\mathbb{K}, \sigma)$.

Definition 3.9. A weighted formula $\varphi \in$ MSO$(\mathbb{K}, \sigma)$ is *syntactically restricted* if for all subformulae ϑ of φ the following two conditions hold:

1. If $\vartheta = \forall X.\psi$ for some $\psi \in$ MSO$(\mathbb{K}, \sigma)$, then ψ is syntactically unambiguous.
2. If $\vartheta = \forall x.\psi$ for some $\psi \in$ MSO$(\mathbb{K}, \sigma)$, then $\psi \in$ aUMSO$(\mathbb{K}, \sigma)$.

We collect all syntactically restricted formulae in sRMSO$(\mathbb{K}, \sigma)$.

For this definition it is important to note that conditions (1) and (2) in the last definition are in particular true for all syntactically unambiguous formulae. For this we need to check that formulae of the form $y < x \overset{+}{\rightarrow} \psi(y)^-$ and $Y < X \overset{+}{\rightarrow} \psi(Y)^-$ are syntactically unambiguous. Indeed $y < x \overset{+}{\rightarrow} \psi(y)^- = (\neg(y < x) \vee \neg\psi(y))^+$ and $Y < X \overset{+}{\rightarrow} \psi(Y)^- = (\neg(Y < X) \vee \neg\psi(Y))^+$.

Let sRFO$(\mathbb{K}, \sigma)$ = sRMSO$(\mathbb{K}, \sigma) \cap$ FO$(\mathbb{K}, \sigma)$. In sREMSO$(\mathbb{K}, \sigma) \subset$ sRMSO$(\mathbb{K}, \sigma)$ we collect all formulae of the form $\exists X_1. \ldots \exists X_m.\psi$ such that $\psi \in$ sRFO$(\mathbb{K}, \sigma)$.

Now, let wUMSO$(\mathbb{K}, \sigma)$, the collection of *weakly unambiguous* formulae, be the smallest subset of MSO$(\mathbb{K}, \sigma)$ containing all constants k ($k \in \mathbb{K}$) and all syntactically unambiguous formulae which is closed under conjunction, disjunction and existential quantification (both first- and second-order). We define the fragment swRMSO$(\mathbb{K}, \sigma)$ as follows.

Definition 3.10. A weighted formula $\varphi \in$ MSO$(\mathbb{K}, \sigma)$ is *syntactically weakly restricted* if for all subformulae ϑ of φ the following two conditions hold:

1. If $\vartheta = \forall X.\psi$ for some $\psi \in \mathrm{MSO}(\mathbb{K})$, then ψ is syntactically unambiguous.
2. If $\vartheta = \forall x.\psi$ for some $\psi \in \mathrm{MSO}(\mathbb{K})$, then $\psi \in \mathrm{wUMSO}(\mathbb{K}, \sigma)$.

We collect all syntactically restricted formulae in $\mathrm{swRMSO}(\mathbb{K}, \sigma)$.

Clearly, $\mathrm{aUMSO}(\mathbb{K}, \sigma) \subset \mathrm{wUMSO}(\mathbb{K}, \sigma) \subset \mathrm{sRMSO}(\mathbb{K}, \sigma) \subset \mathrm{swRMSO}(\mathbb{K}, \sigma) \subset \mathrm{MSO}(\mathbb{K}, \sigma)$. Droste and Gastin showed:

Theorem 3.11 (*Droste & Gastin [14]*). *Let $\mathbb{K}$ be a commutative semiring and let $S : \Delta^* \to \mathbb{K}$ be a formal power series. Then S is regular iff S is* sRMSO($\mathbb{K}$)*-definable iff S is* sREMSO($\mathbb{K}$)*-definable. Moreover, if $\mathbb{K}$ is additively locally finite, then S is regular iff S is* swRMSO($\mathbb{K}$)*-definable. If $\mathbb{K}$ is locally finite, then S is regular iff S is* MSO($\mathbb{K}$)*-definable.*

Let us now consider the class of trees $T_\Sigma(\Delta)$. Given a tree $t \in T_\Sigma(\Delta)$, we may order its domain $V(t) \subset \mathbb{N}_+^*$ lexicographically. This order is easily seen to be definable in MSO. Let $(E_i, \leq)$ be the signature which we obtain from (E_i) by adding the binary relation symbol $\leq$ which we interpret by the lexicographic order. Using the signature $(E_i, \leq)$ we obtain the fragments $\mathrm{sRMSO}(\mathbb{K}, (E_i, \leq))$, $\mathrm{swRMSO}(\mathbb{K}, (E_i, \leq))$ and $\mathrm{sREMSO}(\mathbb{K}, (E_i, \leq))$ for trees.

Theorem 3.12 (*Droste & Vogler, Mathissen [18,52]*). *Let $\mathbb{K}$ be a commutative semiring and let $S : T_\Sigma(\Delta) \to \mathbb{K}$ be a tree series. Then S is regular iff S is* sRMSO($\mathbb{K}$)*-definable iff S is* sREMSO($\mathbb{K}$)*-definable. Moreover, if $\mathbb{K}$ is additively locally finite, then S is regular iff S is* swRMSO($\mathbb{K}$)*-definable. If $\mathbb{K}$ is locally finite, then S is regular iff S is* MSO($\mathbb{K}$)*-definable.*

Note that the proof is constructive. Given an effectively given semiring $\mathbb{K}$ and an sRMSO($\mathbb{K}$) sentence φ, we can compute a weighted tree automaton $\mathcal{A}$ such that $\|\mathcal{A}\| = [\![\varphi]\!]$ and vice versa.

3.4. Definable transductions

Transductions realized by different machine models on different structures play an important role in theoretical computer science; maybe most notable are rational transductions of words and transductions realized by different kinds of tree transducers. Using logical interpretations, a common notion of model theory, Courcelle [10] introduced a new kind of transductions between classes of relational structures, so-called MSO-definable transductions. Here one derives a new structure from a given one by interpreting it in m copies of the given structure for some fixed $m \in \mathbb{N}$, that is by describing it in m copies of a given structure using logical formulae. In this subsection let σ' be a second relational signature and let $\mathcal{C}'$ be a class of finite σ'-structures. In order to define $\mathrm{sRMSO}(\mathbb{K}, \sigma')$, $\mathrm{sREMSO}(\mathbb{K}, \sigma')$ and $\mathrm{swRMSO}(\mathbb{K}, \sigma')$, we assume that there is a binary relation symbol $\leq' \in \sigma'$ such that $\leq'^{s'}$ is a linear order for all $s' \in \mathcal{C}'$.

Definition 3.13. Let $m \in \mathbb{N}_+$. A (σ', σ)-*m-copying definition scheme with parameters* $X_1, \ldots, X_n$ is a tuple

$$\mathcal{D} = (\vartheta, (\delta_j)_{1 \leq j \leq m}, (\varphi_l)_{l \in \sigma \star m}), \quad \text{where } \sigma \star m = \{(R, \vec{j}) \mid R \in \sigma, \vec{j} \in [m]^{\rho(R)}\},$$

of formulae in $\mathrm{MSO}(\sigma')$ such that $\mathrm{Free}(\vartheta) \subseteq \{X_1, \ldots, X_n\}$, $\mathrm{Free}(\delta_j) \subseteq \{x_1, X_1, \ldots, X_n\}$ and $\mathrm{Free}(\varphi_l) \subseteq \{x_1, \ldots, x_{\rho(R)}, X_1, \ldots X_n\}$ (where $l = (R, \vec{j}) \in \sigma \star m$) for some first-order variables x_i.

Now, let $\mathcal{D}$ be as in Definition 3.13, let $s' \in \mathcal{C}'$ and let $T_1, \ldots, T_n \subseteq V(s')$ such that $s' \models \vartheta[T_1, \ldots, T_n]$. Then define the σ-structure $\mathbf{def}_{\mathcal{D}}(s', T_1, \ldots, T_n) = s = (V(s), (R^s)_{R \in \sigma})$ with $V(s) \subseteq V(s') \times [m]$ as follows:

$$(v, j) \in V(s) \iff s' \models \delta_j[v, T_1, \ldots, T_n] \quad \text{for all } v \in V(s') \text{ and all } j \in [m].$$

$$((v_1, j_1), \ldots, (v_r, j_r)) \in R^s \iff s' \models \varphi_{R,(j_1,\ldots,j_r)}[v_1, \ldots, v_r, T_1, \ldots, T_n] \quad \text{for all } R \in \sigma$$
$$\text{and all } ((v_1, j_1), \ldots, (v_r, j_r)) \in V(s)^r \text{ where } r = \rho(R).$$

By abusing notation, we define the relation $\mathbf{def}_{\mathcal{D}}$ by letting $(s', s) \in \mathbf{def}_{\mathcal{D}}$ if $s' \in \mathcal{C}'$ and there are sets $T_1, \ldots, T_n \subseteq V(s')$ with $s' \models \vartheta[T_1, \ldots, T_n]$ such that $s = \mathbf{def}_{\mathcal{D}}(s')$.

Definition 3.14. A relation $\Phi \subseteq \mathcal{C}' \times \mathcal{C}$ is called a *transduction*. Now, let $Z \subseteq \mathrm{MSO}(\sigma')$. A *transduction* $\Phi \subseteq \mathcal{C}' \times \mathcal{C}$ is *Z-definable* if there is a m-copying definition scheme $\mathcal{D} = (\vartheta, (\delta_j)_{1 \leq j \leq m}, (\varphi_l)_{l \in \sigma \star m})$ for some $m \in \mathbb{N}_+$ with $\vartheta, \delta_j, \varphi_l \in Z$ for all $1 \leq j \leq m$ and $l \in \sigma \star m$ such that $\Phi = \mathbf{def}_{\mathcal{D}}$.

Example 3.15. Without giving proofs, the following transductions were stated to be MSO-definable in [10]. Let A, B be alphabets and let $h : A^* \to B^*$ be a homomorphism.

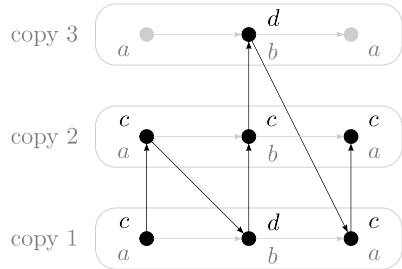
1. We show that h is an MSO-definable transduction. For this let m be the maximal length of an element of the finite set $h(A)$. We give an m-copying definition scheme $\mathcal{D} = (\vartheta, (\delta_j)_{j \in [m]}, (\varphi_{\mathrm{Lab}_b, j})_{b \in B, j \in [m]}, (\varphi_{\leq, j_1, j_2})_{j_1, j_2 \in [m]})$ without parameters for h. Since h is a total function we let ϑ be some tautology. Moreover, we let

$$\delta_j(x) = \bigvee_{|h(a)| \geq j} \mathrm{Lab}_a(x), \qquad \varphi_{\mathrm{Lab}_b, j}(x) = \bigvee_{a \in A, h(a)_j = b} \mathrm{Lab}_a(x),$$

$$\varphi_{\leq, j_1, j_2}(x, y) = \begin{cases} x \leq y & \text{if } j_1 \leq j_2 \\ x < y & \text{if } j_1 > j_2. \end{cases}$$

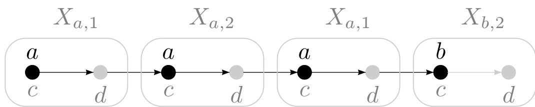
Here $h(a)_j$ denotes the label of the j-th position of the word $h(a)$. Intuitively the i-th position of the j-th copy of a word $w = a_1 \ldots a_n$ corresponds to the j-th position of $h(a_i)$. See the following picture for the transduction $h : \{a, b\}^* \to \{c, d\}^*$ given by $h(a) = cc$ and $h(b) = dcd$.

The definition of $ccdcdcc = h(aba)$ in aba.



2. Let h be non-erasing, i.e. a homomorphism such that $h^{-1}(\varepsilon) = \{\varepsilon\}$. We show that $h^{-1} \subseteq B^* \times A^*$, given by $(v, w) \in h^{-1}$ iff $h(w) = v$, is an MSO-definable transduction. We give a 1-copying definition scheme with parameters $(X_{a,k})_{a \in A, k=1,2}$. The idea is to use the parameters to partition a word into non-empty subwords by coloring these subwords with 1 and 2 and to assign to each subword a preimage $a \in A$ under h provided it exists. See the following picture for the transduction $h : \{a, b\}^* \to \{c, d\}^*$ given by $h(a) = h(b) = cd$.

The definition of $aaab \in h^{-1}(cdcdcdcd)$ in $cdcdcdcd$.



Let us informally describe some macros which are easily seen to be expressively in MSO. For all $1 \le i \le n$, the macro $\mathrm{Pos}_i(Y, y)$ says that y is the ith position of the subword formed by the set Y, $\mathrm{Eql}_w(Y)$ says that the subword formed by Y is w. The macro $\mathrm{MxCn}(X, Y)$ states that Y is a maximal connected component of the subgraph induced by X in the successor structure of the word. And $\mathrm{Prt}((X_{a,k})_{a,k})$ expresses that the $X_{a,k}$'s form a partition. Now, a 1-copying definition scheme $\mathcal{D} = (\vartheta, \delta, (\varphi_{\mathrm{Lab}_a})_{a \in A}, \varphi_{\le})$ for h^{-1} is given by

$$\vartheta((X_{a,k})_{a,k}) = \mathrm{Prt}((X_{a,k})_{a,k}) \wedge \bigwedge_{a,k} \forall Y.\, \mathrm{MxCn}(X_{a,k}, Y) \to \mathrm{Eql}_{h(a)}(Y),$$

$$\delta(x, (X_{a,k})_{a,k}) = \bigvee_{a,k} \exists Y. \mathrm{MxCn}(X_{a,k}, Y) \wedge \mathrm{Pos}_1(Y, x),$$

$$\varphi_{\mathrm{Lab}_a}(x, (X_{a,k})_{a,k}) = x \in X_{a,1} \vee x \in X_{a,2},$$

$$\varphi_{\le}(x, y) = x \le y.$$

Proposition 3.16 (*Courcelle [10]*). *Let $\Phi \subseteq \mathcal{C}' \times \mathcal{C}$ be an* MSO*-definable transduction and let $L \subseteq \mathcal{C}$ be* MSO*-definable. Then $\Phi^{-1}(L) = \{s' \in \mathcal{C}' \mid \exists s \in L.\, (s', s) \in \Phi\}$ is* MSO*-definable.*

We show a similar result for series. Let $\Phi \subseteq \mathcal{C}' \times \mathcal{C}$ be a transduction such that $\Phi(s') = \{s \in \mathcal{C} \mid (s', s) \in \Phi\}$ is finite for every $s' \in \mathcal{C}'$, and let $S : \mathcal{C} \to \mathbb{K}$. We define $\Phi^{-1}(S) : \mathcal{C}' \to \mathbb{K}$ by letting $(\Phi^{-1}(S), s') = \sum_{s \in \Phi(s')} (S, s)$. If the transduction $\Phi^{-1} = \{(s, s') \mid (s', s) \in \Phi\}$ given by the inverse of the relation Φ also has only finite images, then we denote by $\Phi(S)$ the series $(\Phi^{-1})^{-1}(S)$ for any series $S : \mathcal{C}' \to \mathbb{K}$. Clearly, for any MSO-definable transduction $\Phi \subseteq \mathcal{C}' \times \mathcal{C}$ we have $\Phi(s')$ is finite, since the parameter can only assume a finite number of values, as the domain of s' was assumed to be finite for all $s' \in \mathcal{C}'$. Let us call a definition scheme $\mathcal{D}$ with parameters $X_1, \ldots, X_n$ *unambiguous* if for any pair $(s', s) \in \mathbf{def}_{\mathcal{D}}$ there is at most one assignment of parameters $\gamma : \{X_1, \ldots, X_n\} \to \mathscr{P}(V(s'))$ such that $\mathbf{def}_{\mathcal{D}}(s', \gamma(X_1), \ldots, \gamma(X_n)) = s$. For $Z \subseteq \mathrm{MSO}(\sigma')$ we call any transduction which can be defined by an unambiguous definition scheme where all the formulae are in Z *unambiguously Z-definable transduction*. In particular, any definition scheme $\mathcal{D}$ without parameters, which hence defines a partial function, is unambiguous.

Let us remark that functions $f : \Delta^* \to \Delta'^*$ and functions $f : T_\Sigma(\Delta) \to T_{\Sigma'}(\Delta')$ which are definable by definition schemes without parameters were characterized in [27,3,25]; (see also [26] for a characterization of MSO-definable transductions $\Phi \subseteq \Delta^* \times \Delta'^*$). There it is shown that these functions are precisely those which can be realized by deterministic two-way transducers for words and by finite-copying deterministic macro tree transducers for trees.

Theorem 3.17 (*Transfer Theorem*). *Let $\Phi \subseteq \mathcal{C}' \times \mathcal{C}$ be unambiguously* MSO*-definable and let $S : \mathcal{C} \to \mathbb{K}$.*

1. *If S is* $\mathrm{MSO}(\mathbb{K})$*-definable, then $\Phi^{-1}(S)$ is* $\mathrm{MSO}(\mathbb{K})$*-definable.*
2. *If S is* $\mathrm{sRMSO}(\mathbb{K})$*-definable, then $\Phi^{-1}(S)$ is* $\mathrm{sRMSO}(\mathbb{K})$*-definable.*
3. *If S is* $\mathrm{swRMSO}(\mathbb{K})$*-definable, then $\Phi^{-1}(S)$ is* $\mathrm{swRMSO}(\mathbb{K})$*-definable.*
4. *If Φ is unambiguously* FO*-definable and S is* $\mathrm{sREMSO}(\mathbb{K})$*-definable, then $\Phi^{-1}(S)$ is* $\mathrm{sREMSO}(\mathbb{K})$*-definable.*

Proof. (1.) Let $\mathcal{D} = (\vartheta, (\delta_j)_{1\le j\le m}, (\varphi_l)_{l\in I\star m})$ be an unambiguous (σ', σ)-m-copying definition scheme such that $\mathbf{def}_{\mathcal{D}} = \Phi$ and $\psi \in \mathrm{MSO}(\mathbb{K}, \sigma)$. Let $\psi \in \mathrm{MSO}(\mathbb{K}, \sigma)$ and let $\mathcal{V}_1$ and $\mathcal{V}_2$ be finite sets of first-order and second-order variables, respectively, such that $\mathcal{V} = \mathcal{V}_1 \uplus \mathcal{V}_2 \supseteq \mathrm{Free}(\psi)$. Let $\mathcal{F} : \mathcal{V}_1 \to [m]$. We agree to write $\mathcal{F}[x \to j]$ for the mapping that equals $\mathcal{F}$ on $\mathcal{V}_1 \setminus \{x\}$ and assumes j for x. Let

$$\chi_{\mathcal{F}} = \left(\bigwedge_{1\le j\le m} \bigwedge_{y\in\mathcal{F}^{-1}(j)} \delta_j(y, X_1, \ldots, X_n)^+ \right).$$

By induction on the structure of ψ we now define the formula $\psi^{\mathcal{F}} \in \mathrm{MSO}(\mathbb{K}, \sigma')$.

$$k^{\mathcal{F}} = k,$$
$$(x = y)^{\mathcal{F}} = \begin{cases} x = y & \text{if } \mathcal{F}(x) = \mathcal{F}(y) \\ 0 & \text{otherwise,} \end{cases}$$
$$(x \in X)^{\mathcal{F}} = x \in X^j \text{ for } j = \mathcal{F}(x),$$
$$R(x_1 \ldots x_{\rho(R)})^{\mathcal{F}} = \varphi_{R,\vec{j}}(x_1 \ldots x_{\rho(R)}, X_1, \ldots, X_n)^+ \quad \text{for } \vec{j} = (\mathcal{F}(x_1), \ldots, \mathcal{F}(x_{\rho(R)})).$$

If ψ is $x = y$, $x \in X$ or $R(x_1, \ldots, x_{\rho(R)})$, let $(\neg\psi)^{\mathcal{F}} = (\psi^{\mathcal{F}})^-$ with the convention $0^- = 1$. Moreover, let

$$(\psi_1 \wedge \psi_2)^{\mathcal{F}} = \psi_1^{\mathcal{F}} \wedge \psi_2^{\mathcal{F}}$$
$$(\psi_1 \vee \psi_2)^{\mathcal{F}} = \psi_1^{\mathcal{F}} \vee \psi_2^{\mathcal{F}}$$
$$(\exists x.\psi)^{\mathcal{F}} = \bigvee_{1\le j\le m} \exists x.(\delta_j(x, X_1, \ldots, X_n)^+ \wedge \psi^{\mathcal{F}[x\to j]})$$
$$(\exists X.\psi)^{\mathcal{F}} = \exists X^1 \ldots . \exists X^m . \Big[\bigwedge_{1\le j\le m} \forall x.(x \in X^j \to \delta_j(x, X_1, \ldots, X_n))^+ \wedge \psi^{\mathcal{F}} \Big]$$
$$(\forall x.\psi)^{\mathcal{F}} = \bigwedge_{1\le j\le m} \forall x.\delta_j(x, X_1, \ldots, X_n) \xrightarrow{+} \psi^{\mathcal{F}[x\to j]}$$
$$(\forall X.\psi)^{\mathcal{F}} = \forall X^1 \ldots . \forall X^m . \left(\bigwedge_{1\le j\le m} \forall x.x \in X^j \to \delta_j(x, X_1, \ldots, X_n) \right) \xrightarrow{+} \psi^{\mathcal{F}}.$$

This concludes the inductive definition of $\psi^{\mathcal{F}}$. Now, let $s' \in \mathcal{C}'$, $T_1, \ldots, T_n \subseteq V(s')$ and $s \in \mathcal{C}$ such that $s = \mathbf{def}_{\mathcal{D}}(s', T_1, \ldots, T_n)$. Let $\mathcal{V}_2' = \bigcup_{j=1}^m \{X^j \mid X \in \mathcal{V}_2\}$. Moreover, let $\gamma : \mathcal{V}' = \mathcal{V}_1 \cup \mathcal{V}_2' \cup \{X_1, \ldots, X_n\} \to V(s') \cup \mathscr{P}(V(s'))$ be a $(\mathcal{V}', s')$-assignment such that $\gamma(X_i) = T_i$ for $1 \le i \le n$ and $s' \models \delta_j[\gamma(x), T_1, \ldots, T_n]$ for all $x \in \mathcal{V}_1$ with $j = \mathcal{F}(x)$. Observe that $\mathcal{V}' \supseteq \mathrm{Free}(\psi^{\mathcal{F}})$. We now define the $(\mathcal{V}, s)$-assignment $\gamma^{\mathcal{F}}$ as follows. Let $\gamma^{\mathcal{F}}(x) = (\gamma(x), \mathcal{F}(x))$ for all $x \in \mathcal{V}_1$ and let $\gamma^{\mathcal{F}}(X) = \{(v, j) \mid 1 \le j \le m,\ v \in \gamma(X^j) \text{ and } s' \models \delta_j[v, T_1, \ldots, T_n]\}$ for all $X \in \mathcal{V}_2$. By induction on the structure of ψ, we show that

$$(\llbracket \psi^{\mathcal{F}} \rrbracket_{\mathcal{V}'}, (s', \gamma)) = (\llbracket \psi \rrbracket_{\mathcal{V}}, (s, \gamma^{\mathcal{F}})). \tag{1}$$

For the formula k ($k \in \mathbb{K}$) we have $(\llbracket k^{\mathcal{F}} \rrbracket_{\mathcal{V}'}, (s', \gamma)) = k = (\llbracket k \rrbracket_{\mathcal{V}}, (s, \gamma^{\mathcal{F}}))$. Note that the translation of an atomic formula and its negation take on either 0 or 1 as values. We get

$$\begin{aligned}
(\llbracket (x = y)^{\mathcal{F}} \rrbracket_{\mathcal{V}'}, (s', \gamma)) = 1 \quad & \text{iff } \mathcal{F}(x) = \mathcal{F}(y) \text{ and } \gamma(x) = \gamma(y) \\
& \text{iff } \gamma^{\mathcal{F}}(x) = \gamma^{\mathcal{F}}(y) \text{ iff } (\llbracket x = y \rrbracket_{\mathcal{V}}, (s, \gamma^{\mathcal{F}})) = 1, \\
(\llbracket (x \in X)^{\mathcal{F}} \rrbracket_{\mathcal{V}'}, (s', \gamma)) = 1 \quad & \text{iff } \gamma(x) \in \gamma(X^j) \text{ for } j = \mathcal{F}(x) \\
& \text{iff } \gamma^{\mathcal{F}}(x) \in \gamma^{\mathcal{F}}(X) \text{ iff } (\llbracket x \in X \rrbracket_{\mathcal{V}}, (s, \gamma^{\mathcal{F}})) = 1 \\
(\llbracket R_i(x_1, \ldots, x_{\rho(i)})^{\mathcal{F}} \rrbracket_{\mathcal{V}'}, (s', \gamma)) = 1 \quad & \text{iff } (s', \gamma) \models \varphi_{R_i,\vec{j}}(x_1, \ldots, x_{\rho(i)}, X_1, \ldots, X_n) \text{ for } \vec{j} = (\mathcal{F}(x_1), \ldots, \mathcal{F}(x_{\rho(i)})). \\
& \text{iff } R_i^s(\gamma^{\mathcal{F}}(x_1), \ldots, \gamma^{\mathcal{F}}(x_{\rho(i)})) \\
& \text{iff } (\llbracket R_i(x_1, \ldots, x_{\rho(i)}) \rrbracket_{\mathcal{V}}, (s, \gamma^{\mathcal{F}})) = 1.
\end{aligned}$$

If ψ equals $x = y$, $x \in X$ or $R_i(x_1, \ldots, x_{\rho(i)})$, we have

$$\begin{aligned}
(\llbracket (\neg\psi)^{\mathcal{F}} \rrbracket_{\mathcal{V}'}, (s', \gamma)) = 1 \quad & \text{iff } (\llbracket \psi^{\mathcal{F}} \rrbracket_{\mathcal{V}'}, (s', \gamma)) = 0 \\
& \text{iff } (\llbracket \psi \rrbracket_{\mathcal{V}}, (s, \gamma^{\mathcal{F}})) = 0 \text{ iff } (\llbracket \neg\psi \rrbracket_{\mathcal{V}}, (s, \gamma^{\mathcal{F}})) = 1.
\end{aligned}$$

Now, for disjunction we get

$$\begin{aligned}
(\llbracket (\psi_1 \vee \psi_2)^{\mathcal{F}} \rrbracket_{\mathcal{V}'}, (s', \gamma)) &= (\llbracket \psi_1^{\mathcal{F}} \rrbracket_{\mathcal{V}'}, (s', \gamma)) + (\llbracket \psi_2^{\mathcal{F}} \rrbracket_{\mathcal{V}'}, (s', \gamma)) \\
&= (\llbracket \psi_1 \rrbracket_{\mathcal{V}}, (s, \gamma^{\mathcal{F}})) + (\llbracket \psi_2 \rrbracket_{\mathcal{V}}, (s, \gamma^{\mathcal{F}})) = (\llbracket \psi_1 \vee \psi_2 \rrbracket_{\mathcal{V}}, (s, \gamma^{\mathcal{F}})).
\end{aligned}$$

Similarly for conjunction. For universal first-order quantification we have

$$\begin{aligned}
([\![(\forall x.\psi)^{\mathcal{F}}]\!]_{\mathcal{V}'}, (s', \gamma)) &= \prod_{1\le j\le m} \prod_{v\in V(s')} ([\![\delta_j(x, X_1, \dots, X_n) \xrightarrow{+} \psi^{\mathcal{F}[x\to j]}]\!]_{\mathcal{V}'\cup\{x\}}, (s', \gamma[x \to v])) \\
&= \prod_{\substack{1\le j\le m\\ v\in V(s')}} \Big(([\![\delta_j(x, X_1, \dots, X_n)^+]\!]_{\mathcal{V}'\cup\{x\}}, (s', \gamma[x \to v])) \cdot ([\![\psi^{\mathcal{F}[x\to j]}]\!]_{\mathcal{V}'\cup\{x\}}, (s', \gamma[x \to v])) \\
&\qquad + ([\![\delta_j(x, X_1, \dots, X_n)^-]\!]_{\mathcal{V}'\cup\{x\}}, (s', \gamma[x \to v]))\Big) \\
&= \prod_{1\le j\le m} \prod_{\substack{v\in V(s')\\ s'\models\delta_j[v,T_1,\dots,T_n]}} ([\![\psi^{\mathcal{F}[x\to j]}]\!]_{\mathcal{V}'\cup\{x\}}, (s', \gamma[x \to v])) \\
&= \prod_{1\le j\le m} \prod_{\substack{v\in V(s')\\ s'\models\delta_j[v,T_1,\dots,T_n]}} ([\![\psi]\!]_{\mathcal{V}\cup\{x\}}, (s, (\gamma[x \to v])^{\mathcal{F}[x\to j]}))
\end{aligned}$$

and using crucially the commutativity of $\mathbb{K}$ we get

$$= \prod_{v\in V(s)} ([\![\psi]\!]_{\mathcal{V}\cup\{x\}}, (s, \gamma^{\mathcal{F}}[x \to v])) = ([\![\forall x.\psi]\!]_{\mathcal{V}}, (s, \gamma^{\mathcal{F}})).$$

Similarly for existential quantification. Moreover, using the abbreviation $\mathcal{X} = \{X^1, \dots, X^m\}$ we get, for $\exists X.$,

$$\begin{aligned}
&([\![(\exists X.\psi)^{\mathcal{F}}]\!]_{\mathcal{V}'}, (s', \gamma)) \\
&= \sum_{T^1\subseteq V(s')} \cdots \sum_{T^m\subseteq V(s')} \Big([\![\bigwedge_{j=1}^{m} \forall x.(x \in X^j \to \delta_j(x, X_1, \dots, X_n))^+ \wedge \psi^{\mathcal{F}}]\!]_{\mathcal{V}'\cup\mathcal{X}}, (s', \gamma[X^1 \to T^1, \dots, X^m \to T^m])\Big) \\
&= \sum_{\substack{T^1\subseteq V(s')\\ T^1\times\{1\}\subseteq V(s)}} \cdots \sum_{\substack{T^m\subseteq V(s')\\ T^m\times\{m\}\subseteq V(s)}} ([\![\psi^{\mathcal{F}}]\!]_{\mathcal{V}'\cup\mathcal{X}}, (s', \gamma[X^1 \to T^1, \dots, X^m \to T^m])) \\
&= \sum_{\substack{T^1\subseteq V(s')\\ T^1\times\{1\}\subseteq V(s)}} \cdots \sum_{\substack{T^m\subseteq V(s')\\ T^m\times\{m\}\subseteq V(s)}} ([\![\psi]\!]_{\mathcal{V}\cup\{X\}}, (s, \gamma[X^1 \to T^1, \dots, X^m \to T^m]^{\mathcal{F}})) \\
&= \sum_{T\subseteq V(s)} ([\![\psi]\!]_{\mathcal{V}\cup\{X\}}, (s, \gamma^{\mathcal{F}}[X \to T])) \\
&= ([\![\exists X.\psi]\!]_{\mathcal{V}}, (s, \gamma^{\mathcal{F}})).
\end{aligned}$$

Again similarly for $\forall X$.

Thus, we have $([\![\psi^{\mathcal{F}}]\!]_{\mathcal{V}'}, (s', \gamma)) = ([\![\psi]\!]_{\mathcal{V}}, (s, \gamma^{\mathcal{F}}))$. Now assume that $\psi \in \mathrm{MSO}(\mathbb{K}, \sigma)$ is a sentence such that $S = [\![\psi]\!]$. As before, let $s' \in \mathcal{C}'$, let $T_1, \dots, T_n \subseteq V(s')$ and let $s \in \mathcal{C}$ such that $s = \mathbf{def}_{\mathcal{D}}(s', T_1, \dots, T_n)$. Moreover, let $\gamma : \{X_1, \dots, X_n\} \to \mathscr{P}(V(s'))$ be given by $\gamma(X_i) = T_i$ for all $1 \le i \le n$. By what we just showed we get $([\![\psi^{\emptyset}]\!], (s', \gamma)) = ([\![\psi]\!], s)$. Hence, we get

$$\begin{aligned}
(\Phi^{-1}(S), s') &= \sum_{s\in\Phi(s')} (S, s) \overset{\mathcal{D}\text{ unambiguous}}{=} \sum_{\substack{T_1,\dots,T_n\subseteq V(s')\\ s'\models\vartheta[T_1,\dots,T_n]}} (S, \mathbf{def}_{\mathcal{D}}(s', T_1, \dots, T_n)) \\
&= \sum_{\substack{T_1,\dots,T_n\subseteq V(s')\\ s'\models\vartheta[T_1,\dots,T_n]}} ([\![\psi]\!], \mathbf{def}_{\mathcal{D}}(s', T_1, \dots, T_n)) = ([\![\exists X_1, \dots, X_n.\vartheta(X_1, \dots, X_n)^+ \wedge \psi^{\emptyset}]\!], s').
\end{aligned}$$

We conclude that $[\![\exists X_1, \dots, X_n.\vartheta(X_1, \dots, X_n)^+ \wedge \psi^{\emptyset}]\!] = \Phi^{-1}(S)$.

(2. & 3.) As in the proof of (1), let $\mathcal{D} = (\vartheta, (\delta_j)_{1\le j\le m}, (\varphi_l)_{l\in I\star m})$ be an unambiguous (σ', σ)-m-copying definition scheme such that $\mathbf{def}_{\mathcal{D}} = \Phi$ and $\psi \in \mathrm{MSO}(\mathbb{K}, \sigma)$. In addition to the proof of (1), we adapt the inductive definition of $\psi^{\mathcal{F}}$ in the

following cases:

$$(\psi_1 \vee \psi_2)^{\mathcal{F}} = \begin{cases} (\psi_1^{\mathcal{F}} \vee \psi_2^{\mathcal{F}})^+ & \text{if } \psi_1 \vee \psi_2 \text{ is syntactically unambiguous} \\ \psi_1^{\mathcal{F}} \vee \psi_2^{\mathcal{F}} & \text{otherwise} \end{cases}$$

$$(\exists x.\psi)^{\mathcal{F}} = \begin{cases} \left(\bigvee_{j=1}^{m} \exists x.(\delta_j(x, X_1, \ldots, X_n) \wedge \psi^{\mathcal{F}[x \to j]})\right)^+ & \text{if } \exists x.\psi \text{ is syntactically unambiguous} \\ \bigvee_{j=1}^{m} \exists x.(\delta_j(x, X_1, \ldots, X_n)^+ \wedge \psi^{\mathcal{F}[x \to j]}) & \text{otherwise} \end{cases}$$

$$(\exists X.\psi)^{\mathcal{F}} = \begin{cases} \left(\exists X^1, \ldots, X^m.\left(\bigwedge_{j \in [m]} \forall x.(x \in X^j \to \delta_j(x, X_1, \ldots, X_n)) \wedge \psi^{\mathcal{F}}\right)\right)^+ & \text{if } \exists X.\psi \text{ is syntactically unambiguous} \\ \exists X^1, \ldots, X^m.\left(\bigwedge_{j \in [m]} \forall x.(x \in X^j \to \delta_j(x, X_1, \ldots, X_n))^+ \wedge \psi^{\mathcal{F}}\right) & \text{otherwise.} \end{cases}$$

Now, for the inductive proof of Eq. (1) it remains to consider the cases where ψ is syntactically unambiguous and of one of the three forms above. Hence, let ψ be of this kind. Observe that $\psi^{\mathcal{F}}$ as well as ψ only takes on 0 and 1 as values. Let $\psi_{\text{old}}^{\mathcal{F}}$ be the translation of ψ we would obtain if we had not adapted the inductive definition of $\psi^{\mathcal{F}}$, i.e. the definition we would obtain in the proof of (1). Then $\psi^{\mathcal{F}} = (\psi_{\text{old}}^{\mathcal{F}})^+$. Observe that, since ψ does not contain constant formulae, using the Boolean semiring $\mathbb{B}$ Eq. (1) gives $(s', \gamma) \models \psi_{\text{old}}^{\mathcal{F}}$ iff $(s, \gamma^{\mathcal{F}}) \models \psi$. Hence

$$\begin{aligned}(\llbracket(\psi)^{\mathcal{F}}\rrbracket_{\mathcal{V}'}, (s', \gamma)) = (\llbracket(\psi_{\text{old}}^{\mathcal{F}})^+\rrbracket_{\mathcal{V}'}, (s', \gamma)) = 1 \ &\text{iff } (s', \gamma) \models \psi_{\text{old}}^{\mathcal{F}} \\ &\text{iff } (s, \gamma^{\mathcal{F}}) \models \psi \text{ iff } (\llbracket\psi\rrbracket_{\mathcal{V}}, (s, \gamma^{\mathcal{F}})) = 1.\end{aligned}$$

Now assume that $\varphi \in \text{sRMSO}(\mathbb{K}, \sigma)$ (respectively $\varphi \in \text{swRMSO}(\mathbb{K}, \sigma)$) is a sentence such that $S = \llbracket\varphi\rrbracket$. As in the proof of (1), we conclude that $\llbracket\exists X_1, \ldots, X_n.\vartheta(X_1, \ldots, X_n)^+ \wedge \varphi^{\emptyset}\rrbracket = \Phi^{-1}(S)$. Analyzing the translation we obtain that a syntactically unambiguous formula ψ is translated to an syntactically unambiguous $\psi^{\mathcal{F}}$ and hence we get that formulae in aUMSO($\mathbb{K}$) are translated to formulae in aUMSO($\mathbb{K}$). The fact that wUMSO($\mathbb{K}$) is preserved is also easy to see since $\forall x.(x \in X^j \wedge \delta_j(x, X_1, \ldots, X_n))^+$ is a syntactically unambiguous formula. It is now clear that $\varphi^{\emptyset} \in \text{sRMSO}(\mathbb{K}, \sigma')$ (respectively $\varphi^{\emptyset} \in \text{swRMSO}(\mathbb{K}, \sigma')$). Thus $\Phi^{-1}(S)$ is sRMSO($\mathbb{K}$)-definable (respectively swRMSO($\mathbb{K}$)-definable).

(4.) From the last proof and using the distributivity of the semiring in order to pull the existential quantifiers to the front this immediately follows. □

Remark 3.18. For Theorem 3.17(1) the assumption that $\mathcal{C}'$ is equipped with a linear order $\leq'$ is needed to guarantee the existence of φ^+ for certain formulae φ in the definition of $\psi^{\mathcal{F}}$. There might be other reasons why this is possible. For example, if $\mathbb{K}$ is idempotent, then Theorem 3.17(1) holds even if there is no such a linear order $\leq'$. A more general transfer theorem can be found in [52].

The author believes that Transfer Theorem 3.17 provides a powerful tool not only to transfer definability results between different structures, as we will do in the following sections, but also to show that certain transformations on formal power series or on tree series realized by transductions preserve regularity. For example, for words we immediately get from the result of [25] that if $S : \Delta'^* \to \mathbb{K}$ is regular and $g : \Delta^* \to \Delta'^*$ is a function realized by a deterministic two-way transducer, then $g^{-1}(S)$ is regular. Or for trees we immediately get from the result of [27,3] that if $S : T_\Sigma(\Delta) \to \mathbb{K}$ is regular and $g : T_{\Sigma'}(\Delta') \to T_\Sigma(\Delta)$ is a function realized by a finite-copying deterministic macro tree transducer, then $g^{-1}(S)$ is regular. To provide some more evidence we show how to obtain Theorem 3.1 of [19] in this context. We note however that the proof in [19] is much more elementary.

Corollary 3.19 (*Droste & Zhang [19, Theorem 3.1]*). *Let A, B be alphabets with $B \subseteq A$ and let $h : A^* \to B^*$ be a non-erasing homomorphism. If $S : A^* \to \mathbb{K}$ is regular, then so is $\widetilde{h}S : B^* \to \mathbb{K}$ given by $(\widetilde{h}S, v) = \sum_{w \in h^{-1}(v)} (S, vw)$ for all $v \in B^*$.*

Proof. Let $S : A^* \to \mathbb{K}$ be regular and thus sRMSO($\mathbb{K}$)-definable by Theorem 3.11. By Example 3.15(2) the transduction $h^{-1} \subseteq B^* \times A^*$ is MSO-definable. Let $S(x, y) \in \text{FO}$ define the successor relation. If we now replace in the definition scheme of Example 3.15(2) the formula ϑ by the following formula

$$\begin{aligned}\vartheta \ &\wedge \exists y.\left(\forall z.y \leq z \wedge \bigvee_{a \in A} y \in X_{a,1}\right) \\ &\wedge \bigwedge_{a \in A} \forall x.\,\big(x \in X_{a,1} \wedge \exists z.S(x, z) \wedge \neg(z \in X_{a,1})\big) \to \bigvee_{a' \in A} z \in X_{a',2} \\ &\wedge \bigwedge_{a \in A} \forall x.\,\big(x \in X_{a,2} \wedge \exists z.S(x, z) \wedge \neg(z \in X_{a,2})\big) \to \bigvee_{a' \in A} z \in X_{a',1},\end{aligned}$$

we obtain an unambiguous definition scheme. From this it is easy to deduce that the transduction $g = \{(v, vw) \mid \underset{\sim}{w} \in h^{-1}(v)\} \subseteq B^* \times A^*$ is again unambiguously MSO-definable. From Transfer Theorem 3.17 we obtain that $g^{-1}(S) = \widetilde{h}S$ is sRMSO($\mathbb{K}$)-definable and hence, again by Theorem 3.11, regular. □

In [14] weighted logics over words have also been considered for non-commutative semirings. For this it is assumed that whenever we have a conjunction $\varphi \wedge \psi$ which is not in the scope of a universal first-order quantifier, the values of the constant subformulae k ($k \in \mathbb{K}$) in φ all commute with the values of the constant subformulae in ψ. For the fragment sRMSO($\mathbb{K}$) which characterizes regular series it is further assumed that for the semantics of the universal quantification the product is taken along the order of positions and along the lexicographic order of the subsets, respectively. The translation of the first-order universal quantification in the proof of our Transfer Theorem 3.17 does not work for this fragment since we cannot assure that the product is taken along the order of positions. In fact, a counterexample showing that Theorem 3.17(2) and (3) do not hold for non-commutative semirings can be obtained from [19, p.379]. There, the semiring $\mathbb{K} = (\mathscr{P}(\{a, b\}^*), \cup, \cdot, \emptyset, \{\varepsilon\})$ and the regular series $S \in \mathbb{K}\langle\langle \{a, b\}^* \rangle\rangle$ given by $(S, w) = \{w\}$ were considered. It was shown that for the identity mapping $\mathrm{id} : \{a, b\}^* \to \{a, b\}^*$ the series id$S$ (as defined in Corollary 3.19) is not regular any more. Hence from the proof of Corollary 3.19 above we conclude that Theorem 3.17(2) and (3) do not hold for non-commutative semirings.

4. Characterizations of recognizable text series

Texts, introduced by Ehrenfeucht and Rozenberg [21], extend the model of words by a second linear order. Think for example of an algebraic term which could be written in infix notation or in polish notation and hence is an object with two linear orders on it. Or think of a list of web pages in alphabetical order which contain a certain keyword and should be ranked by relevance. This ranking gives a second linear order. These considerations lead naturally to the notion of a text. The theory of texts originates in the theory of 2-structures, which was developed by Rozenberg and Ehrenfeucht (cf. [20]). They also proposed texts as a well-suited model for natural texts that may carry in its structure grammatical information [22, p.264].

A number of authors [23,38,39] investigated classes of text languages such as the families of context-free, equational or recognizable text languages and developed a language theory. In particular, the fundamental result of Büchi and Elgot (Theorem 3.2) was extended to texts by Hoogeboom and ten Pas [39]. In this section we generalize this result to a weighted setting by using and adapting definable transductions of Hoogeboom and ten Pas. Furthermore, we characterize recognizable series in terms of automata.

4.1. Definable and recognizable text series

Let Δ be a finite alphabet. A text is a word over Δ equipped with an additional linear order; more precisely it is defined as follows:

Definition 4.1. A *text* over Δ is a tuple $(V, \lambda, \leq_1, \leq_2)$, where $\leq_1$ and $\leq_2$ are linear orders over the finite but non-empty domain V and $\lambda : V \to \Delta$ is a labeling function.

We consider texts as relational structures where the relations are given by the labeling and by $\leq_1$ and $\leq_2$. More formally, a text is a relational structure over the signature $\mathrm{txtsig}_\Delta = \{(\mathrm{Lab}_a)_{a\in\Delta}, \leq_1, \leq_2\}$, where for all $a \in \Delta$ we let Lab_a be a unary symbol interpreted by the set of elements labeled with a and where $\leq_1$ and $\leq_2$ are binary symbols interpreted by the two linear orders. We define sRMSO($\mathbb{K}$, txtsig_Δ) and swRMSO($\mathbb{K}$, txtsig_Δ) with respect to the linear order $\leq_1$. We collect all texts over Δ in TXT(Δ), where as usual we identify isomorphic texts. For this reason, if not stated otherwise, we assume that for a text $(V, \lambda, \leq_1, \leq_2)$ we have $V = [n]$ for some positive integer n and that the first order $\leq_1$ coincides with the usual order on $[n]$. We may thus represent a text with domain $[n]$ by the pair $(\lambda(1) \ldots \lambda(n), (i_1, \ldots, i_n))$, where $i_1, \ldots, i_n \in [n]$ such that $i_1 <_2 \ldots <_2 i_n$. When visualizing a text in a picture we will often omit the first order and assume the nodes to be ordered from the left to the right. By $[i, j]_1$ we denote the set $\{k \in V \mid i \leq_1 k \leq_1 j\}$ for some $i, j \in V$ with $i \leq_1 j$. For $i, j \in V$ with $i \leq_2 j$ the subset $[i, j]_2$ of V is defined analogously. Subsets of V of this form are called *intervals* of the first and second order, respectively.

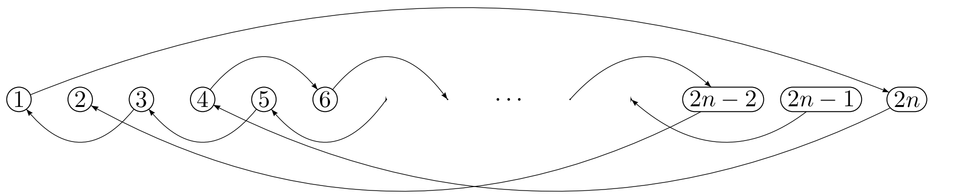
Let us start by defining an algebraic structure on the set of texts following [39]. A *biorder* is a pair of two linear orders over a common finite domain, i.e. a text without labeling. Again we identify isomorphic biorders and assume that the domain equals $[n]$ for some positive integer n. Consequently we represent a biorder with domain $[n]$ by the successor structure $(i_1, \ldots, i_n)$ of its second order. When visualizing a biorder we again often omit its first order. Each biorder π with domain $[n]$ defines an n-ary operation on texts — we obtain a new text $\pi(\tau_1, \ldots, \tau_n)$ by substituting given texts $\tau_1, \ldots, \tau_n$ into the nodes of the biorder. That is, we consider the disjoint union of the domains where, given two elements of the union, if they belong to the same text τ_i, their order is determined by τ_i; otherwise, if they belong to τ_i and τ_j for some $i \neq j$, then their order is given by the order of i and j in π. More formally, for all $i \in [n]$, let τ_i be a text with domain $[d_i]$ for some $d_i \in \mathbb{N}$. Define $\tau = \pi(\tau_1, \ldots, \tau_n)$ by letting $V(\tau) = \bigcup_{i\in[n]}[d_i] \times \{i\}$, by letting $(k, i) \in \mathrm{Lab}_a^\tau$ iff $k \in \mathrm{Lab}_a^{\tau_i}$ for all $a \in \Delta$ and by letting $(k, i) \leq_j (k', i')$ iff either $i <_j^\pi i'$ or $i = i'$ and $k \leq_j^{\tau_i} k'$ for $j \in \{1, 2\}$. The texts $\tau_1, \ldots, \tau_n$ then become intervals of the new text of both the first and the second order. This kind of operations for graphs is known as modular decomposition and has been rediscovered several times (cf. [56]).

Example 4.2. There are two biorders $\circ = (1, 2)$, $\bullet = (2, 1)$ of cardinality two.

Consider the texts $\tau_1 = (ab, (2, 1))$, $\tau_2 = (cd, (1, 2))$ and $\tau_3 = (ca, (2, 1))$. Then $\bullet(\tau_1, \tau_2) = (abcd, 3421)$ and $\circ(\bullet(\tau_1, \tau_2), \tau_3) = (abcdca, (3, 4, 2, 1, 6, 5))$. In the following, we will also use infix notation and write $\tau_1 \circ \tau_2$ for $\circ(\tau_1, \tau_2)$ and $\tau_1 \bullet \tau_2$ for $\bullet(\tau_1, \tau_2)$.

A subset of the domain of some text being an interval of both orders is called a *clan*. A biorder is *primitive* if it contains at least two elements and has only trivial clans, i.e. the singletons and the domain itself. Clearly, the two biorders $\circ$ and $\bullet$ of cardinality two (cf. Example 4.2) are both primitive. Let Σ be a set of primitive biorders and let $\mathrm{TXT}_\Sigma(\Delta)$ be the set of all texts generated from the singleton texts, i.e. from Δ, using Σ. If Σ comprises all primitive biorders, then $\mathrm{TXT}_\Sigma(\Delta) = \mathrm{TXT}(\Delta)$ [39]. We consider $\mathrm{TXT}_\Sigma(\Delta)$ as a Σ-algebra. Let $\mathrm{txt} : T_\Sigma(\Delta) \to \mathrm{TXT}_\Sigma(\Delta)$ be the natural epimorphism assigning to each term over Σ and Δ its value.

Example 4.3. Let $n \geq 3$ and let $\pi_n = (2n-1, 2n-3, \ldots, 1, 2n, 4, 6, 8, \ldots, 2n-2, 2)$ a biorder of length $2n$.



Observe that for any two vertices $i, i+1$ of π_n the smallest clan containing $i, i+1$ contains 1 and $2n$ since either $i \leq_2 1 \leq_2 2n \leq_2 i+1$ or $i+1 \leq_2 1 \leq_2 2n \leq_2 i$. Thus, for any $n \geq 3$, π_n does not contain non-trivial clans and is hence primitive. This shows that the cardinality of the set of all primitive biorders is $\aleph_0$.

Applying the theory of 2-structures developed by Ehrenfeucht and Rozenberg [21], one obtains that $\mathrm{TXT}_\Sigma(\Delta)$ is the free algebra in the class of all Σ-algebras where $\circ$ and $\bullet$ satisfy the associativity law. Thus, different preimages of a text $\tau \in \mathrm{TXT}(\Delta)$ under txt only differ with respect to these two associativity laws [39]. Let $\mathrm{sh}(\tau)$ be the preimage of τ where the brackets are in the rightmost form, that is, which contains neither $\circ(\circ(t_1, t_1), t_3)$ nor $\bullet(\bullet(t_1, t_1), t_3)$ as a subterm for all terms t_1, t_2, t_3. Hoogeboom and ten Pas called $\mathrm{sh}(\tau)$ the *r-shape* of τ. They considered finite sets Σ and called $L \subseteq \mathrm{TXT}_\Sigma(\Delta)$ in this case a language of *bounded primitivity*. *We now fix a finite set of primitive biorders* Σ. We will only consider the semantics of formulae relatively to $\mathrm{TXT}_\Sigma(\Delta)$.

Proposition 4.4 (*Hoogeboom & ten Pas [39]*). *The functions* $\mathrm{sh} : \mathrm{TXT}_\Sigma(\Delta) \to T_\Sigma(\Delta)$ *and* $\mathrm{txt} : T_\Sigma(\Delta) \to \mathrm{TXT}_\Sigma(\Delta)$ *are unambiguously* MSO*-definable.*

Since clearly $\mathrm{sh}^{-1}(\mathrm{txt}^{-1}(L)) = L$ for any $L \subseteq \mathrm{TXT}_\Sigma(\Delta)$, Hoogeboom and ten Pas deduce from Proposition 3.3, Proposition 3.16 and Proposition 2.2 the following theorem.

Theorem 4.5 (*Hoogeboom & ten Pas [39]*). *Let Σ be a finite set of primitive biorders. A text language $L \subseteq \mathrm{TXT}_\Sigma(\Delta)$ is recognizable iff L is* MSO*-definable.*

We will now consider formal text series, i.e. functions $S : \mathrm{TXT}_\Sigma(\Delta) \to \mathbb{K}$, and extend the latter result.

From Proposition 4.4 and the results of the previous sections we can immediately extend Theorem 4.5 to text series. Recall that a text series $S : \mathrm{TXT}_\Sigma(\Delta) \to \mathbb{K}$ is recognizable if there is a $\mathbb{K}$-Σ-algebra $\mathscr{A}$ of finite rank, a Σ-homomorphism $\varphi : \mathrm{TXT}_\Sigma(\Delta) \to \mathscr{A}$ and a linear form $\gamma : \mathscr{A} \to \mathbb{K}$ such that $\gamma \circ \varphi = S$.

Theorem 4.6. *Let $\mathbb{K}$ be a commutative ring or let $\mathbb{K}$ be a commutative and locally finite semiring. Let Σ be a finite set of primitive biorders. Then $S : \mathrm{TXT}_\Sigma(\Delta) \to \mathbb{K}$ is recognizable iff S is* sRMSO($\mathbb{K}$)*-definable. Moreover, if $\mathbb{K}$ is additively locally finite, then S is recognizable iff S is* swRMSO($\mathbb{K}$)*-definable. Furthermore, if $\mathbb{K}$ is locally finite, then S is recognizable iff S is* MSO($\mathbb{K}$)*-definable.*

Proof. Since the lexicographic order of the nodes of a tree is MSO(E_i)-definable, by Proposition 4.4 the function sh is MSO(txtsig_Δ)-definable even if we add the lexicographic order to MSO logic on trees. Hence, by Transfer Theorem 3.17(2) we get that $S = \mathrm{sh}^{-1}(\mathrm{txt}^{-1}(S))$ is sRMSO($\mathbb{K}$, txtsig_Δ)-definable if $\mathrm{txt}^{-1}(S)$ is sRMSO($\mathbb{K}$, $(E_i, \leq)$)-definable. Conversely, since txt is MSO(E_i)-definable, we get, again by Theorem 3.17(2), that $\mathrm{txt}^{-1}(S)$ is sRMSO($\mathbb{K}$, $(E_i, \leq)$)-definable if S is sRMSO($\mathbb{K}$)-definable and thus S is sRMSO($\mathbb{K}$, txtsig_Δ)-definable iff $\mathrm{txt}^{-1}(S)$ is sRMSO($\mathbb{K}$, $(E_i, \leq)$)-definable.

Now, we have

$$\begin{aligned} S \text{ is sRMSO}(\mathbb{K}, \mathrm{txtsig}_\Delta)\text{-definable} &\iff \mathrm{txt}^{-1}(S) \text{ is sRMSO}(\mathbb{K}, (E_i, \leq))\text{-definable} \\ &\overset{\text{Theorem 3.12}}{\iff} \mathrm{txt}^{-1}(S) \text{ is regular} \\ &\overset{\text{Proposition 2.11}}{\iff} \mathrm{txt}^{-1}(S) \text{ is recognizable} \\ &\overset{\text{Corollary 2.18}}{\iff} S \text{ is recognizable} \end{aligned}$$

where in the last step we used the assumption on $\mathbb{K}$. This proves the first assertion. The second and third assertions can be shown analogously. □

By observing additional properties of sh, in the following, we will extend the latter result also to the fragment sREMSO($\mathbb{K}$).

Proposition 4.7. *The function* sh : $\mathrm{TXT}_\Sigma(\Delta) \to T_\Sigma(\Delta)$ *is an unambiguously* FO*-definable function. This holds even if we equip trees with the lexicographic order.*

Proof. Let $\tau \in \mathrm{TXT}_\Sigma(\Delta)$ with domain $[n]$. Clearly, the leaves of sh(τ) are in bijection with $[n]$. (We choose the bijection such that the lexicographic order of the leaves corresponds to the order $\leq_1$ on $[n]$.) Moreover, since any primitive biorder has cardinality at least two, sh(τ) has at most $n-1$ inner nodes. In [39] a 2-copying definition scheme (without parameters) for sh was given where the first copy of the text τ corresponds to the leaves of sh(τ) and the second copy to the inner nodes. More precisely, if we identify the nodes $[n]$ of τ with the leaves of sh(τ) and if we let v be an inner node of sh(τ), then the node of τ which corresponds to v is given by moving to the last child of v and then repeatedly moving to the first child until a leaf $i_v \in [n]$ is reached.

Now, let $I_v \subseteq [n]$ be the set of leaves of the subtree of sh(τ) rooted at v. Crucial for the definition scheme of Hoogeboom and ten Pas is the formula assoc(x, X) for which we have $\tau \models$ assoc$[i, I]$ iff $i = i_v$ and $I = I_v$ for some inner node v of sh(τ). There are other formulae involved in the definition scheme of Hoogeboom and ten Pas which contain nested universal quantification over sets. However, on analyzing the formulae it turns out that any quantification only concerns intervals of the first order. Hence, we can transform them into equivalent first-order formulae by identifying an interval with its first and its last elements. So, for example, we transform assoc(x, X) into a formula assoc(x, x_1, x_2) such that we have $\tau \models$ assoc$[i, i_1, i_2]$ iff $i = i_v$ and $\{i \in [n] \mid i_1 \leq_1 i \leq_1 i_2\} = I_v$ for some inner node v of sh(τ). This way we transform the definition scheme into one consisting of first-order formulae only.

If we now add the lexicographic order $\leq$ to MSO logic on trees, we also have to give interpreting formulae for it. The formulae are as follows:

$$\varphi_{\leq}^{1,1}(x, y) = x \leq_1 y$$
$$\varphi_{\leq}^{1,2}(x, y) = \exists y_1, y_2.\mathrm{assoc}(y, y_1, y_2) \wedge x <_1 y_1$$
$$\varphi_{\leq}^{2,1}(x, y) = \exists x_1, x_2.\mathrm{assoc}(x, x_1, x_2) \wedge x_1 \leq_1 y$$
$$\varphi_{\leq}^{2,2}(x, y) = \exists x_1, x_2, y_1, y_2.\ \mathrm{assoc}(x, x_1, x_2) \wedge \mathrm{assoc}(y, y_1, y_2) \wedge (x_1 <_1 y_1 \vee (x_1 = y_1 \wedge y_2 \leq x_2)).$$ □

Hence, similarly to Theorem 4.6, using Proposition 4.7 and Transfer Theorem 3.17(4), we obtain the following.

Theorem 4.8. *Let $\mathbb{K}$ be a commutative ring or let $\mathbb{K}$ be a commutative and locally finite semiring. Let Σ be a finite set of primitive biorders and let S : $\mathrm{TXT}_\Sigma(\Delta) \to \mathbb{K}$. Then S is recognizable iff S is* sREMSO($\mathbb{K}$)*-definable.*

Let us call a field *computable* if all operations $(+, -, \cdot, ^{-1})$ are computable. So, for example, the rationals $\mathbb{Q}$ form a computable field.

Corollary 4.9. *Let $\mathbb{K}$ be a computable field and let Σ be a finite set of primitive biorders. It is decidable whether two given sentences $\varphi, \psi \in$* sRMSO($\mathbb{K}$, txtsig$_\Delta$) *satisfy $[\![\varphi]\!] = [\![\psi]\!]$.*

Proof. The proofs of Proposition 4.4 and Transfer Theorem 3.17 are effective. Thus, given $\varphi, \psi \in$ sRMSO($\mathbb{K}$, txtsig$_\Delta$) we can construct tree formulae $\varphi', \psi' \in$ sRMSO($\mathbb{K}$, $(E_i, \leq)$) from which in turn we can effectively construct two weighted tree automata $\mathcal{A}, \mathcal{A}'$ such that $\|\mathcal{A}\| = \mathrm{txt}^{-1}([\![\varphi]\!])$ and $\|\mathcal{A}'\| = \mathrm{txt}^{-1}([\![\psi]\!])$. Clearly, $[\![\varphi]\!] = [\![\psi]\!]$ iff $\|\mathcal{A}\| = \|\mathcal{A}'\|$. The latter can be decided by Theorem 4.1 of [62]. □

Similarly, for locally finite semirings the proof of Theorem 3.12 is also effective; hence, given a formula $\varphi \in$ MSO($\mathbb{K}$) we can construct a weighted tree automaton $\mathcal{A}$ such that $[\![\varphi]\!] = \|\mathcal{A}\|$, and we obtain:

Corollary 4.10. *Let $\mathbb{K}$ be a computable locally finite semiring and let Σ be a finite set of primitive biorders. It is decidable whether two given sentences $\varphi, \psi \in$* MSO($\mathbb{K}$, txtsig$_\Delta$) *satisfy $[\![\varphi]\!] = [\![\psi]\!]$.*

Using the Boolean semiring $\mathbb{B}$, we obtain from Theorem 4.8 the following corollary, which sharpens one implication of the result of Hoogeboom and ten Pas (Theorem 4.5).

Corollary 4.11. *Let Σ be a finite set of primitive biorders. A language $L \subseteq \mathrm{TXT}_\Sigma(\Delta)$ is* MSO*-definable iff it is* EMSO*-definable.*

Example 4.12. Let $\mathbb{K} = \mathbb{Z}$ be the ring of integers. Let Clan(x_1, x_2) be a first-order formula saying that the interval $[x_1, x_2]_1$ is a proper clan:

$$\mathrm{Clan}(x_1, x_2) = x_1 <_1 x_2 \ \wedge\ \exists z_1, z_2.\ \forall x_2'.\ (x_1 \leq_1 x_2' \leq_1 x_2) \to (z_1 \leq_2 x_2' \leq_2 z_2).$$

Consider

$$\varphi = \exists x_1, x_2.\mathrm{Clan}(x_1, x_2)^+ \wedge \forall x, y.\ x_1 \leq_1 x, y \leq_1 x_2 \xrightarrow{+} (x \leq_1 y \leftrightarrow y \leq_2 x)^+.$$

For a text τ, $([\![\varphi]\!], \tau)$ gives the number of proper clans generated only from the biorder •. By Theorem 4.6, $[\![\varphi]\!]$ is recognizable.

4.2. Automata over texts

Next, we discuss an automaton model inspired by the model of branching automata of Lodaya and Weil [46] and the model of parenthesizing automata of Ésik and Németh [30]. We combine both models into a generalized automaton model which again is generalized using weights. The automata will enable us to extend Theorems 4.6 and 4.8 to arbitrary commutative semirings. Note that in [51] it was demonstrated that, when considered for texts, Thomas graph acceptors [65] are weaker than the automata presented now.

For the rest of this section let Π be a finite set of primitive biorders of cardinality at least four[6] *and let $\Sigma = \Pi \cup \{\circ, \bullet\}$.*

All the following definitions can be easily adapted to the cases where $\Sigma = \Pi \cup \{\circ\}$, $\Sigma = \Pi \cup \{\bullet\}$ or $\Sigma = \Pi$ such that all the results hold true. However, we will only consider the case $\Sigma = \Pi \cup \{\circ, \bullet\}$, which is the most complicated one.

Definition 4.13. A *weighted branching and parenthesizing automaton* (WBPA) $\mathcal{A}$ is a tuple $\mathcal{A} = (\mathcal{H}, \mathcal{V}, \Omega, \lambda, \bar{\mu}, \gamma)$, where $\bar{\mu} = (\mu_{\mathrm{op}}, \mu_{\mathrm{cl}}, \mu, (\mu_\pi^{\mathrm{fo}})_{\pi\in\Pi}, (\mu_\pi^{\mathrm{jo}})_{\pi\in\Pi})$ such that

- $\mathcal{H}$ and $\mathcal{V}$ are finite, disjoint sets of *horizontal* and *vertical states*, respectively,
- Ω is a finite set of *parentheses*, [7]
- $\mu : (\mathcal{H} \times \Delta \times \mathcal{H}) \cup (\mathcal{V} \times \Delta \times \mathcal{V}) \to \mathbb{K}$ is the *transition function*,
- $\mu_{\mathrm{op}}, \mu_{\mathrm{cl}} : (\mathcal{H} \times \Omega \times \mathcal{V}) \cup (\mathcal{V} \times \Omega \times \mathcal{H}) \to \mathbb{K}$ are the *opening* and *closing parenthesizing functions*, respectively,
- $\mu_\pi^{\mathrm{fo}}, \mu_\pi^{\mathrm{jo}} : (\mathcal{H} \cup \mathcal{V})^{k+1} \to \mathbb{K}$, where $k = \mathrm{rk}(\pi)$, are the *fork* and *join transition functions* and
- $\lambda, \gamma : \mathcal{H} \cup \mathcal{V} \to \mathbb{K}$ are the *initial* and *final weight functions*, respectively.

We now come to the notion of a run r of $\mathcal{A}$. We will given an inductive definition where we also define its *label* $\mathrm{lab}(r) \in \mathrm{TXT}_\Sigma(\Delta)$, its *weight* $\mathrm{wgt}_\mathcal{A}(r) \in \mathbb{K}$, its *initial state* $\mathrm{init}(r) \in \mathcal{H} \cup \mathcal{V}$ and its *final state* $\mathrm{fin}(r) \in \mathcal{H} \cup \mathcal{V}$. Formally, the set of *runs* of $\mathcal{A}$ is the smallest set of words over the alphabet $\Delta \cup \Omega \cup \mathcal{H} \cup \mathcal{V} \cup \{(, [,],)\} \cup \{, \}$ such that:

1. The word (q_1, a, q_2) is a run for all $(q_1, q_2) \in (\mathcal{H} \times \mathcal{H}) \cup (\mathcal{V} \times \mathcal{V})$ and $a \in \Delta$. We set

$$\mathrm{lab}((q_1, a, q_2)) = a \in \mathrm{TXT}_\Sigma(\Delta), \qquad \mathrm{wgt}_\mathcal{A}((q_1, a, q_2)) = \mu(q_1, a, q_2),$$
$$\mathrm{init}((q_1, a, q_2)) = q_1 \quad \text{and} \quad \mathrm{fin}((q_1, a, q_2)) = q_2.$$

2. If r_1 and r_2 are runs such that $\mathrm{fin}(r_1) = \mathrm{init}(r_2) \in \mathcal{H}$ (respectively, such that $\mathrm{fin}(r_1) = \mathrm{init}(r_2) \in \mathcal{V}$), then $r = r_1r_2$ is a run having

$$\mathrm{lab}(r) = \mathrm{lab}(r_1) \circ \mathrm{lab}(r_2), \qquad (\text{resp. } \mathrm{lab}(r) = \mathrm{lab}(r_1) \bullet \mathrm{lab}(r_2)),$$
$$\mathrm{wgt}_\mathcal{A}(r) = \mathrm{wgt}_\mathcal{A}(r_1) \cdot \mathrm{wgt}_\mathcal{A}(r_2), \qquad \mathrm{init}(r) = \mathrm{init}(r_1) \quad \text{and} \quad \mathrm{fin}(r) = \mathrm{fin}(r_2).$$

3. If r is a run resulting from (2) having $\mathrm{init}(r) \in \mathcal{H}$ (respectively, $\mathrm{init}(r) \in \mathcal{V}$) and if $q_1, q_2 \in \mathcal{V}$ (respectively, if $q_1, q_2 \in \mathcal{H}$) and $s \in \Omega$, then $r' = (q_1, (_s, \mathrm{init}(r))\ r\ (\mathrm{fin}(r),)_s, q_2)$ is a run. We set

$$\mathrm{lab}(r') = \mathrm{lab}(r), \qquad \mathrm{init}(r') = q_1 \quad \text{and} \quad \mathrm{fin}(r') = q_2,$$
$$\mathrm{wgt}_\mathcal{A}(r') = \mu_{\mathrm{op}}((q_1, (_s, \mathrm{init}(r))) \cdot \mathrm{wgt}_\mathcal{A}(r) \cdot \mu_{\mathrm{cl}}((\mathrm{fin}(r),)_s, q_2)).$$

4. If $\pi \in \Pi$ with $\mathrm{rk}(\pi) = k$ and if $r_1, \ldots, r_k$ are runs, moreover, if $q, p \in \mathcal{H}$ (respectively, $q, p \in \mathcal{V}$), then $r = (q, \pi, \mathrm{init}(r_1), \ldots, \mathrm{init}(r_k))[r_1, \ldots, r_k](\mathrm{fin}(r_1), \ldots, \mathrm{fin}(r_k), \pi, p)$ is a run having

$$\mathrm{lab}(r) = \pi(\mathrm{lab}(r_1), \ldots, \mathrm{lab}(r_k)), \qquad \mathrm{init}(r) = q \quad \text{and} \quad \mathrm{fin}(r) = p,$$
$$\mathrm{wgt}_\mathcal{A}(r) = \mu_\pi^{\mathrm{fo}}(q, \mathrm{init}(r_1), \ldots, \mathrm{init}(r_k)) \cdot \mathrm{wgt}_\mathcal{A}(r_1) \cdot \ldots \cdot \mathrm{wgt}_\mathcal{A}(r_k) \cdot \mu_\pi^{\mathrm{jo}}(\mathrm{fin}(r_1), \ldots, \mathrm{fin}(r_k), p).$$

Now, let $\tau \in \mathrm{TXT}_\Sigma(\Delta)$. Since in (3) above we require that the run r we start with results from (2), we do not allow repeated application of (3), and therefore there are only finitely many runs r of $\mathcal{A}$ with label τ. Intuitively we do not allow for doubled parentheses. If r is a run of $\mathcal{A}$ with $\mathrm{lab}(r) = \tau$, $\mathrm{init}(r) = q_1$, $\mathrm{fin}(r) = q_2$, we write $r : q_1 \xrightarrow{\tau} q_2$. Observe that q_1 and q_2 are either both in $\mathcal{H}$ or both in $\mathcal{V}$. The behavior of $\mathcal{A}$ is a text series $\|\mathcal{A}\|$: $\mathrm{TXT}_\Sigma(\Delta) \to \mathbb{K}$. It is given by

$$(\|\mathcal{A}\|, \tau) = \sum_{q_1, q_2 \in \mathcal{H}\cup\mathcal{V}} \lambda(q_1) \cdot \sum_{r : q_1 \xrightarrow{\tau} q_2} \mathrm{wgt}_\mathcal{A}(r) \cdot \gamma(q_2).$$

A text series S is *regular* if there is a WBPA $\mathcal{A}$ such that $\|\mathcal{A}\| = S$.

Remark 4.14. We note that a WBPA could also be defined in the same manner for any free algebra over a finite family of operations Π and two associative operations. It is even straightforward to extend the definition to more than two associative

[6] Observe that there is no primitive biorder of cardinality three.

[7] We let $s \in \Omega$ represent both the opening and the closing parentheses. To help the intuition we also write $(_s$ or $)_s$ for s.

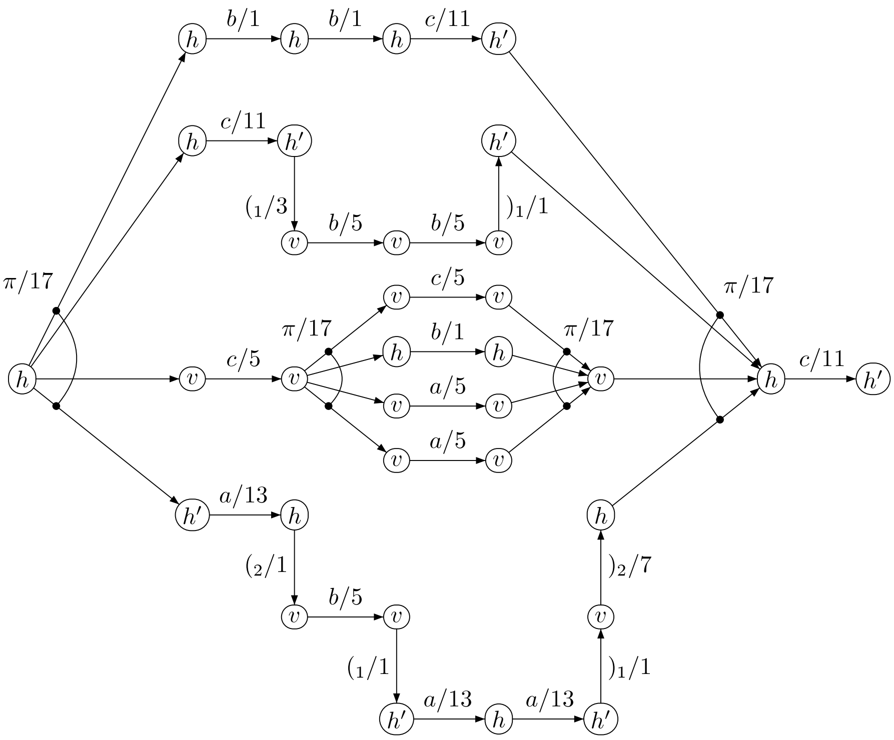


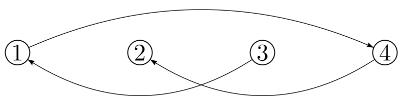
Fig. 1. The visualization of a sample run of $\mathcal{A}$ on τ as given in Example 4.15.

operations. However, if we admit unary operations in Π, then the proof of Proposition 4.17 below does not work. In order to make it work, one has to introduce "dummy waiting states" (cf. [46, p.274]).

Let us further remark that WBPAs are related to the weighted branching automata of Kuske and Meinecke [44]. These are automata processing elements of free algebras over two associative operations, one of which is commutative. This extra commutativity makes the objects more complicated to deal with. Moreover, Kuske and Meinecke used so-called bisemirings rather than semirings as a weight structure. Bisemirings are essentially commutative semirings with a second multiplication which also distributes over addition. Bisemiring-weighted automata for the free algebra over a finite family of operations Π and a single associative operation were considered by Heger [37].

Example 4.15. A run r of a WBPA can be visualized as a graph. The nodes correspond to the states that appear in the run. The edges are given as follows. If $r = (q_1, a, q_2)$, then we have an edge between q_1 and q_2 labeled with $a/\mu(a)$. If r is of the form r_1r_2 for two runs r_1, r_2, then we identify the nodes corresponding to $\text{init}(r_1)$ and $\text{fin}(r_2)$. If r is of the form $(q_1, (_s, \text{init}(r'))\, r'\, (\text{fin}(r'),)_s, q_2)$, then we introduce two new nodes for q_1 and q_2 and two new edges, one from q_1 to $\text{init}(r')$ labeled $(_s$ and $\mu(q_1, (_s, \text{init}(r'))$, and one from $\text{fin}(r')$ to q_2 labeled with $)_s$ and $\mu(\text{fin}(r'),)_s, q_2)$. Last, if r is of the form $(q, \pi, \text{init}(r_1), \ldots, \text{init}(r_k))[r_1, \ldots, r_k](\text{fin}(r_1), \ldots, \text{fin}(r_k), \pi, p)$, then we introduce two new nodes for q and p and two new edges for each r_i $(1 \le i \le k)$, one from q to $\text{init}(r_i)$ and one from $\text{fin}(r_i)$ to p.

Consider for example the following primitive biorder π of cardinality 4.



The text $\tau = (bbccbbccbaaabaac, (10, 8, 11, 9, 7, 1, 2, 3, 12, 14, 15, 13, 4, 6, 5, 16))$ is given by

$$\pi(b \circ (b \circ c), c \circ (b \bullet b), c \bullet \pi(c, b, a, a), a \circ (b \bullet (a \circ a))) \circ c.$$

Now, let the WBPA $\mathcal{A} = (\mathcal{H}, \mathcal{V}, \Omega, \lambda, \bar{\mu}, \gamma)$ be given as follows. We have $\mathcal{H} = \{h, h'\}$, $\mathcal{V} = \{v\}$ and $\Omega = \{1, 2\}$. Moreover, we let $\mu_{\text{op}}(h, (_2, v) = \mu_{\text{op}}(v, (_1, h') = 1$, $\mu_{\text{op}}(h', (_1, v) = 3$, $\mu_{\text{cl}}(v,)_2, h) = 7$ and $\mu_{\text{cl}}(v,)_1, h') = \mu_{\text{cl}}(h',)_1, v) = 1$. Additionally, let $\mu(h, b, h) = 1$, $\mu(h, c, h') = 11$, $\mu(h', a, h) = \mu(h, a, h') = 13$ and $\mu(v, \delta, v) = 5$ for all $\delta \in \{a, b, c\}$. Last, let $\mu_\pi^{\text{fo}}(h, h, h, v, h') = \mu_\pi^{\text{fo}}(v, v, h, v, v) = \mu_\pi^{\text{jo}}(v, h, v, v, v) = \mu_\pi^{\text{jo}}(h', h', v, h, h) = 17$. We set any other value to 0. Now, Fig. 1 shows a sample run of $\mathcal{A}$ on τ.

Proposition 4.16. *Let $S : \text{TXT}_\Sigma(\Delta) \to \mathbb{K}$. If S is regular then S is recognizable.*

Proof. Let $\mathcal{P} = (\mathcal{H}, \mathcal{V}, \Omega, \lambda, \bar{\mu}, \gamma)$, where $\bar{\mu} = (\mu_{\text{op}}, \mu_{\text{cl}}, \mu, (\mu_\pi^{\text{fo}})_{\pi\in\Pi}, (\mu_\pi^{\text{jo}})_{\pi\in\Pi})$ be a WBPA. We start by constructing a tree automaton $\mathcal{A} = (Q, \delta, \kappa)$ over $T_\Sigma(\Delta)$ such that $\|\mathcal{A}\| = \text{txt}^{-1}(\|\mathcal{P}\|)$. For this, let $Q = (\mathcal{H} \times \mathcal{H}) \cup (\mathcal{V} \times \mathcal{V})$. The idea is that the runs of $\mathcal{A}$ on some tree t where the root is assigned state (q, p) correspond to the runs of $\mathcal{P}$ on $\mathrm{txt}(t)$ from q to p. We now define formally $\delta_a \in \mathbb{K}^Q$ for all $a \in \Delta$, $\delta_\circ : Q \times Q \to \mathbb{K}^Q$ and $\delta_\bullet : Q \times Q \to \mathbb{K}^Q$ as well as $\delta_\pi : Q^k \to \mathbb{K}^Q$ for all $\pi \in \Pi$, where $k = \mathrm{rk}(\pi)$ and $\kappa : Q \to \mathbb{K}$. Let, for all $h_1, h_2, h_3 \in \mathcal{H}$ and $v_1, v_2, v_3 \in \mathcal{V}$ as well as for all $(q, p), (q_1, p_1), \ldots (q_k, p_k) \in Q$,

$$\begin{aligned}
&(\delta_a)_{(q,p)} = \mu(q, a, p),\\
&\delta_\circ((h_1, h_3), (h_3, h_2))_{(h_1,h_2)} = 1 = \delta_\bullet((v_1, v_3), (v_3, v_2))_{(v_1,v_2)},\\
&\delta_\circ((h_1, h_2), (h_2, h_3))_{(v_1,v_2)} = \sum_{s\in\Omega} \mu_{\mathrm{op}}(v_1, (_s, h_1) \cdot \mu_{\mathrm{cl}}(h_3,)_s, v_2),\\
&\delta_\bullet((v_1, v_2), (v_2, v_3))_{(h_1,h_2)} = \sum_{s\in\Omega} \mu_{\mathrm{op}}(h_1, (_s, v_1) \cdot \mu_{\mathrm{cl}}(v_3,)_s, h_2),\\
&\delta_\pi((q_1, p_1), (q_2, p_2), \ldots, (q_k, p_k))_{(q,p)} = \mu^{\mathrm{fo}}_\pi(q, q_1, \ldots, q_k) \cdot \mu^{\mathrm{jo}}_\pi(p, p_1, \ldots, p_k),\\
&\kappa((q, p)) = \lambda(q) \cdot \gamma(p).
\end{aligned}$$

Any other value is set to 0. As described in Section 2.1 after the definition of weighted tree automata, we can extend these functions $(\delta_f)_{f\in\Delta\cup\Sigma}$ to multilinear operations on $\mathbb{K}^Q$ and hence turn $\mathbb{K}^Q$ into a $\mathbb{K}$-Σ-algebra. Moreover, again as described in Section 2.1, we obtain a Σ-homomorphism $\delta : T_\Sigma(\Delta) \to \mathbb{K}^Q$ and a linear form $\kappa : \mathbb{K}^Q \to \mathbb{K}$. Let $t \in T_\Sigma(\Delta)$. We show that

$$\delta(t)_{(q_1,q_2)} = \sum_{r:q_1 \xrightarrow{\mathrm{txt}(t)} q_2} \mathrm{wgt}_{\mathcal{P}}(r) \quad \text{for all } (q_1, q_2) \in Q. \tag{2}$$

Therefore we proceed by induction on t. For $a \in \Delta$ we have

$$\delta(a)_{(q_1,q_2)} = \mu(q_1, a, q_2) = \sum_{r:q_1 \xrightarrow{a} q_2} \mathrm{wgt}_{\mathcal{P}}(r).$$

Let $t = \circ(t_1, t_2)$, then

$$\begin{aligned}
\delta(t)_{(h_1,h_2)} &= \sum_{(q_1,q_2),(q_3,q_4)\in Q} \delta_\circ((q_1, q_2), (q_3, q_4))_{(h_1,h_2)} \cdot \delta(t_1)_{(q_1,q_2)} \cdot \delta(t_2)_{(q_3,q_4)}\\
&= \sum_{h_3\in\mathcal{H}} \delta_\circ((h_1, h_3), (h_3, h_2))_{(h_1,h_2)} \cdot \delta(t_1)_{(h_1,h_3)} \cdot \delta(t_2)_{(h_3,h_2)}\\
&= \sum_{h_3\in\mathcal{H}} \sum_{r_1:h_1 \xrightarrow{\mathrm{txt}(t_1)} h_3} \mathrm{wgt}_{\mathcal{P}}(r_1) \cdot \sum_{r_2:h_3 \xrightarrow{\mathrm{txt}(t_2)} h_2} \mathrm{wgt}_{\mathcal{P}}(r_2)\\
&= \sum_{r:h_1 \xrightarrow{\mathrm{txt}(t)} h_2} \mathrm{wgt}_{\mathcal{P}} r)
\end{aligned}$$

and

$$\begin{aligned}
\delta(t)_{(v_1,v_2)} &= \sum_{(q_1,q_2),(q_3,q_4)\in Q} \delta_\circ((q_1, q_2), (q_3, q_4))_{(v_1,v_2)} \cdot \delta(t_1)_{(q_1,q_2)} \cdot \delta(t_2)_{(q_3,q_4)}\\
&= \sum_{h_1,h_2,h_3\in\mathcal{H}} \delta_\circ((h_1, h_2), (h_2, h_3))_{(v_1,v_2)} \cdot \delta(t_1)_{(h_1,h_2)} \cdot \delta(t_2)_{(h_2,h_3)}\\
&= \sum_{h_1,h_2,h_3\in\mathcal{H}} \sum_{s\in\Omega} \mu_{\mathrm{op}}(v_1, (_s, h_1) \cdot \sum_{r_1:h_1 \xrightarrow{\mathrm{txt}(t_1)} h_2} \mathrm{wgt}_{\mathcal{P}}(r_1) \cdot \sum_{r_2:h_2 \xrightarrow{\mathrm{txt}(t_2)} h_3} \mathrm{wgt}_{\mathcal{P}}(r_2) \cdot \mu_{\mathrm{cl}}(h_3,)_s, v_2)\\
&= \sum_{r:v_1 \xrightarrow{\mathrm{txt}(t)} v_2} \mathrm{wgt}_{\mathcal{P}}(r).
\end{aligned}$$

The case where $t = \bullet(t_1, t_2)$ can be shown analogously, since the definition of δ is symmetric. Now, let $t = \pi(t_1, \ldots, t_k)$ for some $\pi \in \Pi$ with $\mathrm{rk}(\pi) = k$ and some $t_1, \ldots, t_k \in T_\Sigma(\Delta)$. Then

$$\begin{aligned}
\delta(t)_{(q,p)} &= \sum_{(q_1,p_1),\ldots,(q_k,p_k)\in Q} \delta_\pi((q_1, p_1), \ldots, (q_k, p_k))_{(q,p)} \cdot \delta(t_1)_{(q_1,p_1)} \cdots \delta(t_k)_{(q_k,p_k)}\\
&= \sum_{(q_1,p_1),\ldots,(q_k,p_k)\in Q} \mu^{\mathrm{fo}}_\pi(q, q_1, \ldots, q_k) \cdot \sum_{r_1:q_1 \xrightarrow{\mathrm{txt}(t_1)} p_1} \mathrm{wgt}_{\mathcal{P}}(r_1) \cdots \sum_{r_k:q_k \xrightarrow{\mathrm{txt}(t_k)} p_k} \mathrm{wgt}_{\mathcal{P}}(r_k) \cdot \mu^{\mathrm{jo}}_\pi(p, p_1, \ldots, p_k)\\
&= \sum_{r:q \xrightarrow{\mathrm{txt}(t)} p} \mathrm{wgt}_{\mathcal{P}}(r).
\end{aligned}$$

This concludes the proof of Eq. (2). Together with the definition of κ we hence get $\|\mathcal{A}\| = \text{txt}^{-1}(\|\mathcal{P}\|)$. Clearly, from Eq. (2) we also get $\delta(\circ(\circ(t_1, t_2), t_3)) = \delta(\circ(t_1, \circ(t_2, t_3)))$ and $\delta(\bullet(\bullet(t_1, t_2), t_3)) = \delta(\bullet(t_1, \bullet(t_2, t_3)))$ for all $t_1, t_2, t_3 \in T_\Sigma(\Delta)$. We conclude that $\ker(\text{txt}) \subseteq \ker(\delta)$. There is thus a Σ-homomorphism $\varphi : \text{TXT}_\Sigma(\Delta) \to \mathbb{K}^Q$ such that $\varphi \circ \text{txt} = \delta$. We conclude that $\|\mathcal{P}\| = \text{txt}^{-1}(\|\mathcal{P}\|) \circ \text{sh} = \kappa \circ \delta \circ \text{sh} = \kappa \circ \varphi \circ \text{txt} \circ \text{sh} = \kappa \circ \varphi$. Hence, (φ, κ) is a representation of $\|\mathcal{P}\|$. □

Proposition 4.17. *Let $S : T_\Sigma(\Delta) \to \mathbb{K}$ be a regular tree series. Then $\text{sh}^{-1}(S)$ is regular.*

Proof. Let $\mathcal{A} = (Q, (\delta_g)_{g\in\Sigma}, \kappa)$ be a weighted tree automaton over $T_\Sigma(\Delta)$ and let $\delta : T_\Sigma(\Delta) \to \mathbb{K}^Q$ be defined as in Section 2.1. Let f be a fresh symbol not in Q and let $Q' = \{f\} \uplus Q$. We set $\mathcal{H} = \{q^{\mathcal{H}} \mid q \in Q'\} \times Q'$, $\mathcal{V} = \{q^{\mathcal{V}} \mid q \in Q'\} \times Q'$ and $\Omega = Q'$. We construct a WBPA $\mathcal{P} = (\mathcal{H}, \mathcal{V}, \Omega, \lambda, (\mu_{\text{op}}, \mu_{\text{cl}}, \mu, (\mu_\pi^{\text{fo}})_{\pi\in\Pi}, (\mu_\pi^{\text{jo}})_{\pi\in\Pi}), \gamma)$ such that $\|\mathcal{P}\| = \text{sh}^{-1}(\|\mathcal{A}\|)$.

Given some $\tau \in \text{TXT}_\Sigma(\Delta)$, the idea of our construction will be as follows. The WBPA will simulate the (top-down) behavior of $\mathcal{A}$ on $\text{sh}(\tau)$. The traversal of $\text{sh}(\tau)$ will be such that at nodes labeled $\circ$ or $\bullet$ first the tree rooted at the right child is processed and then the tree rooted at the left child is processed. For a node labeled with some $\pi \in \Pi$ the WBPA will use its ability to fork and process the trees rooted at the children in parallel. Let us consider a small example and look at the text τ given by $(a \bullet b) \circ \pi(a, b, c, d) \circ e$ and $\text{sh}(\tau) = \circ(\bullet(a, b), \circ(\pi(a, b, c, d), e))$, where π is some 4-ary operation in Π. Recall that we identify the nodes of $\text{sh}(\tau)$ with words in $(\mathbb{N}_+)^*$ as described in Example 3.1(2). Let us assume that $\mathcal{P}$ starts in state $(q^{\mathcal{H}}, p)$. It will first use an opening parenthesizing transition to simulate a transition of $\mathcal{A}$ at the root of $\text{sh}(\tau)$. Suppose that $\mathcal{A}$ assigns states q_1 and q_2 to nodes 1 and 2 of $\text{sh}(\tau)$, respectively. Then $\mathcal{P}$ will change to state $(q_1^{\mathcal{V}}, p)$ while storing q_2 in the parenthesis (i.e. opening the parenthesis $(_{q_2}$). The weight of this transition will equal the weight of the transition of $\mathcal{A}$, i.e. $\delta_\circ(q_1, q_2)_q$. In the next step when processing the first a, the WBPA will simulate transitions of $\mathcal{A}$ at both nodes 1 and 11. Next, $\mathcal{P}$ will simulate a transition of $\mathcal{A}$ at node 12. We ask the reader to observe in the formal definition below that the fresh state f is used to distinguish the last two cases. Next, $\mathcal{P}$ will close the parenthesis and recover q_2. Now, $\mathcal{P}$ will execute a forking transition and simulate transitions of $\mathcal{A}$ at nodes 2 and 21. Assume that $\mathcal{A}$ assigns states $p_1, \dots, p_4$ to nodes 211, 212, 213, 214, respectively, and state q_3 to node 22. Then $\mathcal{P}$ will fork in states $(p_1^{\mathcal{H}}, p)$, $(p_2^{\mathcal{H}}, p)$, $(p_3^{\mathcal{H}}, p)$ and $(p_4^{\mathcal{H}}, q_3)$, storing state q_3 in the second component of the fourth state. It will recover this state when joining again. Observe that since forks and joins can be nested, we also have to memorize p and hence the latter trick only works as $\text{rk}(\pi) \geq 2$ for all $\pi \in \Pi$. A similar trick was used in [46]. Now, after joining, $\mathcal{P}$ will finally simulate the behavior of $\mathcal{A}$ at node 22 and terminate.

We now give the definitions more formally. We define $\mu, \mu_{\text{op}}, \mu_{\text{cl}}, \mu^{\text{fo}}, \mu^{\text{jo}}, \lambda, \gamma$. For all $a \in \Delta, \pi \in \Pi$ with $\text{rk}(\pi) = k$, for all $q, \tilde{q}, q_1, q_2, q_3, \dots, q_k \in Q$ and $q', q'' \in Q'$, let

$$\mu((q^{\mathcal{V}}, q'), a, (f^{\mathcal{V}}, q')) = \mu((q^{\mathcal{H}}, q'), a, (f^{\mathcal{H}}, q')) = (\delta_a)_q,$$
$$\mu((q_1^{\mathcal{H}}, q'), a, (q_2^{\mathcal{H}}, q')) = \sum_{s\in Q} \delta_\circ(s, q_2)_{q_1} \cdot (\delta_a)_s,$$
$$\mu((q_1^{\mathcal{V}}, q'), a, (q_2^{\mathcal{V}}, q')) = \sum_{s\in Q} \delta_\bullet(s, q_2)_{q_1} \cdot (\delta_a)_s,$$
$$\mu_{\text{op}}((q_1^{\mathcal{H}}, q'), (_{q_3}, (q_2^{\mathcal{V}}, q')) = \delta_\circ(q_2, q_3)_{q_1},$$
$$\mu_{\text{op}}((q_1^{\mathcal{V}}, q'), (_{q_3}, (q_2^{\mathcal{H}}, q')) = \delta_\bullet(q_2, q_3)_{q_1},$$
$$\mu_{\text{op}}((q^{\mathcal{H}}, q'), (_f, (q^{\mathcal{V}}, q')) = \mu_{\text{op}}((q^{\mathcal{V}}, q'), (_f, (q^{\mathcal{H}}, q')) = 1,$$
$$\mu_{\text{cl}}((f^{\mathcal{H}}, q'),)_q, (q^{\mathcal{V}}, q')) = \mu_{\text{cl}}((f^{\mathcal{V}}, q'),)_q, (q^{\mathcal{H}}, q')) = 1,$$
$$\mu_{\text{cl}}((f^{\mathcal{H}}, q'),)_f, (f^{\mathcal{V}}, q')) = \mu_{\text{cl}}((f^{\mathcal{V}}, q'),)_f, (f^{\mathcal{H}}, q')) = 1,$$
$$\mu_\pi^{\text{fo}}((q^{\mathcal{H}}, q'), (q_1^{\mathcal{H}}, q'), \dots, (q_{k-1}^{\mathcal{H}}, q'), (q_k^{\mathcal{H}}, \tilde{q})) = \sum_{s\in Q} \delta_\circ(s, \tilde{q})_q \cdot \delta_\pi(q_1, \dots, q_k)_s$$
$$\mu_\pi^{\text{fo}}((q^{\mathcal{V}}, q'), (q_1^{\mathcal{V}}, q'), \dots, (q_{k-1}^{\mathcal{V}}, q'), (q_k^{\mathcal{V}}, \tilde{q})) = \sum_{s\in Q} \delta_\bullet(s, \tilde{q})_q \cdot \delta_\pi(q_1, \dots, q_k)_s$$
$$\mu_\pi^{\text{fo}}((q^{\mathcal{H}}, q'), (q_1^{\mathcal{H}}, q'), \dots, (q_{k-1}^{\mathcal{H}}, q'), (q_k^{\mathcal{H}}, f)) = \delta_\pi(q_1, \dots, q_k)_q$$
$$\mu_\pi^{\text{fo}}((q^{\mathcal{V}}, q'), (q_1^{\mathcal{V}}, q'), \dots, (q_{k-1}^{\mathcal{V}}, q'), (q_k^{\mathcal{V}}, f)) = \delta_\pi(q_1, \dots, q_k)_q$$
$$\mu_\pi^{\text{jo}}((f^{\mathcal{H}}, q'), \dots, (f^{\mathcal{H}}, q'), (f^{\mathcal{H}}, q''), (q''^{\mathcal{H}}, q')) = 1$$
$$\mu_\pi^{\text{jo}}((f^{\mathcal{V}}, q'), \dots, (f^{\mathcal{V}}, q'), (f^{\mathcal{V}}, q''), (q''^{\mathcal{V}}, q')) = 1$$
$$\gamma((f^{\mathcal{H}}, f)) = 1,$$
$$\lambda((q^{\mathcal{H}}, f)) = \kappa(q).$$

In all other cases set the values to 0. Observe that there is no run r of $\mathcal{P}$ with $\text{init}(r) = (f^{\mathcal{V}}, q')$ or $\text{init}(r) = (f^{\mathcal{H}}, q')$ and $\text{wgt}_{\mathcal{P}}(r) \neq 0$ for some $q' \in Q'$. Moreover, for all $p, p' \in \{q^{\mathcal{H}} \mid q \in Q'\} \cup \{q^{\mathcal{V}} \mid q \in Q'\}$ and for all $q, q' \in Q'$, if $r : (p, q) \xrightarrow{\tau} (p', q')$ is a run of $\mathcal{P}$ on some text τ with $\text{wgt}_{\mathcal{P}}(r) \neq 0$, then $q = q'$. Indeed, if $\tau = a$ for some $a \in \Delta$, then this is trivial. If $\tau = \tau' \bullet \tau''$, then there are either runs $r_1 : (p, q) \xrightarrow{\tau'} (p', q')$, $r_2 : (p', q') \xrightarrow{\tau''} (p'', q'')$ of weight $\neq 0$ and we can conclude that $q = q' = q''$ by induction hypothesis, or there are transitions $\mu_{\mathrm{op}}((p, q), (_s, (p_1, q_1)), \mu_{\mathrm{cl}}((p_4, q_4,)_s, (p', q'))$ and runs $r_1 : (p_1, q_1) \xrightarrow{\tau'} (p_3, q_3), r_2 : (p_3, q_3) \xrightarrow{\tau''} (p_4, q_4)$ all of weight $\neq 0$ and we can conclude that $q = q_1 = \ldots = q_4 = q'$ by induction hypothesis and definition of μ_{op} and μ_{cl}. Similar arguments can be used for the case $\tau = \tau' \circ \tau''$. Now, if $\tau = \pi(\tau_1, \ldots, \tau_2)$ for some $\pi \in \Pi$ with $\mathrm{rk}(\tau) = k$ and some $\tau_1, \ldots, \tau_k \in \mathrm{TXT}_\Sigma(\Delta)$, then there are runs $r_i : (p_i, q_i) \xrightarrow{\tau_i} (p'_i, q'_i)$ for $1 \leq i \leq k$ and there are transitions $\mu^{\mathrm{fo}}_\pi((p, q), (p_1, q_1), \ldots, (p_k, q_k))$ and $\mu^{\mathrm{jo}}_\pi((p'_1, q'_1), \ldots, (p'_k, q'_k), (p', q'))$ all of weight $\neq 0$. From induction hypothesis and from the definition of μ^{fo} and μ^{jo} we conclude that $q = q_1 = q'_1 = q'$.

Now, let $\tau \in \mathrm{TXT}_\Sigma(\Delta)$, let $q \in Q$ and let $q' \in Q'$. By induction on the structure of τ we show, for all $\tau \in \mathrm{TXT}_\Sigma(\Delta)$,

$$\sum_{r:(q^{\mathcal{V}},q')\xrightarrow{\tau}(f^{\mathcal{V}},q')} \mathrm{wgt}_{\mathcal{P}}(r) = \sum_{r:(q^{\mathcal{H}},q')\xrightarrow{\tau}(f^{\mathcal{H}},q')} \mathrm{wgt}_{\mathcal{P}}(r) = \delta(\mathrm{sh}(\tau))_q. \tag{3}$$

For $a \in \Delta$, we have

$$\begin{aligned}\sum_{r:(q^{\mathcal{H}},q')\xrightarrow{a}(f^{\mathcal{H}},q')} \mathrm{wgt}_{\mathcal{P}}(r) &= \mu((q^{\mathcal{H}}, q'), a, (f^{\mathcal{H}}, q')) = \delta(a)_q \\ &= \mu((q^{\mathcal{V}}, q'), a, (f^{\mathcal{V}}, q')) = \sum_{r:(q^{\mathcal{V}},q')\xrightarrow{a}(f^{\mathcal{V}},q')} \mathrm{wgt}_{\mathcal{P}}(r).\end{aligned}$$

Now, let $\tau = \pi(\tau_1, \ldots, \tau_k)$ for some $\pi \in \Pi$ with $\mathrm{rk}(\pi) = k$ and some $\tau_1, \ldots, \tau_k \in \mathrm{TXT}_\Sigma(\Delta)$. Then

$$\begin{aligned}\sum_{r:(q^{\mathcal{V}},q')\xrightarrow{\tau}(f^{\mathcal{V}},q')} \mathrm{wgt}_{\mathcal{P}}(r) &= \sum_{q_1,\ldots,q_k\in Q} \mu^{\mathrm{fo}}_\pi((q^{\mathcal{V}}, q'), (q^{\mathcal{V}}_1, q'), \ldots, (q^{\mathcal{V}}_{k-1}, q'), (q^{\mathcal{V}}_k, f)) \\ &\quad \cdot \Bigg(\sum_{r_1:(q^{\mathcal{V}}_1,q')\xrightarrow{\tau_1}(f^{\mathcal{V}},q')} \cdots \sum_{r_{k-1}:(q^{\mathcal{V}}_{k-1},q')\xrightarrow{\tau_{k-1}}(f^{\mathcal{V}},q')} \sum_{r_k:(q^{\mathcal{V}}_k,f)\xrightarrow{\tau_k}(f^{\mathcal{V}},f)} \mathrm{wgt}_{\mathcal{P}}(r_1) \cdot \ldots \cdot \mathrm{wgt}_{\mathcal{P}}(r_k) \Bigg) \\ &\quad \cdot \mu^{\mathrm{jo}}_\pi((f^{\mathcal{V}}, q'), \ldots, (f^{\mathcal{V}}, q'), (f^{\mathcal{V}}, f), (f^{\mathcal{V}}, q')) \\ &= \sum_{q_1,\ldots,q_k\in Q} \delta_\pi(q_1, \ldots, q_k)_q \cdot \delta(\mathrm{sh}(\tau_1))_{q_1} \cdots \delta(\mathrm{sh}(\tau_k))_{q_k} \\ &= \delta(\mathrm{sh}(\tau))_q.\end{aligned}$$

The calculations showing $\sum_{r:(q^{\mathcal{H}},q')\xrightarrow{\tau}(f^{\mathcal{H}},q')} \mathrm{wgt}_{\mathcal{P}}(r) = \delta(\mathrm{sh}(\tau))_q$ are exactly the same; simply replace $\mathcal{V}$ by $\mathcal{H}$.

Now, let $\tau = \tau' \bullet \tau''$ for some $\tau', \tau'' \in \mathrm{TXT}_\Sigma(\Delta)$. We may assume that τ' is not a $\bullet$-product. First, we show that $\sum_{r:(q^{\mathcal{V}},q')\xrightarrow{\tau}(f^{\mathcal{V}},q')} \mathrm{wgt}_{\mathcal{P}}(r) = \delta(\mathrm{sh}(\tau))_q$. For this we consider three subcases. First, let $\tau' = a$ for some $a \in \Delta$. We have

$$\begin{aligned}\sum_{r:(q^{\mathcal{V}},q')\xrightarrow{\tau}(f^{\mathcal{V}},q')} \mathrm{wgt}_{\mathcal{P}}(r) &= \sum_{q_1\in Q'} \sum_{r_1:(q^{\mathcal{V}},q')\xrightarrow{a}(q^{\mathcal{V}}_1,q')} \mathrm{wgt}_{\mathcal{P}}(r_1) \cdot \sum_{r_2:(q^{\mathcal{V}}_1,q')\xrightarrow{\tau''}(f^{\mathcal{V}},q')} \mathrm{wgt}_{\mathcal{P}}(r_2) \\ &= \sum_{q_1\in Q} \sum_{s\in Q} \delta_\bullet(s, q_1)_q \cdot (\delta_a)_s \cdot \delta(\mathrm{sh}(\tau''))_{q_1} = \delta(\mathrm{sh}(\tau))_q.\end{aligned}$$

Second, let $\tau' = \dot{\tau} \circ \ddot{\tau}$ for some $\dot{\tau}, \ddot{\tau} \in \mathrm{TXT}_\Sigma(\Delta)$. Then

$$\begin{aligned}&\sum_{r:(q^{\mathcal{V}},q')\xrightarrow{\tau}(f^{\mathcal{V}},q')} \mathrm{wgt}_{\mathcal{P}}(r) = \sum_{q_1\in Q'} \sum_{r_1:(q^{\mathcal{V}},q')\xrightarrow{\tau'}(q^{\mathcal{V}}_1,q')} \mathrm{wgt}_{\mathcal{P}}(r_1) \cdot \sum_{r_2:(q^{\mathcal{V}}_1,q')\xrightarrow{\tau''}(f^{\mathcal{V}},q')} \mathrm{wgt}_{\mathcal{P}}(r_2) \\ &= \sum_{q_1\in Q} \sum_{r_1:(q^{\mathcal{V}},q')\xrightarrow{\tau'}(q^{\mathcal{V}}_1,q')} \mathrm{wgt}_{\mathcal{P}}(r_1) \cdot \delta(\mathrm{sh}(\tau''))_{q_1} \\ &= \sum_{\substack{q_1\in Q \\ q_2,q_3,q_4\in Q'}} \mu_{\mathrm{op}}((q^{\mathcal{V}}, q'), (_{q_4}, (q^{\mathcal{H}}_2, q')) \cdot \sum_{r_1:(q^{\mathcal{H}}_2,q')\xrightarrow{\tau'}(q^{\mathcal{H}}_3,q')} \mathrm{wgt}_{\mathcal{P}}(r_1) \cdot \mu_{\mathrm{cl}}((q^{\mathcal{H}}_3, q'),)_{q_4}, (q^{\mathcal{V}}_1, q')) \cdot \delta(\mathrm{sh}(\tau''))_{q_1} \\ &= \sum_{q_1,q_2\in Q} \mu_{\mathrm{op}}((q^{\mathcal{V}}, q'), (_{q_1}, (q^{\mathcal{H}}_2, q')) \cdot \sum_{r_1:(q^{\mathcal{H}}_2,q')\xrightarrow{\tau'}(f^{\mathcal{H}},q')} \mathrm{wgt}_{\mathcal{P}}(r_1) \cdot \mu_{\mathrm{cl}}((f^{\mathcal{H}}, q'),)_{q_1}, (q^{\mathcal{V}}_1, q')) \cdot \delta(\mathrm{sh}(\tau''))_{q_1} \\ &= \sum_{q_1,q_2\in Q} \delta_\bullet(q_2, q_1)_q \cdot \delta(\mathrm{sh}(\tau'))_{q_2} \cdot \delta(\mathrm{sh}(\tau''))_{q_1} \\ &= \delta(\mathrm{sh}(\tau))_q.\end{aligned}$$

And third, let $\tau' = \pi(\tau_1, \ldots, \tau_k)$ for some $\pi \in \Pi$ with $\mathrm{rk}(\pi) = k$ and some $\tau_1, \ldots, \tau_k \in \mathrm{TXT}_\Sigma(\Delta)$. Then, we have

$$\begin{aligned}
\sum_{r:(q^{\mathcal{V}},q')\xrightarrow{\tau}(f^{\mathcal{V}},q')} \mathrm{wgt}_{\mathcal{P}}(r) &= \sum_{\tilde{q}\in Q'} \sum_{r_1:(q^{\mathcal{V}},q')\xrightarrow{\tau'}(\tilde{q}^{\mathcal{V}},q')} \mathrm{wgt}_{\mathcal{P}}(r_1) \cdot \sum_{r_2:(\tilde{q}^{\mathcal{V}},q')\xrightarrow{\tau''}(f^{\mathcal{V}},q')} \mathrm{wgt}_{\mathcal{P}}(r_2) \\
&= \sum_{\tilde{q}\in Q} \sum_{r_1:(q^{\mathcal{V}},q')\xrightarrow{\tau'}(\tilde{q}^{\mathcal{V}},q')} \mathrm{wgt}_{\mathcal{P}}(r_1) \cdot \delta(\mathrm{sh}(\tau''))_{\tilde{q}} \\
&= \sum_{\tilde{q}\in Q} \sum_{q_1,\ldots,q_k\in Q} \mu^{\mathrm{fo}}_{\pi}((q^{\mathcal{V}}, q'), (q_1^{\mathcal{V}}, q'), \ldots, (q_{k-1}^{\mathcal{V}}, q'), (q_k^{\mathcal{V}}, \tilde{q})) \\
&\quad \cdot \left(\sum_{r_1:(q_1^{\mathcal{V}},q')\xrightarrow{\tau_1}(f^{\mathcal{V}},q')} \cdots \sum_{r_{k-1}:(q_{k-1}^{\mathcal{V}},q')\xrightarrow{\tau_{k-1}}(f^{\mathcal{V}},q')} \sum_{r_k:(q_k^{\mathcal{V}},\tilde{q})\xrightarrow{\tau_k}(f^{\mathcal{V}},\tilde{q})} \mathrm{wgt}_{\mathcal{P}}(r_1) \cdot \ldots \cdot \mathrm{wgt}_{\mathcal{P}}(r_k) \right) \\
&\quad \cdot \mu^{\mathrm{jo}}_{\pi}((f^{\mathcal{V}}, q'), \ldots, (f^{\mathcal{V}}, q'), (f^{\mathcal{V}}, \tilde{q}), (\tilde{q}^{\mathcal{V}}, q')) \cdot \delta(\mathrm{sh}(\tau''))_{\tilde{q}} \\
&= \sum_{\tilde{q}\in Q} \sum_{q_1,\ldots,q_k\in Q} \sum_{s\in Q} \delta_{\bullet}(s, \tilde{q})_q \cdot \delta_\pi(q_1, \ldots, q_k)_s \cdot \delta(\mathrm{sh}(\tau_1))_{q_1} \cdot \cdots \cdot \delta(\mathrm{sh}(\tau_j))_{q_k} \cdot \delta(\mathrm{sh}(\tau''))_{\tilde{q}} \\
&= \delta(\mathrm{sh}(\tau))_q .
\end{aligned}$$

We thus have $\sum_{r:(q^{\mathcal{V}},q')\xrightarrow{\tau}(f^{\mathcal{V}},q')} \mathrm{wgt}_{\mathcal{P}}(r) = \delta(\mathrm{sh}(\tau))_q$ if $\tau = \tau' \bullet \tau''$ for some $\tau', \tau'' \in \mathrm{TXT}_\Sigma(\Delta)$. In this case, we can calculate

$$\sum_{r:(q^{\mathcal{H}},q')\xrightarrow{\tau}(f^{\mathcal{H}},q')} \mathrm{wgt}_{\mathcal{P}}(r) = \mu_{\mathrm{op}}((q^{\mathcal{H}}, q'), (_f, (q^{\mathcal{V}}, q')) \cdot \sum_{r:(q^{\mathcal{V}},q')\xrightarrow{\tau}(f^{\mathcal{V}},q')} \mathrm{wgt}_{\mathcal{P}}(r) \cdot \mu_{\mathrm{cl}}((f^{\mathcal{V}}, q'),)_f, (f^{\mathcal{H}}, q')) = \delta(\mathrm{sh}(\tau))_q .$$

Thus, we showed Eq. (3) in the case where $\tau = \pi(\tau_1, \ldots, \tau_k)$ and in the case where $\tau = \tau' \bullet \tau''$ for some $\tau', \tau'' \in \mathrm{TXT}_\Sigma(\Delta)$. The case where $\tau = \tau' \circ \tau''$ can be shown analogously; one just has to swap $\mathcal{H}$ and $\mathcal{V}$, h and v, and $\circ$ and $\bullet$ in the calculations above. We conclude that, for all $\tau \in \mathrm{TXT}_\Sigma(\Delta)$,

$$\begin{aligned}
(\|\mathcal{P}\|, \tau) &= \sum_{q\in Q} \lambda((q^{\mathcal{H}}, f)) \cdot \sum_{r:(q^{\mathcal{H}},f)\xrightarrow{\tau}(f^{\mathcal{H}},f)} \mathrm{wgt}_{\mathcal{P}}(r) \cdot \gamma((f^{\mathcal{H}}, f)) \\
&= \sum_{q\in Q} \delta(\mathrm{sh}(\tau))_q \cdot \kappa(q) = (\|\mathcal{A}\|, \mathrm{sh}(\tau)). \quad \square
\end{aligned}$$

With the help of the last two propositions we are now ready to extend Theorems 4.6 and 4.8 to arbitrary commutative semirings and conclude the main theorem of this section.

Theorem 4.18. *Let $\mathbb{K}$ be any commutative semiring, let Σ be a finite set of primitive biorders and let $S : \mathrm{TXT}_\Sigma(\Delta) \to \mathbb{K}$. Then the following are equivalent.*

1. *S is recognizable.*
2. *S is regular.*
3. *S is* sRMSO($\mathbb{K}$)*-definable.*
4. *S is* sREMSO($\mathbb{K}$)*-definable.*

Proof. (1) ⇒ (4). Let S be recognizable. By Lemma 2.12(2), $\mathrm{txt}^{-1}(S)$ is recognizable. As in the proof of Theorem 4.6, we conclude that S is sRMSO($\mathbb{K}$)-definable and with Proposition 4.7 that S is sREMSO($\mathbb{K}$)-definable.

(4) ⇒ (3). Trivial.

(3) ⇒ (2). Let S be sRMSO($\mathbb{K}$)-definable. As in the proof of Theorem 4.6, we conclude that $\mathrm{txt}^{-1}(S)$ is regular. We conclude from Proposition 4.17 that $\mathrm{sh}^{-1}(\mathrm{txt}^{-1}(S)) = S$ is regular.

(2) ⇒ (1). This is Proposition 4.16. □

We note that the equivalence of (1) and (2) is a generalization of a result of Ésik and Németh [30, Theorem 3.8] and of a result of Lodaya and Weil [46, Theorem 3.1]. Similarly to the last theorem, we get:

Theorem 4.19. *Let $\mathbb{K}$ be additively locally finite and let $S : \mathrm{TXT}_\Sigma(\Delta) \to \mathbb{K}$. Then S is regular iff S is* swRMSO($\mathbb{K}$)*-definable.*

Note that again all proofs are constructive. Hence, given an sRMSO($\mathbb{K}$) (respectively, swRMSO($\mathbb{K}$)) sentence φ and an effectively given semiring $\mathbb{K}$ we can compute a WBPA $\mathcal{A}$ such that $\|\mathcal{A}\| = [\![\varphi]\!]$. Also the converse is constructive, and given a WBPA $\mathcal{A}$ we can construct an sREMSO($\mathbb{K}$) sentence φ such that $\|\mathcal{A}\| = [\![\varphi]\!]$. From a result of Maletti [48] we therefore get the following corollary:

Corollary 4.20. *Let $\mathbb{K}$ be a computable zero-sum free semiring and let $S : \mathrm{TXT}_\Sigma(\Delta) \to \mathbb{K}$ be regular. It is decidable whether $(S, \tau) = 0$ for all $\tau \in \mathrm{TXT}_\Sigma(\Delta)$.*

Proof. Let $\mathcal{P}$ be a WBPA such that $\|\mathcal{P}\| = S$. The proof of Proposition 4.16 is effective and gives a tree automaton $\mathcal{A}$ such that $\|\mathcal{A}\| = \mathrm{txt}^{-1}(S)$. Clearly, $(S, \tau) = 0$ for all $\tau \in \mathrm{TXT}_\Sigma(\Delta)$ iff $(\|\mathcal{A}\|, t) = 0$ for all $t \in T_\Sigma(\Delta)$. The latter can be decided by [48, Corollary 3]. □

4.3. A note on alternating texts and SP-biposets

Now, let $\Sigma = \{\circ, \bullet\}$ be the set of the two biorders of cardinality two. Then $\mathrm{TXT}_\Sigma(\Delta)$, which is called the set of alternating texts [22, p. 261], is the free bisemigroup over Δ, where a bisemigroup is a set together with two associative operations. Several authors have investigated the free bisemigroup as a fundamental, two-dimensional extension of classical automaton theory; see, for example, Ésik and Németh [30] and Hashiguchi et al. [34–36]. Ésik and Németh considered the so-called *sp-biposets* as a representation for the free bisemigroup.

A *Δ-labeled biposet* is a finite non-empty set V of vertices equipped with two partial orders $\le_h$ and $\le_v$ and a labeling function $\lambda : V \to \Delta$. Suppose two biposets $p_1 = (V_1, \lambda_1, \le_h^1, \le_v^1)$, $p_2 = (V_2, \lambda_2, \le_h^2, \le_v^2)$, where we assume that V_1 and V_2 are disjoint. We define $p_1 \circ p_2 = (V_1 \uplus V_2, \lambda_1 \cup \lambda_2, \le_h^\circ, \le_v^\circ)$ as follows:

$$\le_h^\circ = \le_h^1 \cup \le_h^2 \cup (V_1 \times V_2) \quad \text{and} \quad \le_v^\circ = \le_v^1 \cup \le_v^2 .$$

The operation $\bullet$ is defined analogously: $p_1 \bullet p_2 = (V_1 \uplus V_2, \lambda_1 \cup \lambda_2, \le_h^\bullet, \le_v^\bullet)$ is then given by

$$\le_h^\bullet = \le_h^1 \cup \le_h^2 \quad \text{and} \quad \le_v^\bullet = \le_v^1 \cup \le_v^2 \cup (V_1 \times V_2).$$

Clearly, both products are associative. Of course we consider biposets only up to isomorphism. The set of biposets generated from the singletons using $\circ$ and $\bullet$ is denoted $\mathrm{SPB}(\Delta)$. Its elements are called *sp-biposets*.

Proposition 4.21 (*Ésik [28]*). *$\mathrm{SPB}(\Delta)$ is the free bisemigroup over Δ.*

Hence, $\mathrm{SPB}(\Delta)$ and $\mathrm{TXT}_{\{\circ,\bullet\}}(\Delta)$ are isomorphic, and since the notion of recognizability depends on the algebraic structure of $\mathrm{SPB}(\Delta)$ it is clear that a set $L \subseteq \mathrm{SPB}(\Delta)$ is $(\{\circ, \bullet\})$-recognizable iff it is recognizable considered as a set of alternating texts. Likewise, the notion of regularity for texts which we introduced only depends on the algebraic Σ-structure of $\mathrm{TXT}_\Sigma(\Delta)$ and, hence, it makes sense to define that $S : \mathrm{SPB}(\Delta) \to \mathbb{K}$ is regular if it is regular as a set of alternating texts. This is exactly how Ésik and Németh defined regular subsets of $\mathrm{SPB}(\Delta)$. In fact, WBPAs were inspired by Ésik and Németh's parenthesizing automata and generalize them. An isomorphism between $\mathrm{SPB}(\Delta)$ and $\mathrm{TXT}_{\{\circ,\bullet\}}(\Delta)$ is given by mapping an sp-biposet $p = (V, \lambda \le_h, \le_v)$ to the alternating text $(V, \lambda, \le_h \cup \le_v, \le_h \cup \le_v^{-1})$, where $\le_v^{-1}$ is the inverse relation of $\le_v$ [30]. From this it is clear that $\le_h \cup \le_v$ is a linear order on V. We define $\mathrm{MSO}(\mathbb{K})$, over the relation symbols $(\mathrm{Lab}_a)_{a\in\Delta}, \le_h, \le_v$, which are interpreted as one expects. For $\mathrm{sRMSO}(\mathbb{K})$ and $\mathrm{sREMSO}(\mathbb{K})$ we add the binary symbol $\le$, which is interpreted by the linear order $\le_h \cup \le_v$. Clearly, both the isomorphism given above and its inverse are FO-definable. In fact, the interpreting formulae may be chosen propositional. Hence, we get immediately from Transfer Theorem 3.17 together with Theorem 4.18 and Theorem 4.6 the following extension of Theorem 5.2 in [30].

Theorem 4.22. *Let $\mathbb{K}$ be any commutative semiring and let $S : \mathrm{SPB}(\Delta) \to \mathbb{K}$. Then S is regular (recognizable) iff S is $\mathrm{sRMSO}(\mathbb{K})$-definable iff it is $\mathrm{sREMSO}(\mathbb{K})$-definable. If $\mathbb{K}$ is additively locally finite, then S is regular (recognizable) iff S is $\mathrm{swRMSO}(\mathbb{K})$-definable. Moreover, if $\mathbb{K}$ is locally finite, then S is regular (recognizable) iff S is $\mathrm{MSO}(\mathbb{K})$-definable.*

Corollary 4.23. *Let $\mathbb{K}$ be a computable field. It is decidable whether two given sentences $\varphi, \psi \in \mathrm{sRMSO}(\mathbb{K})$ over sp-biposets satisfy $[\![\varphi]\!] = [\![\psi]\!]$.*

Corollary 4.24. *Let $\mathbb{K}$ be a computable locally finite commutative semiring. It is decidable whether two given sentences $\varphi, \psi \in \mathrm{MSO}(\mathbb{K})$ over sp-biposets satisfy $[\![\varphi]\!] = [\![\psi]\!]$.*

Corollary 4.25. *A language $L \subseteq \mathrm{SPB}(\Delta)$ is MSO-definable iff it is definable in EMSO.*

5. Conclusion and open problems

We introduced an algebraic concept of weighted recognizability and the notion of the syntactic algebra of a series from a general algebra into a semiring. These notions extend the corresponding concepts of Reutenauer [58] and Bozapalidis et al. [8,7,6] for words and trees, respectively. We then considered weighted logics of Droste and Gastin [13] over arbitrary relational structures and showed a transfer theorem making use of Courcelle's MSO-definable transductions [10]. In the last section we applied this transfer theorem to text series. Utilizing the transductions of Hoogeboom and ten Pas [39] and the results of the first section we could immediately deduce a characterization of recognizable text series in terms of weighted logics; however, only under the assumption that the underlying semiring is a ring or locally finite. After that we introduced a new weighted automaton model operating on texts. The model was inspired by Lodaya and Weil's branching automata [46] and Ésik and Németh's parenthesizing automata [30]. We succeeded in showing that these automata precisely describe the recognizable series and the ones definable in a certain fragment of weighted logics. This way we were able to drop the restrictions on the semiring mentioned above and showed a characterization valid for any commutative semiring.

Let us conclude by pointing to some open problems and future work:

1. In Section 2 we showed that if the semiring we consider is locally finite or a ring, then a series over a general algebra is recognizable iff its syntactic algebra is of finite rank. Furthermore, we gave counterexamples showing that this characterization does not hold for the natural numbers and the arctic semiring.
   Can one characterize the semirings which offer the above-mentioned characterization?

2. Similarly to [14], we defined φ^+ for $\varphi \in$ MSO in case the class of relational structures under consideration is equipped with a linear order. Due to the definition of $(\exists x.\psi)^+$, the length of φ^+ in terms of φ might be exponential. Hence, the translation preserving sRMSO($\mathbb{K}$), which we used in the proof of our transfer theorem, also gives an exponential blow-up in the length of the formulae. In contrast, note that for the automaton constructions we gave we only had a polynomial blow-up in the number of states.
   Is it possible to syntactically adapt the definition of sRMSO($\mathbb{K}$) and to prove a transfer theorem for this new sRMSO($\mathbb{K}$), avoiding an exponential blow-up of the size of the formulae?
3. To obtain our logical characterization of regular series we used an interpretation (in trees) and made use of a general transfer theorem. This admits the advantage that decidability results can be deduced straightforwardly from the corresponding results for trees. However, we paid the price that we had to restrict ourselves to *commutative* semirings. For words and trees, weighted logic over non-commutative semirings was investigated in [14,18] and regular series were characterized.
   Can one give a logical characterization of regular text series in case the semiring is non-commutative?

Acknowledgments

The author would like to thank Manfred Droste, Dietrich Kuske, Magnus Steinby and Pascal Weil for their helpful comments. Moreover, the author is indebted to the anonymous referees of both the conference version and the journal version. Their remarks resulted in many improvements.

Furthermore, the author acknowledges the financial support of the German Research Foundation (DFG) within the Gaduiertenkolleg 446.

References

[1] A. Beimel, F. Bergadano, N.H. Bshouty, E. Kushilevitz, S. Varricchio, Learning functions represented as multiplicity automata, Journal of the ACM 47 (3) (2000) 506–530.
[2] J. Berstel, C. Reutenauer, Rational Series and Their Languages, EATCS Monographs on Theoretical Computer Science, Springer, 1988.
[3] R. Bloem, J. Engelfriet, A comparison of tree transductions defined by monadic second order logic and by attribute grammars, Journal of Computer and System Sciences 61 (1) (2000) 1–50.
[4] B. Bollig, I. Meinecke, Weighted distributed systems and their logics, in: Proc. of LFCS 2007, New York, in: Lecture Notes in Computer Science, vol. 4514, 2007, pp. 54–68.
[5] B. Borchardt, A pumping lemma and decidability problems for recognizable tree series, Acta Cybernetica 16 (4) (2004) 509–544.
[6] S. Bozapalidis, Effective construction of the syntactic algebra of a recognizable series on trees, Acta Informatica 28 (4) (1991) 351–363.
[7] S. Bozapalidis, A. Alexandrakis, Représentations matricielles des séries d'arbre reconnaissables, Theoretical Informatics and Applications 23 (4) (1989) 449–459.
[8] S. Bozapalidis, O. Louscou-Bozapalidou, The rank of a formal tree power series, Theoretical Computer Science 27 (1983) 211–215.
[9] J.R. Büchi, Weak second-order arithmetic and finite automata, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik 6 (1960) 66–92.
[10] B. Courcelle, Monadic second-order definable graph transductions: A survey, Theoretical Computer Science 126 (1994) 53–75.
[11] B. Courcelle, Basic notions of universal algebra for language theory and graph grammars, Theoretical Computer Science 163 (1996) 1–54.
[12] J. Doner, Tree acceptors and some of their applications, Journal of Computer and System Sciences 4 (1970) 406–451.
[13] M. Droste, P. Gastin, Weighted automata and weighted logics, Theoretical Computer Science 380 (2007) 69–86.
[14] M. Droste, P. Gastin, Weighted automata and weighted logics. In Droste et al. [15] (Chapter 5).
[15] M. Droste, W. Kuich, H. Vogler (Eds.), Handbook of Weighted Automata. EATCS Monographs on Theoretical Computer Science, Springer, 2009.
[16] M. Droste, G. Rahonis, Weighted automata and weighted logics on infinite words, in: Proc. of the 10th DLT, Santa Barbara, in: Lecture Notes in Computer Science, vol. 4036, 2006, pp. 49–58.
[17] M. Droste, G. Rahonis, Weighted automata and weighted logics with discounting, in: Proc. of the 12th CIAA, Prague, in: Lecture Notes in Computer Science, vol. 4783, 2007, pp. 73–84.
[18] M. Droste, H. Vogler, Weighted logics for unranked tree automata. Theory of Computing Systems, 2009 (in press).
[19] M. Droste, G.Q. Zhang, On transformations of formal power series, Information and Computation 184 (2) (2003) 369–383.
[20] A. Ehrenfeucht, T. Harju, G. Rozenberg, The Theory of 2-Structures: A Framework for Decomposition and Transformation of Graphs, World Scientific, 1999.
[21] A. Ehrenfeucht, G. Rozenberg, Theory of 2-structures. I and II, Theoretical Computer Science 70 (1990) 277–342.
[22] A. Ehrenfeucht, G. Rozenberg, T-structures, T-functions, and texts, Theoretical Computer Science 116 (1993) 227–290.
[23] A. Ehrenfeucht, P. ten Pas, G. Rozenberg, Context-free text grammars, Acta Informatica 31 (2) (1994) 161–206.
[24] C.C. Elgot, Decision problems of finite automata design and related arithmetics, Transactions of the AMS 98 (1961) 21–51.
[25] J. Engelfriet, H.J. Hoogeboom, MSO definable string transductions and two-way finite-state transducer, ACM Transactions on Computational Logic 2 (2) (2001) 216–254.
[26] J. Engelfriet, H.J. Hoogeboom, Finitary compositions of two-way finite-state transductions, Fundamenta Informaticae 80 (2007) 111–123.
[27] J. Engelfriet, S. Maneth, Macro tree transducers, attribute grammars, and MSO definable tree translations, Information and Computation 154 (1) (1999) 34–91.
[28] Z. Ésik, Free algebras for generalized automata and language theory, RIMS Kokyuroku 1166 (2000) 52–58.
[29] Z. Ésik, W. Kuich, Formal tree series, Journal of Automata, Languages and Combinatorics 8 (2) (2003) 219–285.
[30] Z. Ésik, Z.L. Németh, Higher dimensional automata, Journal of Automata, Languages and Combinatorics 9 (1) (2004) 3–29.
[31] Z. Fülöp, M. Steinby, Formal series of general algebras over a field and their syntactic algebras, in: Z. Ésik and Z. Fülöp, (Eds.), Automata, Formal Languages, and Related Topics, Institute of Informatics, University of Szeged, pp. 55–78, 2009.
[32] F. Gécseg, M. Steinby, Tree languages, in: G. Rozenberg, A. Salomaa (Eds.), Word, Language, Grammar, in: Handbook of Formal Languages, vol. 3, Springer, 1997, pp. 1–68 (Chapter 1).
[33] A. Habrard, J. Oncina, Learning multiplicity tree automata, in: Proc. of the 8th ICGI, Tokyo, in: Lecture Notes in Computer Science, vol. 4201, 2006, pp. 268–280.
[34] K. Hashiguchi, S. Ichihara, S. Jimbo, Formal languages over free binoids, Journal of Automata, Languages and Combinatorics 5 (3) (2000) 219–234.
[35] K. Hashiguchi, S. Jimbo, T. Kunai, Finite codes over free binoids, Journal of Automata, Languages and Combinatorics 7 (4) (2002) 505–518.
[36] K. Hashiguchi, S. Jimbo, Y. Wada, Regular binoid expressions and regular binoid languages, Theoretical Computer Science 1–3 (304) (2003) 291–313.

[37] C. Heger, A simulation of weighted branching automata by weighted tree automata. Belegarbeit, Technische Universität Dresden, 2006.
[38] H.J. Hoogeboom, P. ten Pas, Text languages in an algebraic framework, Fundamenta Informaticae 25 (3) (1996) 353–380.
[39] H.J. Hoogeboom, P. ten Pas, Monadic second-order definable text languages, Theory of Computing Systems 30 (1997) 335–354.
[40] N. Immerman, Descriptive complexity and nested words. Talk given at 1st LATA, Tarragona, 2007.
[41] W. Kuich, Semirings and formal power series, in: G. Rozenberg, A. Salomaa (Eds.), Word, Language, Grammar, in: Handbook of Formal Languages, vol. 1, Springer, 1997, ch. 9.
[42] W. Kuich, Formal series over sorted algebras, Discrete Mathematics 254 (1–3) (2002) 231–258.
[43] W. Kuich, A. Salomaa, Semirings, Automata, Languages, in: EATCS Monographs on Theoretical Computer Science, vol. 5, Springer, 1986.
[44] D. Kuske, I. Meinecke, Branching automata with costs — A way of reflecting parallelism in costs star, Theoretical Computer Science 328 (2004) 53–75.
[45] S. Lang, Algebra, Springer, 2002.
[46] K. Lodaya, P. Weil, Rationality in algebras with a series operation, Information and Computation 171 (2) (2001) 269–293.
[47] O. Louscou-Bozapalidou, Some remarks on recognizable tree series, International Journal of Computer Mathematics 70 (1999) 649–655.
[48] A. Maletti, Relating tree series transducers and weighted tree automata, International Journal of Foundations of Computer Science 16 (4) (2005) 723–741.
[49] C. Mathissen, Definable transductions and weighted logics for texts, in: Proc. of the 11th DLT, Turku, in: Lecture Notes in Computer Science, vol. 4588, 2007, pp. 324–336.
[50] C. Mathissen, Weighted logics for nested words and algebraic formal power series, in: Proc. of the 35th ICALP, Reykjavík, Part II, in: Lecture Notes in Computer Science, vol. 5126, 2008, pp. 221–232.
[51] C. Mathissen, Existential MSO over two successors is strictly weaker than over linear orders, Theoretical Computer Science 410 (2009) 3982–3987.
[52] C. Mathissen, Weighted automata and weighted logics for tree-like structures. Ph.D. Thesis, Universität Leipzig, 2009.
[53] I. Mäurer, Weighted picture automata and weighted logics, in: Proc. of the 23rd STACS, Marseille, in: Lecture Notes in Computer Science, vol. 3884, 2006, pp. 313–324.
[54] I. Meinecke, Weighted logics for traces, in: Proc. of the 1st CSR, St. Petersburg, in: Lecture Notes in Computer Science, vol. 3967, 2006, pp. 235–246.
[55] J. Mezei, J.B. Wright, Algebraic automata and context-free sets, Information and Control 11 (1967) 3–29.
[56] R.H. Möhring, F.J. Radermacher, Substitution decomposition for discrete structures and connections with combinatorial optimization, Annals Discrete Mathematics 19 (1984) 257–356.
[57] G. Rahonis, Weighted muller tree automata and weighted logics, Journal of Automata, Languages and Combinatorics 12 (4) (2007) 455–483.
[58] C. Reutenauer, Séries formelles et algèbres syntactiques, Journal of Algebra 66 (1980) 448–483.
[59] A. Salomaa, M. Soittola, Automata-Theoretic Aspects of Formal Power Series, in: Texts and Monographs in Computer Science, Springer, 1978.
[60] M.P. Schützenberger, On the definition of a family of automata, Information and Control 4 (1961) 245–270.
[61] S. Schwarz, Łukasiewicz logics and weighted logics over MV-semirings, Journal of Automata, Languages and Combinatorics 12 (4) (2007) 485–499.
[62] H. Seidl, Deciding equivalence of finite tree automata, SIAM Journal of Computing 19 (3) (1990) 424–437.
[63] E.D. Sontag, On some questions of rationality and decidability, Journal of Computer and System Sciences 11 (3) (1975) 375–381.
[64] J.W. Thatcher, J.B. Wright, Generalized finite automata theory with application to a decision problem of second-order logic, Mathematical Systems Theory 2 (1) (1968) 57–81.
[65] W. Thomas, On logics, tilings, and automata, in: Proc. of the 18th ICALP, Madrid, in: Lecture Notes in Computer Science, vol. 510, 1991, pp. 441–454.
[66] H. Wang, On rational series and rational languages, Theoretical Computer Science 205 (1–2) (1998) 329–336.
[67] P. Weil, Algebraic recognizability of languages, in: Proc. of the 29th MFCS, Prague, in: Lecture Notes in Computer Science, vol. 3153, 2004, pp. 149–175.