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On the Semi-Reflexivity of Biprojective Tensor Product Spaces

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1. INTRODUCTION

Let E and F be Banach spaces. If both E and F contain a Schauder basis, it is proved by Holub [5] that the Banach space $\mathscr{L}_b(E, F)$ of continuous linear maps from E into F, carried the topology of bounded convergence in E, is reflexive if, and only if, all the maps of it are compact. This result is later extended by Ruckle [10] to the case both E and F have the approximation property and by Holub [6] to that either E or F has the approximation property. Heinrich [4] has also proved it under a weaker assumption than the approximation property of E or F. In all the preceding considerations tensor product techniques are essentially applied. On the other hand, by a conditional weak compactness criterium of Lewis [9], it follows that the reflexivity of certain tensor products of reflexive Banach Spaces is equivalent to their weak sequential completeness (ibid.). In [15] an extension of the above criterium to a fairly large class of locally convex spaces is obtained, which yields Theorem 3.1 below, so that the techniques and the results of [16] actually supply the present work.

Thus, let E and F be complete locally convex spaces, $E \hat{\otimes}_\epsilon F$ the respective completed (biprojective) ϵ-tensor product and let T be the (vector) subspace of $(E' \otimes F')^*$, consisting of the $\sigma((E' \otimes F')^*, E' \otimes F')$-limits of all weakly Cauchy sequences in $E \hat{\otimes}_\epsilon F$. First, for certain semi-reflexive spaces E and F, it is therein verified that $E \hat{\otimes}_\epsilon F$ is semi-reflexive if, and only if, it is weakly sequentially complete (Theorem 3.1). On the other hand, under some mild restrictions on E and F, it is also proved that $E \hat{\otimes}_\epsilon F$ is semi-reflexive if, and only if, both E and F are semi-reflexive and moreover, every linear map $u \in T$ transfers equicontinuous subsets of E' into relatively compact subsets of F (Theorem 3.2). From these results and certain results of [16] a number of corollaries, referred to the semi-reflexivity of several (locally convex) spaces of vector-valued maps are derived. In particular, they have, among otherthings, a special bearing, into the case under consideration, on the reflexivity criterium for the Banach space $\mathscr{L}_b(E, F)$, stated above.

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2. Notations and Preliminaries

All vector spaces considered in the following are over the field $\mathbb{C}$ of complex numbers. The topological spaces involved are assumed to be Hausdorff. For a dual pair $\langle X, Y\rangle$ of vector spaces we denote by $\sigma(X, Y)$, $\tau(X, Y)$ and $b(X, Y)$ the weak, the Mackey and the strong topology of X respectively. The locally convex spaces thus obtained are denoted by X_σ, X_τ and X_b. If E is a locally convex space, we denote by E^* and E' the algebraic and the topological dual of E respectively. Moreover, we consider the topology $c(E', E)$ of the absolutely convex compact convergence in E and denote by E'_c the respective locally convex space.

Now, we state, for the sake of references, the following terminology (cf. [15]). A locally convex space E is said to be an *(s)-space*, if every separable (topological vector) subspace G of it has a (weakly) $\sigma(G', G)$-separable topological dual G'. By the techniques of [8, p. 311, 24.1(3)], a locally convex space (E, t) such that there exists a metrizable locally convex (vector space) topology on E, which is coarser than t, is an (s)-space. In particular, if E is a metrizable locally convex space, or a strict (LF)-space, or if E'_σ is separable, then E is an (s)-space. Now, by a *Šmulian space* we mean a locally convex space E with the property that each relatively weakly countably compact subset of it is relatively weakly sequentially compact. By the arguments of [8, p. 311, 24.1(3)], every (s)-space is Šmulian.

3. On Semi-Reflexive Tensor Products

Let E and F be locally convex spaces. We denote by $L(E,F)$ and $\mathscr{L}(E,F)$ the (vector) space of linear maps and continuous linear maps from E into F respectively. We consider the (vector) space $\mathscr{L}(E'_\tau, F)$ of continuous linear maps of E'_τ into F, which, of course, coincides with the space $\mathscr{L}(E'_\sigma, F_\sigma)$ of (weakly) continuous linear maps of E'_σ into F_σ (cf. [14, p. 429, Proposition 42.2]). Furthermore, we denote by $\mathscr{L}_e(E'_\tau, F)$ the space $\mathscr{L}(E'_\tau, F)$, equipped with the topology of equicontinuous convergence in E' and consider the (topological vector) subspace $\mathscr{L}_e(E'_c, F)$ of it, where $\mathscr{L}(E'_c, F)$ is the (vector) space of continuous linear maps from E'_c into F. On the other hand, we consider the (vector) space $K(E',F)$ of all linear maps from E' into F, which transfers equicontinuous subsets of E' into relatively compact subsets of F and denote by $\mathscr{K}(E',F)$ its (vector) subspace of all weakly $(\sigma(E', E), \sigma(F, F'))$ continuous linear maps. By [12, p. 35, Proposition 5], $\mathscr{L}(E'_c, F) = \mathscr{K}(E',F)$. For a map $u \in L(E,F)$ we denote by ${}^t u$ its transpose. If $u \in \mathscr{L}(E'_\tau, F)$, then ${}^t u \in \mathscr{L}(F'_\tau, E)$. Now, let $E \otimes F$ be the tensor product (vector) space of E and F. We denote by $E \widehat{\otimes}_\epsilon F$ the completion of the locally convex space $E \otimes_\epsilon F$, the (vector) space $E \otimes F$ equipped with the biprojective tensor product topology ϵ. $E \widehat{\otimes}_\epsilon F$ is contained in $\mathscr{L}(E'_c, F)$, whenever the last space is complete. Examples of locally convex spaces E and F with $\mathscr{L}(E'_c, F)$ complete are treated in [1] and [2]. Now, an element u of $E \widehat{\otimes}_\epsilon F$ is said to be of *separable range* (abbreviated to (sr)-element), if there exists a separable (topological vector) subspace G of F such that u belong to $E \widehat{\otimes}_\epsilon G$. We denote by $E \overset{\vee}{\otimes} F$ the set of (sr)-elements of $E \widehat{\otimes}_\epsilon F$. For E and F complete, $E \overset{\vee}{\otimes} F$ is a vector subspace of $E \widehat{\otimes}_\epsilon F$ (cf. [15, Lemma 4.2]). On the other hand, for any subset M of $E \overset{\vee}{\otimes} F$, $x' \in E'$ and $y' \in F'$ we will consider the sets $M(x') = \{u(x') \colon u \in M\} \subseteq F$ and ${}^tM(y') = \{{}^tu(y') \colon u \in M\} \subseteq E$.

Now, if (u_n) is a (weakly) $\sigma(E \widehat{\otimes}_\epsilon F, (E \widehat{\otimes}_\epsilon F)')$-Cauchy sequence in $E \widehat{\otimes}_\epsilon F$, then, by the Grothendieck's completeness theorem, (u_n) is contained in $(E' \otimes F')^*$ and it is also clearly $\sigma((E' \otimes F')^*, E' \otimes F')$-Cauchy, so that by the completeness of $((E' \otimes F')^*, \sigma((E' \otimes F')^*, E' \otimes F'))$, there exists the $\sigma((E' \otimes F')^*, E' \otimes F')$-limit of (u_n). Thus, let T denote the (vector) subspace of $(E' \otimes F')^* = L(E', F'^*)$, consisting of the $\sigma((E' \otimes F')^*, E' \otimes F')$-limits of all (weakly) $\sigma(E \widehat{\otimes}_\epsilon F, (E \widehat{\otimes}_\epsilon F)')$-Cauchy sequences in $E \widehat{\otimes}_\epsilon F$. For complete and weakly sequentially complete (locally convex) spaces E and F, T is contained in $\mathscr{L}(E'_\tau, F)$ (cf. [16, p. 121, Lemma 3.1]).

We are now in a position to state and prove the main theorems of this note. That is, we first have

THEOREM 3.1. *Let E and F be semi-reflexive complete locally convex spaces such that E be Šmulian and F be (s)-space. Suppose, moreover, that $E \overset{\vee}{\otimes} F = E \widehat{\otimes}_\epsilon F$. Then, the following assertions are equivalent:*

(1) *$E \widehat{\otimes}_\epsilon F$ is semi-reflexive.*

(2) *$E \widehat{\otimes}_\epsilon F$ is weakly sequentially complete.*

Proof. (1) implies (2). It follows by [11, p. 144, Section 5.5]. Now, (2) implies (1). In fact, if M is a bounded subset of $E \widehat{\otimes}_\epsilon F$, then all the sets $M(x')$ and ${}^tM(y')$ are obviously bounded and hence, by the semi-reflexivity of E and F, weakly relatively compact. On the other hand, by hypothesis that E is a Šmulian space, each ${}^tM(y')$ is conditionally weakly compact, so that, by [15, Theorem 4.3], the set M is conditionally weakly compact. Thus, by hypothesis (2), the completeness of $E \widehat{\otimes}_\epsilon F$ and the theorem of Eberlein (cf. [8, p. 313, 24.2(1)]), M is clearly weakly relatively compact and the proof is finished.

On the other hand, we also get

THEOREM 3.2. *Let E and F be complete locally convex spaces with E Šmulian and F(s)-space. Suppose, moreover, that $E \overset{\vee}{\otimes} F = E \widehat{\otimes}_\epsilon F$ and consider the following statements* (1) *and* (2):

(1) (a) *Both E and F are semi-reflexive.*

(b) *T is contained in $K(E', F)$.*

(2) *$E \widehat{\otimes}_\epsilon F$ is semi-reflexive.*

Then, (2) *implies* (1). *If, moreover, one of the spaces* E, F, E'_c *and* F'_c *has the approximation property,* (1) *implies also* (2).

Proof. (1) implies (2). By hypothesis and [16, p. 121, Theorem 3.2], $E \hat{\otimes}_\epsilon F$ is weakly sequentially complete and hence semi-reflexive (Theorem 3.1 above). Now, (2) implies (1). By the completeness of E and F and the arguments of [11, p. 167, Section 9.1], it follows that both E and F are closed subsets of $E \hat{\otimes}_\epsilon F$ and hence semi-reflexive (cf. [7, p. 272, Proposition 2]). On the other hand, (1) (b) follows by [16, p. 121, Theorem 3.2] and the proof is completed.

Now, we get the following corollaries of the preceding theorem.

First, by [1, p. 197, Satz 9] and Theorem 3.2 above, we have

COROLLARY 3.3. *Let E and F be complete locally convex spaces with E Šmulian and F (s)-space such that* $E \check{\otimes} F = E \hat{\otimes}_\epsilon F$ *and* $\mathscr{L}(E'_\sigma, F_\sigma) = \mathscr{K}(E', F)$. *Suppose, moreover, that one of the spaces* E, F, E'_c *and* F'_c *has the approximation property. Then,* $E \hat{\otimes}_\epsilon F$ $(= \mathscr{L}_e(E'_c, F) = \mathscr{L}_e(E'_\tau, F))$ *is semi-reflexive if, and only if, both E and F are.*

Remark 3.4. Let E and F be locally convex spaces. If F has F'_σ separable, or both E and F are metrizable, then, by the techniques of [15, Theorem 4.3], we may clearly suppose that $E \check{\otimes} F = E \hat{\otimes}_\epsilon F$ and hence we have

COROLLARY 3.5. *Let E and F be complete locally convex spaces such that E be Šmulian and* F'_σ *be separable (or both E and F be Fréchet). Moreover, suppose that* $\mathscr{L}(E'_\sigma, F_\sigma) = \mathscr{K}(E', F)$ *and one of the spaces* E, F, E'_c *and* F'_c *has the approximation property. Then,* $E \hat{\otimes}_\epsilon F$ *is semi-reflexive if, and only if, both E and F are.*

We say that a Banach space E has the *metric (s)-approximation property* if, for all $\epsilon > 0$ and $x_1, x_2, ..., x_m \in E$, there is a finite rank (continuous linear) operator u on E with $\| u \| \leqslant 1$ and $\| u(x_i) - x_i \| \leqslant \epsilon$ for all $i = 1, 2, ..., m$ (cf. [13, p. 9]).

By Theorem 3.2 in the foregoing and [16, p. 122, Corollary 3.4], we have the following result, which may be considered as a partial converse of the preceding Corollary. That is, we get

COROLLARY 3.6. *Let E and F be Banach spaces such that E or F have the metric (s)-approximation property and let* $E \hat{\otimes}_\epsilon F$ *be semi-reflexive. Then,* $\mathscr{L}(E'_\tau, F) = \mathscr{K}(E', F)$.

Let E and F be Banach spaces with strong duals E' and F' respectively. If E is reflexive, then E has the approximation property if, and only if, this is the case for E' (cf. [11, p. 198, Exercise 30]). Thus, if both E and F are reflexive and one of them has the approximation property, then $E' \hat{\otimes}_\epsilon F$ coinsides with the vector space of all compact linear maps of E into F (cf. [11, p. 113, Theorem 9.5]). From this and Corollary 3.5 above we clearly get the following result, which is the reflexivity criterium for the Banach space $\mathscr{L}_b(E, F)$ stated in Section 1.

COROLLARY 3.7. *Let E and F be Banach spaces such that E or F have the approximation property. Then, the Banach space $\mathscr{L}_b(E, F)$ is reflexive if, and only if, both E and F are reflexive and, moreover, all the maps of it are compact.*

On the other hand, if E is a bornological locally convex space, then both E'_c and E'_b are complete (cf. [8, p. 385, 28.5(1)]. Thus, by Remark 3.4, Theorem 3.2 in the foregoing and [2, p. 7, Satz 6] we also get the following

COROLLARY 3.8. *Let E and F be complete locally convex spaces such that E be bornological (resp. Montel (LF)-space) with the approximation property. Suppose, moreover, that E'_c (resp. E'_b) is Šmulian and F'_σ separable. Then, $\mathscr{L}_c(E, F) = E'_c \widehat{\otimes}_\epsilon F$ (resp. $\mathscr{L}_b(E, F) = E'_b \widehat{\otimes}_\epsilon F$) is semi-reflexive if, and only if, both E'_c and F (resp. E'_b and F) are.*

Furthermore, if E and F are complete locally convex spaces with F a semi-Montel space, which has the approximation property, then, by standard arguments, given, for example, in [14, p. 522, Proposition 50.4], it follows that $E \widehat{\otimes}_\epsilon F = \mathscr{L}_e(E'_\tau, F)$, so that, by Remark 3.4 and Theorem 3.2 we also have

COROLLARY 3.9. *Let E and F be complete locally convex spaces such that E be Šmulian and F be a semi-Montel space with the approximation property (or, in particular, F be a nuclear space) and F'_σ be separable. Then, $E \widehat{\otimes}_\epsilon F = \mathscr{L}_e(E'_\tau, F)$ is semi-reflexive if, and only if, E is semi-reflexive.*

Now, let (X, Σ, μ) be a complex measure space, $1 \leqslant p < +\infty$ and let E be a complete locally convex space. We consider the space $L^p_E(\mu)$ of (classes of) E-valued maps f on X with $q \circ f \in L^p(\mu)$ for every continuous seminorm q on E (cf. [15, Section 5]). If, moreover, E is nuclear, then $L^p_E(\mu) = L^p(\mu) \widehat{\otimes}_\epsilon E$ (as topological vector spaces) (ibid.). Thus, by Remark 3.4 and Theorem 3.2 we get the following

COROLLARY 3.10. *Let (X, Σ, μ) be a complex measure space, $1 < p < +\infty$ and let E be a complete nuclear locally convex space with E'_σ separable (or, let E be a nuclear Fréchet space). Then, $L^p_E(\mu)$ is semi-reflexive.*

After Arens, a (completely regular) topological space X is called *hemicompact*, if there exists a countable fundamental family of compact subsets of it [17]. In this respect, the following final result extends it [17, p. 274, Theorem 10].

COROLLARY 3.11. *Let X be a (completely regular) hemicompact space and let E be a complete locally convex space with E'_σ separable (or, let E be a Fréchet space), which has the approximation property. Then, the space $\mathscr{C}(X, E)$ of all continuous E-valued maps on X, equipped with the topology of compact convergence in X, is semi-reflexive if, and only if, X is descrete and E semi-reflexive.*

Proof. By [3, p. 203, Theorem 3], $\mathscr{C}(X, E) = \mathscr{C}(X) \hat{\otimes}_\epsilon E$. On the other hand, by hypothesis that X is hemicompact and [17, p. 267, Theorem 2], $\mathscr{C}(X)$ is metrizable, so that, by hypotheses for E and Remark 3.4, we may suppose that $\mathscr{C}(X) \overset{\vee}{\otimes} E = \mathscr{C}(X) \hat{\otimes}_\epsilon E$. Suppose now that $\mathscr{C}(X, E)$ is semi-reflexive. Then by Theorem 3.2 in the foregoing, E and $\mathscr{C}(X)$ are semi-reflexive and hence X is descrete (cf. [17, p. 274, Theorem 10]). On the other hand, suppose that X is descrete and E semi-reflexive. Then, by [17, p. 274, Theorem 10], $\mathscr{C}(X)$ is Montel and hence clearly every (weakly) continuous linear map from E' into $\mathscr{C}(X)$ transfers equicontinuous subsets of E' into relatively compact subsets of $\mathscr{C}(X)$, so that the assertion follows by Corollary 3.3 in the foregoing and the proof is finished.

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