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Generalisation of the Perron–Frobenius theory to matrix pencils ☆

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Dedicated to Hans Schneider on the occasion of his 80th birthday

Abstract

We present a new extension of the well-known Perron–Frobenius theorem to regular matrix pairs (E, A). The new extension is based on projector chains and is motivated from the solution of positive differential–algebraic systems or descriptor systems. We present several examples where the new condition holds, whereas conditions in previous literature are not satisfied.
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1. Introduction

The well-known Perron–Frobenius theorem states that for an elementwise nonnegative matrix the spectral radius, i.e., the largest modulus of an eigenvalue is itself an eigenvalue and has a

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nonnegative eigenvector. This result has many applications in all areas of science and engineering, in particular in economics and population dynamics see e.g. [2]. If the dynamics of the system, however, is described by an implicit differential or difference equation, usually called differential–algebraic equation (DAE) or descriptor system such as

$$E\dot{x} = Ax + f, \quad x(t_0) = x_0,$$

or as a discrete time system

$$Ex_{k+1} = Ax_k + f_k, \quad x_0 \text{ given},$$

where E, A are real $n \times n$ matrices, then the dynamics is described by the eigenvalues and eigenvectors associated with the matrix pencil $\lambda E - A$, or just the matrix pair (E, A).

Due to the many applications, several approaches have been presented in the literature to generalise the classical Perron–Frobenius theory to matrix pencils or further to matrix polynomials. In [11] a direct generalisation of the nonnegativity condition for A, $y \geqslant 0 \Rightarrow Ay \geqslant 0$, is given as a sufficient condition, $E^{\mathrm{T}}y \geqslant 0 \Rightarrow A^{\mathrm{T}}y \geqslant 0$, for the existence of a positive eigenvalue and a corresponding nonnegative eigenvector. In [1] a sufficient condition, $(E - A)^{-1}A \geqslant 0$, for the existence of a positive eigenvalue in (0, 1) and a corresponding positive eigenvector if $(E - A)^{-1}A$ is irreducible, is proved. The relationship of the two ideas from [1,11] is studied in [15]. In [14], the condition from [1] is imposed by requiring $E - A$ to be a nonsingular M-matrix and $A \geqslant 0$. There, the structure of nonnegative eigenvectors is studied from the combinatorial point of view. In [16] the Perron–Frobenius theory was extended to matrix polynomials, where the coefficient matrices are entrywise nonnegative. Other extensions concerning matrix polynomials are given in [5].

The main drawbacks of the generalisation in [1] are that on the one hand it is a restrictive condition, since $E - A$ is not necessarily invertible, and on the other hand it does not have the classical Perron–Frobenius theory as a special case, where $E = I$. Furthermore, only the existence of a nonnegative real eigenvalue is guaranteed instead of the spectral radius being an eigenvalue. The condition in [11] has the classical Perron–Frobenius as a special case but the condition is not easy to verify. Furthermore, for the case considered in this paper where E is a singular matrix and (E, A) is a regular matrix pair, it is easy to see that this condition can never hold.

In this paper we propose a new approach to extend the classical Perron–Frobenius theory to matrix pairs (E, A), where a sufficient condition guarantees that the finite spectral radius of (E, A) is an eigenvalue with a corresponding nonnegative eigenvector. As mentioned before, our approach is motivated by the study of positive systems of differential–algebraic equations, see, e.g., [3,8]. It is based on the construction of projector chains as they were introduced in the context in [13]. For the special case $E = I$ our new approach reduces to the classical Perron–Frobenius theorem for matrices. We present several examples where the new condition holds, whereas previous conditions in the literature are not satisfied.

2. Preliminaries

2.1. *The generalised eigenvalue problem*

Let $E, A \in \mathbb{R}^{n\times m}$. A matrix pair (E, A), or matrix pencil $\lambda E - A$, is called *regular* if E and A are square $(n = m)$ and $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{R}$. It is called *singular* otherwise. In this paper we only consider square and regular pencils.

A scalar $\lambda \in \mathbb{C}$ is said to be a *finite eigenvalue* of the matrix pair (E, A) if $\det(\lambda E - A) = 0$. A vector $x \in \mathbb{C}^n \setminus \{0\}$ such that $(\lambda E - A)x = 0$ is called *eigenvector* of (E, A) corresponding to λ.

If E is singular and $v \in \mathbb{C}^n \setminus \{0\}$, such that $Ev = 0$ holds, then v is called eigenvector of (E, A) corresponding to the eigenvalue ∞. The equation

$$\lambda E v = A v \tag{2.1}$$

is called *generalised eigenvalue problem*. The set of all finite eigenvalues is called *finite spectrum* of (E, A) and is denoted by $\sigma_{\mathrm{f}}(E, A)$. The set of all eigenvalues is called *spectrum* of (E, A) and is defined by

$$\sigma(E, A) := \begin{cases} \sigma_{\mathrm{f}}(E, A) & \text{if } E \text{ is invertible,} \\ \sigma_{\mathrm{f}}(E, A) \cup \{\infty\} & \text{if } E \text{ is singular.} \end{cases}$$

If $\sigma_{\mathrm{f}}(E, A) \neq \emptyset$, then we denote by

$$\rho_{\mathrm{f}}(E, A) = \max_{\lambda \in \sigma_{\mathrm{f}}(E,A)} |\lambda|$$

the *finite spectral radius* of (E, A). Vectors $v_1, \ldots, v_k$ form a *Jordan chain* of the matrix pair (E, A) corresponding to an eigenvalue λ if $(\lambda E - A)v_i = -Ev_{i-1}$ for all $1 \leqslant i \leqslant k$ and $v_0 = 0$. A k-dimensional subspace $S_\lambda^{\mathrm{def}} \subset \mathbb{C}^n$ is called *(right) deflating subspace of* (E, A) *corresponding to* λ, if it is spanned by all Jordan chains corresponding to λ and if there exists a k-dimensional subspace $\mathscr{W} \subset \mathbb{C}^n$ such that $ES_\lambda^{\mathrm{def}} \subset \mathscr{W}$ and $AS_\lambda^{\mathrm{def}} \subset \mathscr{W}$. Let $\lambda_1, \ldots, \lambda_p$ be the pairwise distinct finite eigenvalues of (E, A) and let $S_{\lambda_i}^{\mathrm{def}}$, $i = 1, \ldots, p$ be the corresponding deflating subspaces associated with these eigenvalues. We call the subspace defined by

$$S_{\mathrm{f}}^{\mathrm{def}} := S_{\lambda_1}^{\mathrm{def}} \oplus \cdots \oplus S_{\lambda_p}^{\mathrm{def}} \tag{2.2}$$

the *(right) finite deflating subspace* of (E, A).

Two matrix pairs (E, A) and $(\widetilde{E}, \widetilde{A})$ are called equivalent if there exist regular matrices W, T such that

$$E = W\widetilde{E}T, \qquad A = W\widetilde{A}T. \tag{2.3}$$

In this case we write $(E, A) \sim (\widetilde{E}, \widetilde{A})$.

Theorem 2.1 (Kronecker/Weierstraß canonical form). *Let* (E, A) *be a regular matrix pair. Then, we have*

$$(E, A) \sim \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{2.4}$$

where J is a matrix in Jordan canonical form and N is a nilpotent matrix also in Jordan canonical form.

Proof. See, e.g., [6]. □

Lemma 2.2. *Let* (E, A) *be a regular matrix pair. Let* $\hat{\lambda}$ *be chosen such that* $\hat{\lambda}E - A$ *is nonsingular. Then, the matrices*

$$\widehat{E} = (\hat{\lambda}E - A)^{-1}E \quad \text{and} \quad \widehat{A} = (\hat{\lambda}E - A)^{-1}A$$

commute.

Proof. See, e.g., [4,9]. □

2.2. Projectors and index of (E, A)

A matrix Q is called *projector* if $Q^2 = Q$. A projector Q is called projector *onto* a subspace $S \subseteq \mathbb{R}^n$ if im $Q = S$. It is called projector *along* a subspace $S \subseteq \mathbb{R}^n$ if ker $Q = S$. We will make use of the following well-known property of projectors. P_1, P_2 are two projectors onto a subspace S if and only if $P_1 = P_2 P_1$ and $P_2 = P_1 P_2$. P_1, P_2 are two projectors along a subspace S if and only if $P_1 = P_1 P_2$ and $P_2 = P_2 P_1$.

Let (E, A) be a regular matrix pair. As introduced in [13] we define a matrix chain by setting

$$E_0 := E, A_0 := A \quad \text{and} \tag{2.5a}$$
$$E_{i+1} := E_i - A_i \widetilde{Q}_i, \quad A_{i+1} := A_i \widehat{P}_i \quad \text{for } i \geqslant 0, \tag{2.5b}$$

where $\widetilde{Q}_i$ are projectors onto ker E_i and $\widehat{P}_i = I - \widetilde{Q}_i$. Since we have assumed (E, A) to be regular, there exists an index ν such that E_ν is nonsingular and all E_i are singular for $i < \nu$, [12]. Note, that ν is independent of a specific choice of the projectors Q_i. We say that the matrix pair (E, A) has *(tractability) index* ν and denote it by $\operatorname{ind}(E, A) = \nu$. It is well known that for regular pairs (E, A) the tractability index is equal to the Kronecker index that can be determined as the size of the largest Jordan block associated with the eigenvalue infinity in the Kronecker/Weierstraß canonical form of the pair (E, A), see [6,9,12]. In the following we, therefore, only speak of the *index* of the pair (E, A).

Lemma 2.3. *Let* (E, A) *be a matrix pair and define a matrix chain as in* (2.5). *Furthermore, define sets* S_i *by*

$$S_i := \{y \in \mathbb{R}^n : A_i y \in \operatorname{im} E_i\}. \tag{2.6}$$

Then, if E_{i+1} *is nonsingular, we have that*

$$Q_i = -\widetilde{Q}_i E_{i+1}^{-1} A_i$$

is a projector onto ker E *along* S_i.

Proof. See, e.g., [7,12]. □

For the construction of specific projectors in the higher index cases in Section 3.3, we will need the following properties.

Lemma 2.4. *Let* (E, A) *be a regular matrix pair of* $\operatorname{ind}(E, A) = \nu$ *and define a matrix chain as in* (2.5), *where the projectors* $\widetilde{Q}_i$ *are chosen such that* $\widetilde{Q}_j \widetilde{Q}_i = 0$ *holds for all* $0 \leqslant i < j \leqslant \nu - 1$. *For* $0 \leqslant i \leqslant \nu - 1$ *we define projectors* Q_i *onto* ker E_i *by setting* $Q_i = -\widetilde{Q}_i E_\nu^{-1} A_i$ *and* $P_i = I - Q_i$. *Then,* $Q_j Q_i = 0$ *holds for all* $0 \leqslant i < j \leqslant \nu - 1$.

Proof. The matrix $-\widetilde{Q}_i E_\nu^{-1} A_i$ is a projector for all $0 \leqslant i \leqslant \nu - 1$, since

$$\begin{aligned}(-\widetilde{Q}_i E_\nu^{-1} A_i)^2 &= \widetilde{Q}_i E_\nu^{-1} (E_i - E_{i+1}) \widetilde{Q}_i E_\nu^{-1} A_i \\ &= -\widetilde{Q}_i E_\nu^{-1} E_{i+1} \widetilde{Q}_i E_\nu^{-1} A_i = -Q_i E_\nu^{-1} A_i,\end{aligned}$$

where we have used that $E_\nu \widetilde{Q}_i = (E_{i+1} - A_{i+1} \widetilde{Q}_{i+1} - \cdots - A_{\nu-1} Q_{\nu-1}) \widetilde{Q}_i = E_{i+1} \widetilde{Q}_i$.

To show that $Q_j Q_i = 0$ holds for all $0 \leqslant i < j \leqslant \nu - 1$, let i, j be arbitrarily chosen fixed indeces $0 \leqslant i < j \leqslant \nu - 1$. Then, we have that

$$
\begin{aligned}
Q_j Q_i &= \widetilde{Q}_j E_\nu^{-1} A_j \widetilde{Q}_i E_\nu^{-1} A_i = \widetilde{Q}_j E_\nu^{-1} A_i \widehat{P}_i \cdots \widehat{P}_{j-1} \widetilde{Q}_i E_\nu^{-1} A_i \\
&= \widetilde{Q}_j E_\nu^{-1} A_i (I - \widetilde{Q}_i) \cdots (I - \widetilde{Q}_{j-1}) \widetilde{Q}_i E_\nu^{-1} A_i = \widetilde{Q}_j E_\nu^{-1} A_i (\widetilde{Q}_i - \widetilde{Q}_i) \\
&= 0. \qquad \square
\end{aligned}
$$

Note that by definition and Lemma 2.3, we have that $Q_{\nu-1}$ is a projector onto ker $E_{\nu-1}$ along $S_{\nu-1}$. This projector $Q_{\nu-1}$ is called *canonical* in [13].

2.3. Nonnegative matrices

A matrix $T \in \mathbb{R}^{n\times n}$, $T = [t_{ij}]_{i,j=1}^n$ or a vector $v \in \mathbb{R}^n$, $v = [v_i]_{i=1}^n$ is called *nonnegative* and we write $T \geqslant 0$ or $v \geqslant 0$ if all entries t_{ij} or v_i are nonnegative, respectively. We call a vector positive and write $v > 0$ if all entries v_i are positive. The classical Perron–Frobenius theorem, see, e.g., [2, pp. 26/27], states as follows.

Theorem 2.5 (Perron–Frobenius theorem). *Let $T \geqslant 0$ have the spectral radius $\rho(T)$. Then $\rho(T)$ is an eigenvalue of T and T has a nonnegative eigenvector v corresponding to $\rho(T)$. If, in addition, T is irreducible, then $\rho(T)$ is simple and T has a positive eigenvector v corresponding to $\rho(T)$. Furthermore, if $w > 0$ is an eigenvector of T, then $w = \alpha v$, $\alpha \in \mathbb{R}_+$.*

3. Perron–Frobenius theory for matrix pencils

Several generalisations of the classical Perron–Frobenius theorem 2.5 have been presented in the literature. In [1] the condition $(E - A)^{-1}A \geqslant 0$ is shown to be sufficient for the existence of an eigenvalue $\lambda \in (0, 1)$ and a corresponding nonnegative eigenvector. In [11] a direct generalisation of the nonnegativity condition $y \geqslant 0 \Rightarrow Ay \geqslant 0$ of A is given as a sufficient condition $E^{\mathrm{T}}y \geqslant 0 \Rightarrow A^{\mathrm{T}}y \geqslant 0$ for the existence of a positive eigenvalue and a corresponding nonnegative eigenvector.

In the following two subsections we present a different, projector-based extension of the Perron–Frobenius theory to regular matrix pairs (E, A) that has a number of advantages over the existing conditions in the literature. As a motivation, we give in Section 3.1 the generalisation for the case of index one pencils. In Theorem 3.1 we prove an easily computable sufficient condition that guarantees that the finite spectral radius of (E, A) is an eigenvalue with a corresponding nonnegative eigenvector. We present several examples where the new condition holds, whereas the conditions in [1,11] are not satisfied. In the general case (where the index may be arbitrary) presented in Section 3.2, an additional condition on the projectors has to be imposed, see Lemma 3.6, that is satisfied automatically in the index one case. The general sufficient condition that we then prove in Theorem 3.7 is in the index one case the same as in Theorem 3.1 and also guarantees that the finite spectral radius of (E, A) is an eigenvalue with a corresponding nonnegative eigenvector. In Corollary 3.9, we prove three further conditions each of which is equivalent to the condition in Theorem 3.7. All conditions have the classical Perron–Frobenius theory as a special case when $E = I$.

3.1. Regular matrix pairs of index at most one

In this subsection we study regular pairs (E, A) of index at most one. This is a special case of the general result of this paper that we present in the next subsection and it will suit as a motivation for the reader who is not familiar with the projector-based analysis of differential–algebraic equations. The techniques used in the index one case go back to [7].

Theorem 3.1. *Let* (E, A), *with* $E, A \in \mathbb{R}^{n\times n}$, *be a regular matrix pair with* $\operatorname{ind}(E, A) \leqslant 1$. *Let* Q_0 *be a projector onto* $\ker E$ *along the subspace* S_0 *defined as in* (2.6) *for* $i = 0$, *i.e.*,

$$S_0 := \{y \in \mathbb{R}^n : Ay \in \operatorname{im} E\}, \tag{3.1}$$

let $P_0 = I - Q_0$, *and* $A_1 = AP_0$. *Then* $E_1 := E - AQ_0$ *is nonsingular and if*

$$E_1^{-1}A_1 \geqslant 0, \tag{3.2}$$

and $\sigma_{\mathrm{f}}(E, A) \neq \emptyset$, *then the finite spectral radius* $\rho_{\mathrm{f}}(E, A)$ *is an eigenvalue of the matrix pair* (E, A) *and if* $\rho_{\mathrm{f}}(E, A) > 0$, *then there exists a nonnegative eigenvector* v *corresponding to* $\rho_{\mathrm{f}}(E, A)$. *If* $E_1^{-1}A_1$, *in addition, is irreducible, then* $\rho_{\mathrm{f}}(E, A)$ *is simple and* $v > 0$ *is unique up to a scalar multiple.*

Note that, if $\operatorname{ind}(E, A) = 0$, meaning that E is regular, then we have that $Q_0 = 0$ and $E_1 = E$ and the condition in Theorem 3.1 reduces to the one of the classical Perron–Frobenius theorem for $E^{-1}A$.

Proof. Consider the generalised eigenvalue problem (2.1). Since (E, A) is regular of $\operatorname{ind}(E, A) \leqslant 1$, we have that E_1 as defined in (2.5) is nonsingular, see [12], and we can premultiply equation (2.1) by E_1^{-1}. By also using that $P_0 + Q_0 = I$, we obtain

$$\begin{aligned} &E_1^{-1}(\lambda E - A)(P_0 + Q_0)v = 0 \\ &\quad\Leftrightarrow (\lambda E_1^{-1}E - E_1^{-1}AP_0 - E_1^{-1}AQ_0)v = 0. \end{aligned}$$

Furthermore, we have $E_1^{-1}E = P_0$ since $E_1P_0 = (E - AQ_0)P_0 = E$ and $-E_1^{-1}AQ_0 = -Q_0$ since $E_1Q_0 = (E - AQ_0)Q_0 = -AQ_0$. Hence, we obtain

$$[(\lambda I - E_1^{-1}A)P_0 + Q_0]v = 0,$$

which after multiplication by P_0 and Q_0 from the left is equivalent to the system of two equations

$$\begin{cases} P_0[(\lambda I - E_1^{-1}A)P_0 + Q_0]v = 0, \\ Q_0[(\lambda I - E_1^{-1}A)P_0 + Q_0]v = 0. \end{cases} \tag{3.3}$$

We have that Q_0 is a projector onto $\ker E$ along S_0 and by Lemma 2.3 we conclude that $-Q_0E_1^{-1}A$ is also a projector onto $\ker E$ along S_0. Hence, by Lemma 2.3, we have that $(-Q_0E_1^{-1}A)P_0 = Q_0P_0 = 0$. Therefore, by writing $P_0 = I - Q_0$ in the first equation of (3.3), the two equations reduce to

$$\begin{cases} (\lambda I - E_1^{-1}A)P_0v = 0, \\ Q_0v = 0. \end{cases}$$

Since $P_0 = P_0P_0$, this is equivalent to

$$\begin{cases} (\lambda I - E_1^{-1}AP_0)P_0v = 0, \\ Q_0v = 0. \end{cases} \tag{3.4}$$

Setting $x = P_0v$, $y = Q_0v$ and $v = P_0v + Q_0v = x + y$, we obtain a standard eigenvalue problem in the first equation and a linear system in the second equation. From the first equation we know from the Perron–Frobenius theorem that if $E_1^{-1}AP_0 \geqslant 0$, then the spectral radius of $E_1^{-1}AP_0$ is an eigenvalue with a corresponding nonnegative eigenvector. If $E_1^{-1}AP_0$ is in addition irreducible, we have that $\rho(E_1^{-1}AP_0)$ is a simple eigenvalue and there is a corresponding positive eigenvector that is unique up to a multiple. Set $\hat{\lambda} := \rho(E_1^{-1}AP_0)$ and if $\hat{\lambda} \neq 0$, due to (3.4), we can set $\hat{x} = P_0 v$ for the corresponding nonnegative (positive) eigenvector. Then, we obtain

$$\begin{aligned}
&E_1^{-1}AP_0\hat{x} = \hat{\lambda}x \\
&\quad \Leftrightarrow AP_0\hat{x} = \hat{\lambda}E_1\hat{x} \\
&\quad \Leftrightarrow AP_0\hat{x} = \hat{\lambda}(E - AQ_0)\hat{x} \\
&\quad \Leftrightarrow AP_0P_0v = \hat{\lambda}EP_0v - AQ_0P_0v \\
&\quad \Leftrightarrow A(P_0v + Q_0v) = \hat{\lambda}Ev \\
&\quad \Leftrightarrow Av = \hat{\lambda}Ev,
\end{aligned}$$

which is the generalised eigenvalue problem (2.1). Hence, $\rho(E_1^{-1}AP_0) = \rho_f(E, A)$ and if $\rho_f(E, A) \neq 0$, there exists a corresponding nonnegative eigenvector. This completes the proof. □

Example 3.2. Consider the pair (E, A) given by

$$E = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We have $\text{ind}(E, A) = 1$ and the pair has only one finite eigenvalue $\lambda = 0.5$ with eigenvector $v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$, where we may normalise the eigenvector so that $v_1 > 0$.

To check the sufficient condition (3.2) of Theorem 3.1, we choose a projector $\widetilde{Q}_0$ onto $\ker E_0$, e.g.,

$$\widetilde{Q}_0 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix},$$

and get

$$\widetilde{E}_1 = E_0 - A_0\widetilde{Q}_0 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

For the inverse we obtain

$$\widetilde{E}_1^{-1} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix},$$

and a projector Q_0 onto $\ker E$ along S_0 is given by

$$Q_0 = -\widetilde{Q}_0\widetilde{E}_1^{-1}A_0 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

We then have

$$E_1 = E_0 - A_0Q_0 = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}, \quad E_1^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & -1 \end{bmatrix},$$

and we set $P_0 = I - Q_0$. Condition (3.2) then reads

$$E_1^{-1}A_1 = E_1^{-1}AP_0 = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \geqslant 0,$$

and we can apply Theorem 3.1.

For this example, the theories in [1,11] cannot be applied, since $(E-A)^{-1}A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \ngeqslant 0$ and there exists a vector, e.g., $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ such that $E^{\mathrm{T}}y \geqslant 0$ but $A^{\mathrm{T}}y \ngeqslant 0$.

Example 3.3. Consider a pair (E, A) with $\operatorname{ind}(E, A) = 1$ and $E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}$, where E_{11} is nonsingular and $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is partitioned accordingly. For a pencil in this form, $\operatorname{ind}(E, A) = 1$ is equivalent to A_{22} being nonsingular, see, e.g., [9]. We choose any projector $\widetilde{Q}_0$ onto $\ker E$, e.g.

$$\widetilde{Q}_0 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

and compute $\widetilde{E}_1$ and $\widetilde{E}_1^{-1}$. We obtain

$$\widetilde{E}_1 = E - A\widetilde{Q}_0 = \begin{bmatrix} E_{11} & -A_{12} \\ 0 & -A_{22} \end{bmatrix}, \qquad \widetilde{E}_1^{-1} = \begin{bmatrix} E_{11}^{-1} & -E_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & -A_{22}^{-1} \end{bmatrix}.$$

Then, we determine a projector Q_0 onto $\ker E$ along $S_0 = \{y \in \mathbb{R}^n : Ay \in \operatorname{im} E\}$ as

$$Q_0 = -\widetilde{Q}_0 E_1^{-1} A = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E_{11}^{-1} & -E_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & -A_{22}^{-1} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix}.$$

Furthermore, we get $P_0 = \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & 0 \end{bmatrix}$ and then compute E_1 and E_1^{-1}. We obtain

$$E_1 = E - AQ_0 = \begin{bmatrix} E_{11} - A_{12}A_{22}^{-1}A_{21} & -A_{12} \\ -A_{21} & -A_{22} \end{bmatrix},$$

$$E_1^{-1} = \begin{bmatrix} E_{11}^{-1} & -E_{11}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}E_{11}^{-1} & A_{22}^{-1}A_{21}E_{11}^{-1}A_{12}A_{22}^{-1} - A_{22}^{-1} \end{bmatrix}.$$

Condition (3.2) then reads as

$$\begin{aligned} &E_1^{-1}AP_0 \\ &\quad= \begin{bmatrix} E_{11}^{-1} & -E_{11}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}E_{11}^{-1} & A_{22}^{-1}A_{21}E_{11}^{-1}A_{12}A_{22}^{-1} - A_{22}^{-1} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & 0 \end{bmatrix} \\ &\quad= \begin{bmatrix} E_{11}^{-1}A_S & 0 \\ -A_{22}^{-1}A_{21}E_{11}^{-1}A_S & 0 \end{bmatrix} \geqslant 0, \end{aligned}$$

where $A_S = A_{11} - A_{12}A_{22}^{-1}A_{21}$.

Consider again the eigenvalue problem

$$(\lambda E - A)v = 0.$$

In our case we obtain

$$\begin{bmatrix} \lambda E_{11} - A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

Since E_{11} is nonsingular, we can rewrite this system as

$$\begin{cases}(\lambda I - E_{11}^{-1}A_{11})v_1 - E_{11}^{-1}A_{12}v_2 = 0,\\ -A_{21}v_1 - A_{22}v_2 = 0,\end{cases}$$

which is equivalent to

$$\begin{cases}(\lambda I - E_{11}^{-1}A_S)v_1 = 0,\\ v_2 = -A_{22}^{-1}A_{21}v_1,\end{cases} \tag{3.5}$$

where $A_S = A_{11} - A_{12}A_{22}^{-1}A_{21}$. Condition (3.2) gives $E_{11}^{-1}A_S \geqslant 0$ and, by the Perron–Frobenius theorem, we obtain from the first equation of (3.5) that $\rho(E_{11}^{-1}A_S) =: \hat{\lambda}$ is an eigenvalue with a corresponding eigenvector $v_1 \geqslant 0$. Using this, from the second equation of (3.5) we obtain

$$v_2 = -A_{22}^{-1}A_{21}v_1 = -\lambda^{-1}A_{22}^{-1}A_{21}\lambda v_1 = -\lambda^{-1}A_{22}^{-1}A_{21}E_{11}^{-1}A_S v_1 \geqslant 0,$$

since $-A_{22}^{-1}A_{21}E_{11}^{-1}A_S \geqslant 0$ by (3.2) and we have $\hat{\lambda} \geqslant 0$ and $v_1 \geqslant 0$ from the first equation of (3.5).

Remark 3.4. 1. Considering the case $E = I$ in Theorem 3.1, we have $P_0 = I$, and the condition $E_1^{-1}A_1 \geqslant 0$ reduces to the condition $A \geqslant 0$ of the classical Perron–Frobenius theorem.

2. Condition $E_1^{-1}A_1 \geqslant 0$, written out, reads as

$$(E - A(I - P_0))^{-1}AP_0 \geqslant 0,$$

which, without the projectors, would be the condition in [1]:

$$(E - A)^{-1}A \geqslant 0.$$

Yet, whereas $(E - A(I - P_0))$ is nonsingular by construction, the matrix $E - A$ is not necessarily invertible. Hence, the new condition finds a much broader applicability.

3. Consider the case $\sigma_f(E, A) \neq \emptyset$ and $\rho_f(E, A) = 0$. If $E_1^{-1}A_1 \geqslant 0$, then we obtain that $\rho_f(E, A) = 0$ is an eigenvalue of (E, A), however, there is not necessarily a corresponding non-negative eigenvector, as the following example shows.

Example 3.5. Consider the matrices

$$E := T^{-1}\widetilde{E}T = \begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix} = \begin{bmatrix}2 & 2\\ -1 & -1\end{bmatrix},$$

$$A := T^{-1}\widetilde{A}T = \begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}^{-1}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix} = \begin{bmatrix}-1 & -2\\ 1 & 2\end{bmatrix}.$$

It is easy to show that $E_1^{-1}A_1 = 0$ and $\sigma_f(E, A) \neq \emptyset$. Therefore, $\rho_f(E, A) = 0$ is an eigenvalue of (E, A). However, the eigenpairs of (E, A) are $(0, [1 \quad -0.5]^T)$ and $(\infty, [1 \quad -1]^T)$ and there does not exists a nonnegative eigenvector corresponding to $\rho_f(E, A)$.

3.2. *Regular matrix pairs of general index*

In this section we consider a regular matrix pair (E, A) of $\text{ind}(E, A) = \nu$. For $\nu > 1$ we need to define the matrix chain in (2.5) with specific projectors. Lemma 3.6 guarantees the existence of projectors with the required property. The canonical projectors as defined in [13] fulfil the condition of Lemma 3.6. An alternative way to construct these projectors along with some examples is presented in Section 3.3.

Lemma 3.6. *Let* (E, A), *with* $E, A \in \mathbb{R}^{n\times n}$, *be a regular matrix pair of* $\operatorname{ind}(E, A) = \nu$. *Then, a matrix chain as in* (2.5) *can be constructed with specific projectors* Q_i, P_i *such that* $Q_i v = 0$ *holds for all* $v \in S_{\mathrm{f}}^{\mathrm{def}}$ *and for all* $0 \leqslant i < \nu$.

Proof. From [13] we know that for a regular matrix pair (E, A), we have that

$$\ker E_i \cap \ker A_i = \{0\} \tag{3.6}$$

holds for all $0 \leqslant i < \nu$. Furthermore, from (3.6) or, e.g., from [13] we obtain that

$$\ker E_i \cap \ker E_{i+1} = \{0\} \tag{3.7}$$

for all $0 \leqslant i < \nu - 1$. We now show by induction that we can construct projectors Q_i such that $Q_i v = 0$ holds for all $v \in S_{\mathrm{f}}^{\mathrm{def}}$ and for all $0 \leqslant i < \nu$. For the existence of such a Q_0, we have to show that $\ker E_0 \cap S_{\mathrm{f}}^{\mathrm{def}} = \{0\}$. Suppose that $x \in \ker E_0 \cap S_{\mathrm{f}}^{\mathrm{def}}$. Then from $E_0 x = 0$ we obtain that $x = 0$, since otherwise, by definition, x would be an eigenvector of (E, A) corresponding to the eigenvalue ∞. Thus, we can choose the projector Q_0 onto $\ker E_0$ along some subspace M_0 that completely contains $S_{\mathrm{f}}^{\mathrm{def}}$. This ensures $Q_0 v = 0$ for all $v \in S_{\mathrm{f}}^{\mathrm{def}}$. Now, suppose that $Q_i v = 0$ holds for all $v \in S_{\mathrm{f}}^{\mathrm{def}}$ and for all $0 \leqslant i \leqslant k$ for some $k < \nu - 1$. Note that for the complementary projectors $P_i = I - Q_i$, this implies that $P_i v = v$ for all $v \in S_{\mathrm{f}}^{\mathrm{def}}$. To construct a projector Q_{k+1} onto $\ker E_{k+1}$ such that $Q_{k+1} v = 0$ holds for all $v \in S_{\mathrm{f}}^{\mathrm{def}}$, we have to show that $\ker E_{k+1} \cap S_{\mathrm{f}}^{\mathrm{def}} = \{0\}$. For this, suppose that $0 \neq x \in \ker E_{k+1} \cap S_{\mathrm{f}}^{\mathrm{def}}$. Then, by using the assumption, we obtain

$$0 = E_{k+1} x = (E_0 - A_0 Q_0 - \cdots - A_k Q_k) x = E_0 x,$$

from which we again conclude that $x = 0$, since otherwise, by definition, x would be an eigenvector of (E, A) corresponding to the eigenvalue ∞. Thus, we can choose the projector Q_{k+1} onto $\ker E_{k+1}$ along some subspace M_{k+1} that completely contains $S_{\mathrm{f}}^{\mathrm{def}}$. This ensures $Q_{k+1} v = 0$ for all $v \in S_{\mathrm{f}}^{\mathrm{def}}$ and completes the proof. □

Note that for $\nu = 1$, condition $Q_0 v = 0$ holds automatically for all $v \in S_{\mathrm{f}}^{\mathrm{def}}$ and in particular for all eigenvectors, see (3.4).

The following theorem states our main result. We prove a new, broadly applicable Perron–Frobenius-type condition for matrix pairs (E, A) in the general index case.

Theorem 3.7. *Let* (E, A), *with* $E, A \in \mathbb{R}^{n\times n}$, *be a regular matrix pair of* $\operatorname{ind}(E, A) = \nu$. *Let a matrix chain as in* (2.5) *be constructed with projectors* Q_i *as in Lemma* 3.6. *If*

$$E_\nu^{-1} A_\nu \geqslant 0, \tag{3.8}$$

and $\sigma_{\mathrm{f}}(E, A) \neq \emptyset$, *then the finite spectral radius* $\rho_{\mathrm{f}}(E, A)$ *is an eigenvalue of* (E, A) *and if* $\rho_{\mathrm{f}}(E, A) > 0$, *then there exists a corresponding nonnegative eigenvector* $v \geqslant 0$. *If* $E_\nu^{-1} A_\nu$ *is, in addition, irreducible, then we have that* $\rho_{\mathrm{f}}(E, A)$ *is simple and* $v > 0$ *is unique up to a scalar multiple.*

Proof. Consider the generalised eigenvalue problem (2.1)

$$(\lambda E - A)v = 0.$$

If v is an eigenvector corresponding to a finite eigenvalue λ, i.e., $v \in S_{\mathrm{f}}^{\mathrm{def}}$, then we have $Q_i v = 0$ for all $0 \leqslant i < \nu$ and we can equivalently express (2.1) as

$$\begin{aligned}&(\lambda(E - A_0Q_0 - A_1Q_1 - \cdots - A_{\nu-1}Q_{\nu-1}) - A)v = 0\\ &\quad\Leftrightarrow (\lambda E_\nu - A)v = 0\\ &\quad\Leftrightarrow (\lambda I - E_\nu^{-1}A)v = 0.\end{aligned} \tag{3.9}$$

By construction, we have that $v = P_0 \cdots P_{\nu-1}v$ and we obtain that (3.9) is equivalent to

$$\begin{aligned}&(\lambda I - E_\nu^{-1}A)P_0 \cdots P_{\nu-1}v = 0\\ &\quad\Leftrightarrow (\lambda I - E_\nu^{-1}AP_0 \cdots P_{\nu-1})P_0 \cdots P_{\nu-1}v = 0\\ &\quad\Leftrightarrow (\lambda I - E_\nu^{-1}A_\nu)v = 0.\end{aligned} \tag{3.10}$$

Note, that in this way, we have shown that any finite eigenpair of (E, A) is an eigenpair of $E_\nu^{-1}A_\nu$. Conversely, by (3.10), we have that any eigenpair (λ, v) of $E_\nu^{-1}A_\nu$ with $\lambda \neq 0$ is a finite eigenpair of (E, A). Since $E_\nu^{-1}A_\nu \geqslant 0$, by the classical Perron–Frobenius theorem we obtain that $\rho(E_\nu^{-1}A_\nu)$ is an eigenvalue of $E_\nu^{-1}A_\nu$ and there exists a corresponding eigenvector $v \geqslant 0$. Since we have assumed that $\sigma_{\mathrm{f}}(E, A) \neq \emptyset$, we have that $\rho(E_\nu^{-1}A_\nu) = \rho_{\mathrm{f}}(E, A)$ is also an eigenvalue of (E, A). If $\rho(E_\nu^{-1}A_\nu) > 0$, then there exists a corresponding nonnegative eigenvector.[1] □

Remark 3.8. In Theorem 3.7 it is shown that any eigenpair (λ, v) of $E_\nu^{-1}A_\nu$ with $\lambda \neq 0$ is a finite eigenpair of (E, A). However, this is not necessarily the case if $\lambda = 0$, since an eigenvalue 0 of $E_\nu^{-1}A_\nu$ can correspond either to the eigenvalue 0 of (E, A) or to the eigenvalue ∞ of (E, A). One can see this by considering an eigenvector w corresponding to an infinite eigenvalue of (E, A), i.e., $Ew = 0$. Then, we obtain $E_\nu^{-1}A_\nu w = 0$, since $P_0 \cdots P_{\nu-1}w = 0$. Since we have assumed that $\sigma_{\mathrm{f}}(E, A) \neq \emptyset$, we have that $\rho(E_\nu^{-1}A_\nu) = \rho_{\mathrm{f}}(E, A) = 0$ is an eigenvalue of (E, A). However, a corresponding nonnegative eigenvector does not necessarily exist as Example 3.5 shows.

Corollary 3.9. *Let P_r be a projector onto the right finite deflating subspace $S_{\mathrm{f}}^{\mathrm{def}}$, let $\widehat{E}$, $\widehat{A}$ be defined as in Lemma 2.2 and let $\widehat{E}^{\mathrm{D}}$ denote the Drazin inverse of $\widehat{E}$. Under the assumptions of Theorem 3.7 each of the conditions*

$$P_rE_\nu^{-1}A \geqslant 0, \tag{3.11}$$

$$E_\nu^{-1}A\widehat{E}^{\mathrm{D}}\widehat{E} \geqslant 0, \tag{3.12}$$

$$\widehat{E}^{\mathrm{D}}\widehat{A} \geqslant 0 \tag{3.13}$$

is equivalent to condition (3.8), *respectively.*

Proof. From [13, Theorem 3.1, Section 4] we obtain that for projectors as in Lemma 3.6, we have $P_0 \cdots P_{\nu-1} = P_r = \widehat{E}^{\mathrm{D}}\widehat{E}$ and

$$E_\nu^{-1}A_\nu = E_\nu^{-1}AP_r = P_rE_\nu^{-1}A = \widehat{E}^{\mathrm{D}}\widehat{A}. \qquad \square$$

[1] As the referee pointed out, one can modify the ideas in [13] to give a different proof of Theorem 3.7.

Remark 3.10. Condition

$$E_\nu^{-1} A \geqslant 0 \tag{3.14}$$

can also be proved to be sufficient in Theorem 3.7, see Eq. (3.9), yet there is no evidence for it to ever hold.

3.3. Construction of projectors

In Section 3.2, Lemma 3.6, we have proved the existence of specific projectors for constructing the matrix chain in (2.5) in order to establish a sufficient condition in Theorem 3.7 for $\rho_{\mathrm{f}}(E, A)$ to be an eigenvalue with a corresponding nonnegative eigenvector. In [13], projectors with properties as in Lemma 3.6 are called completely decoupling projectors, that are given, for instance, if $Q_i = -Q_i P_{i+1} \cdots P_{\nu-1} E_{\nu-1}^{-1} A_i$ holds for all $i = 0, \ldots, \nu - 1$. It is shown in [13, Theorem 2.2] that such projectors exist and a constructive proof is given. However, to keep the present paper self-contained, we provide a procedure for constructing such completely decoupling projectors.

First, we will formulate the construction procedure in the general case and give a proof by induction. Then, we will exemplarily show how this procedure works in the index $\nu = 2$ case and give two examples for $\nu = 2$.

Consider a regular matrix pair (E, A) of $\mathrm{ind}(E, A) = \nu$. We make the following observations:

1. For fixed projectors $Q_0, \ldots, Q_{\nu-2}$, the projector $Q_{\nu-1}$ is uniquely defined by being a projector onto $\ker E_{\nu-1}$ along $S_{\nu-1}$, see [13].
2. Consider the sets S_i as defined in (2.6). We have that $S_{\mathrm{f}}^{\mathrm{def}} \subseteq S_0$, since for any $v \in S_{\mathrm{f}}^{\mathrm{def}}$ there exists a $w \in S_{\mathrm{f}}^{\mathrm{def}}$ such that $Av = Ew$, and hence, $Av \in \mathrm{im}\, E$, i.e., $v \in S_0$. Furthermore, we have that $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_{\nu-1}$, see [12], and therefore, $S_{\mathrm{f}}^{\mathrm{def}} \subseteq S_{\nu-1}$. From this we conclude that $Q_{\nu-1} v = 0$ holds for all $v \in S_{\mathrm{f}}^{\mathrm{def}}$.

In the following recursive constructions of matrix and projector chains, we denote by $E_j^{(i)}, A_j^{(i)}, Q_j^{(i)}, P_j^{(i)}$ the ith iterate of E_j, A_j, Q_j, P_j in the recursive construction.

With the basic construction of projectors Q_i for $i = 1, \ldots, \nu - 1$ as in [13], i.e., $Q_j Q_i = 0$ for $j > i$, we construct the chain in (2.5) and set $E_j^{(1)} = E_j$, $A_j^{(1)} = A_j$, $Q_j^{(1)} = Q_j$, and $P_j^{(1)} = P_j$. Now, to obtain completely decoupling projectors, we redefine the initial projectors by the procedure in Algorithm 1:

Algorithm 1: Construction of completely decoupling projectors

Input: projectors $Q_i^{(1)}$ for $i = 1, \ldots, \nu - 1$ such that $Q_j Q_i = 0$ for $j > i$
Output: projectors $Q_i^{(2^{i+1})}$ such that $Q_i^{(2^{i+1})} v = 0$ for all $v \in S_f^{def}$

```
for i = 0, ..., ν − 1 do
    for j = 1, ..., ν − i do
        Q_{ν−j}^(new) = −Q_{ν−j}^(old) (E_ν^(old))^{−1} A_{ν−j}^(old), where old denotes an appropriate index;
        redefine the last part of the chain using Q_{ν−j}^(new);
        apply Algorithm 1 to Q_{ν−j+1}, ..., Q_{ν−1} in order to regain the
        completely decoupling property, i.e. Q_{ν−j+1} v = ... = Q_{ν−1} v = 0 for all v ∈ S_f^def.
```

Theorem 3.11. *For the projectors* $Q_i^{(2^{i+1})}$ *computed in Algorithm* 1 *we have* $Q_i^{(2^{i+1})}v = 0$ *for all* $v \in S_{\mathrm{f}}^{\mathrm{def}}$.

Proof. We perform an induction over the length $k = \nu - i$ of the chain in (2.5), where i is the index variable in Algorithm 1.

Let $k = 1$. Without loss of generality, we can consider the index $\nu = 1$ (and $i = 0$) case. We take any projector $Q_0^{(1)}$ onto ker E and having computed $E_1^{(1)}$ we set $Q_0^{(2)} = -Q_0^{(1)}(E_1^{(1)})^{-1}A$, which by Lemma 2.3 fulfils $Q_0^{(2)}v = 0$ for all $v \in S_{\mathrm{f}}^{\mathrm{def}}$.

Suppose that for some chain of length $k > 1$ we can construct completely decoupling projectors and consider a chain of length $k + 1$. Without loss of generality we consider the index $\nu = k + 1$ case, i.e., we have an initial chain with projectors $Q_0^{(1)}, \dots, Q_{\nu-1}^{(1)}$, such that $Q_j^{(1)}Q_i^{(1)} = 0$ holds for $j > i$ and start Algorithm 1. Note, that this is also true for any intermediate chain of length $k + 1$ in a general index $\nu > k + 1$ case due to Lemma 3.6.

Now, we have to subsequently redefine projectors $Q_{\nu-j}$ for $j = 1, \dots, \nu - i$ and have to show that the redefined projectors are completely decoupling. Therefore, we perform an induction over j. Let $j = 1$. We set $Q_{\nu-1}^{(2)} = -Q_{\nu-1}^{(1)}(E_\nu^{(1)})^{-1}A_{\nu-1}^{(1)}$ that by Lemma 2.3 fulfils $Q_{\nu-1}^{(2)}v = 0$ for all $v \in S_{\mathrm{f}}^{\mathrm{def}}$.

Suppose, we have completely decoupling projectors $Q_{\nu-1}, \dots, Q_{\nu-j}$ for some $1 < j < \nu - i$.

Set $Q_{\nu-j-1}^{(2)} = -Q_{\nu-j-1}^{(1)}(E_\nu^{(k)})^{-1}A_{\nu-j-1}^{(1)}$, where k is an appropriate index. This is a projector by Lemma 2.4 and we obtain $Q_{\nu-j-1}^{(2)}v = 0$ for all $v \in S_{\mathrm{f}}^{\mathrm{def}}$, since for all $v \in S_{\mathrm{f}}^{\mathrm{def}}$ there exists $w \in S_{\mathrm{f}}^{\mathrm{def}}$ with $Av = Ew$ and thus

$$\begin{aligned}
Q_{\nu-j-1}^{(2)}v &= -Q_{\nu-j-1}^{(1)}(E_\nu^{(k)})^{-1}A_{\nu-j-1}^{(1)}v = -Q_{\nu-j-1}^{(1)}(E_\nu^{(k)})^{-1}A_{\nu-j-2}^{(1)}P_{\nu-j-2}^{(1)}v \\
&= -Q_{\nu-j-1}^{(1)}(E_\nu^{(k)})^{-1}A_{\nu-j-2}^{(1)}(I - Q_{\nu-j-2}^{(1)})v \\
&= -Q_{\nu-j-1}^{(1)}(E_\nu^{(k)})^{-1}A_{\nu-j-2}^{(1)}v - Q_{\nu-j-1}^{(1)}Q_{\nu-j-2}^{(1)}v \\
&= -Q_{\nu-j-1}^{(1)}(E_\nu^{(k)})^{-1}A_{\nu-j-3}^{(1)}P_{\nu-j-3}^{(1)}v = \cdots = -Q_{\nu-j-1}^{(1)}(E_\nu^{(k)})^{-1}A_0v \\
&= -Q_{\nu-2}^{(1)}(E_\nu^{(k)})^{-1}E_0w \\
&= -Q_{\nu-j-1}^{(1)}(I - Q_0^{(1)} - \cdots - Q_{\nu-j-1}^{(1)} - Q_{\nu-j} - \cdots - Q_{\nu-1})w,
\end{aligned}$$

where $Q_{\nu-j}w = \cdots = Q_{\nu-1}w = 0$, since $Q_{\nu-j}, \dots, Q_{\nu-1}$ are completely decoupling. Furthermore, we have $Q_{\nu-j-1}^{(1)}Q_{i_1}^{(1)} = 0$ for $i_1 = 0, \dots, \nu - j - 2$ and $Q_{\nu-j-1}^{(1)}(I - Q_{\nu-j-1}^{(1)}) = 0$. Hence, we obtain

$$Q_{\nu-j-1}^{(2)}v = 0.$$

This completes the induction over k and we have shown that we can construct a $Q_0^{(2)}$ such that $Q_0^{(2)}v = 0$ for all $v \in S_{\mathrm{f}}^{\mathrm{def}}$.

We redefine the chain starting from $Q_0^{(2)}$ and consider the chain starting from Q_1. The new chain has length k and we can construct completely decoupling projectors by applying the induction assumption. This completes the proof. □

In total, we have to make $\sum_{i=0}^{\nu-1}(2^{i+1} - 1)$ updates of the projectors Q_i. The sufficient condition of Theorem 3.7 is then checked with $E_{\nu-1}^{(2^\nu)}$ instead of $E_{\nu-1}$ and reads

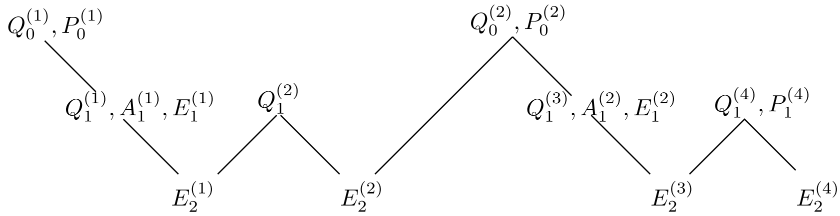

Fig. 3.1. Illustration of the recursive construction of projectors in the index 2 case. Top down, we have the chain matrices in increasing order. From left to right, we have the successive calculation of these.

$$
\begin{aligned}
&E_{\nu-1}^{(2^\nu)} A P_0^{(2)} P_1^{(4)} \cdots P_{\nu-1}^{(2^\nu)} \geqslant 0 \\
&\Leftrightarrow E_{\nu-1}^{(2^\nu)} A_{\nu-1}^{(2^{\nu-1})} P_{\nu-1}^{(2^\nu)} \geqslant 0.
\end{aligned}
$$

So far the described procedure is merely of theoretical value. For a discussion of how to apply this procedure numerically, see [10].

We now show how the projectors are constructed in Algorithm 1 for the index $\nu = 2$ case and give two examples.

We start by choosing any projectors $Q_0^{(1)}$, $Q_1^{(1)}$ onto ker $E_0^{(1)}$, ker $E_1^{(1)}$, respectively. We then determine $E_2^{(1)}$ and set $Q_1^{(2)} = -Q_1^{(1)}(E_2^{(1)})^{-1}A_1^{(1)}$. Then we have $Q_1^{(2)}v = 0$ for all $v \in S_{\mathrm{f}}^{\mathrm{def}}$. By using $Q_1^{(2)}$ compute $E_2^{(2)}$. We proceed by setting $Q_0^{(2)} = -Q_0^{(1)}(E_2^{(2)})^{-1}A_0^{(1)}$, which is a projector by Lemma 2.4. For any $v \in S_{\mathrm{f}}^{\mathrm{def}}$ we have $w \in S_{\mathrm{f}}^{\mathrm{def}}$ such that

$$
Q_0^{(2)}v = -Q_0^{(1)}(E_2^{(2)})^{-1}Av = -Q_0^{(1)}(E_2^{(2)})^{-1}Ew = -Q_0^{(1)}(I - Q_0^{(1)} - Q_1^{(2)})w = 0,
$$

since $Q_1^{(2)}w = 0$. Here we have used the properties $(E_2^{(2)})^{-1}A_i^{(1)}Q_i^{(1)} = -Q_i^{(1)}$ for $i = 0, 1$ and

$$
\begin{aligned}
&E_2^{(2)} = E_0^{(1)} - A_0^{(1)}Q_0^{(1)} - A_1^{(1)}Q_1^{(2)} \\
&\Leftrightarrow I = (E_2^{(2)})^{-1}E_0^{(1)} - (E_2^{(2)})^{-1}A_0^{(1)}Q_0^{(1)} - (E_2^{(2)})^{-1}A_1^{(1)}Q_1^{(2)} \\
&\Leftrightarrow I = (E_2^{(2)})^{-1}E_0^{(1)} + Q_0^{(1)} + Q_1^{(2)}.
\end{aligned}
$$

By using $Q_0^{(2)}$ we compute $E_1^{(2)}$ and $A_1^{(2)}$. Now, we proceed as in the case $\nu = 1$ to define $Q_1^{(3)}$ as a projector onto ker $E_1^{(2)}$. To ensure that it projects along S_1 we again compute $E_2^{(3)}$, set $Q_1^{(4)} = -Q_1^{(3)}(E_2^{(3)})^{-1}A_1^{(2)}$ and obtain that $Q_1^{(4)}v = 0$ for all $v \in S_{\mathrm{f}}^{\mathrm{def}}$. Finally, we compute $E_2^{(4)}$. The sufficient condition of Theorem 3.7 is then checked with $E_2^{(4)}$ instead of E_2 and reads $E_2^{(4)}AP_0^{(2)}P_1^{(4)} \geqslant 0$. For an illustration of the recursive construction of the projectors in the index 2 case with the properties required in Theorem 3.7, see Fig. 3.1.

We now present two index $\nu = 2$ examples, where condition (3.8) of Theorem 3.7 holds, whereas the conditions in [1,11] do not hold.

Example 3.12. Consider the matrix pair (E, A) with

$$
E = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \quad \text{and} \quad A = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 2 \end{array}\right].
$$

We have that (E, A) is regular with $\operatorname{ind}(E, A) = 2$ and there is one finite eigenvalue $\rho_f(E, A) = 2$ and a corresponding eigenvector $\begin{bmatrix} 0 & 0 & v_3 \end{bmatrix}^T$, which can be chosen so that $v_3 > 0$.

We compute the matrix chain by setting, e.g.,

$$Q_0^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_1^{(1)} = E - AQ_0^{(1)} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_1^{(1)} = A_0 P_0^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

We choose, e.g.,

$$Q_1^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P_1^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and compute

$$E_2^{(1)} = E_1^{(1)} - A_1^{(1)} Q_1^{(1)} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad (E_2^{(1)})^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we compute the projector onto $\ker E_1^{(1)}$ along S_1 by setting

$$Q_1^{(2)} = -Q_1^{(1)} (E_2^{(1)})^{-1} A_1^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and determine

$$E_2^{(2)} = E_1^{(1)} - A_1^{(1)} Q_1^{(2)} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad (E_2^{(2)})^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We set

$$Q_0^{(2)} = -Q_0^{(1)} (E_2^{(2)})^{-1} A_0 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P_0^{(2)} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and compute

$$E_1^{(2)} = E_0 - A_0 Q_0^{(2)} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_1^{(2)} = A_0 P_0^{(2)} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Choosing $Q_1^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we determine

$$E_2^{(3)} = E_1^{(2)} - A_1^{(2)} Q_1^{(3)} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2^{(2)} \quad \text{and} \quad (E_2^{(3)})^{-1} = (E_2^{(2)})^{-1},$$

and verify that $Q_1^{(4)} = -Q_1^{(3)} (E_2^{(3)})^{-1} A_1^{(2)} = Q_1^{(3)}$. We finally set $P^{(4)} = I - Q_1^{(4)}$. The sufficient condition (3.8) of Theorem 3.7 then holds, since

$$(E_2^{(4)})^{-1}AP_0^{(2)}P_1^{(4)} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \geqslant 0.$$

The condition in [1], however, is not satisfied, since

$$(E - A)^{-1}A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \ngeqslant 0.$$

Also the condition in [11] does not hold, since, e.g., for $y = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}$ we have $Ey \geqslant 0$ but $Ay \ngeqslant 0$. Note, that we have $P_r = P_0^{(2)}P_1^{(4)}$, yet, condition (3.14) does not hold, since $(E_2^{(4)})^{-1}A \ngeqslant 0$.

Example 3.13. Consider the regular matrix pair (E, A) of $\operatorname{ind}(E, A) = 2$, where

$$E = \begin{bmatrix} E_{11} & E_{12} & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 0 & 0 & A_{14} \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & A_{33} & 0 \\ A_{41} & 0 & 0 & 0 \end{bmatrix}.$$

Note, that every regular matrix pair of index 2 can be equivalently transformed into such a form, where $A_{14}, A_{41}, A_{33}, E_{22}$ are square regular matrices, see [9]. We choose $Q_0^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ and compute

$$P_0^{(1)} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E_1^{(1)} = \begin{bmatrix} E_{11} & E_{12} & 0 & -A_{14} \\ E_{21} & E_{22} & 0 & 0 \\ 0 & 0 & -A_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Choosing

$$Q_1^{(1)} = \begin{bmatrix} I & 0 & 0 & 0 \\ -E_{22}^{-1}E_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{14}^{-1}\widetilde{E}_{11} & 0 & 0 & 0 \end{bmatrix},$$

where $\widetilde{E}_{11} = E_{11} - E_{12}E_{22}^{-1}E_{21}$, we obtain

$$P_1^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ E_{22}^{-1}E_{21} & I & 0 & 0 \\ 0 & 0 & I & 0 \\ -A_{14}^{-1}\widetilde{E}_{11} & 0 & 0 & I \end{bmatrix}, \quad A_1^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{41} & 0 & 0 & 0 \end{bmatrix}, \quad \text{and}$$

$$E_2^{(1)} = \begin{bmatrix} E_{11} & E_{12} & 0 & -A_{14} \\ E_{21} + A_{22}E_{22}^{-1}E_{21} & E_{22} & 0 & 0 \\ 0 & 0 & -A_{33} & 0 \\ A_{41} & 0 & 0 & 0 \end{bmatrix},$$

$$(E_2^{(1)})^{-1} = \begin{bmatrix} 0 & 0 & 0 & -A_{41}^{-1} \\ 0 & E_{22}^{-1} & 0 & E_{22}^{-1}(E_{21} + A_{22}E_{22}^{-1}E_{21})A_{41}^{-1} \\ 0 & 0 & -A_{33}^{-1} & 0 \\ -A_{14}^{-1} & A_{14}^{-1}E_{12}E_{22}^{-1} & 0 & -A_{14}^{-1}(\widetilde{E}_{11} - E_{12}E_{22}^{-1}A_{22}E_{22}^{-1}E_{21})A_{41}^{-1} \end{bmatrix}.$$

We verify that $Q_1^{(2)} = -Q_1^{(1)}(E_2^{(1)})^{-1}A_1^{(1)} = Q_1^{(1)}$ and, hence, $P_1^{(2)} = P_1^{(1)}$, $A_1^{(2)} = A_1^{(1)}$, $E_2^{(2)} = E_2^{(1)}$ and $(E_2^{(2)})^{-1} = (E_2^{(1)})^{-1}$. Setting

$$\begin{aligned} Q_0^{(2)} &= -Q_0^{(1)}(E_2^{(2)})^{-1}A_0 \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ A_{14}(\widetilde{E}_{11} - E_{12}E_{22}^{-1}A_{22}E_{22}^{-1}E_{21}) & -A_{14}E_{12}E_{22}^{-1}A_{22} & 0 & I \end{bmatrix}, \end{aligned}$$

we compute

$$\begin{aligned} P_0^{(2)} &= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -A_{14}(\widetilde{E}_{11} - E_{12}E_{22}^{-1}A_{22}E_{22}^{-1}E_{21}) & A_{14}E_{12}E_{22}^{-1}A_{22} & 0 & 0 \end{bmatrix}, \\ E_1^{(2)} &= E - AQ_0^{(2)} = \begin{bmatrix} E_{12}(I + E_{22}^{-1}A_{22})E_{22}^{-1}E_{21} & E_{12}(I + E_{22}^{-1}A_{22}) & 0 & -A_{14} \\ E_{21} & E_{22} & 0 & 0 \\ 0 & 0 & -A_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ A_1^{(2)} &= AP_0^{(2)} = \begin{bmatrix} -E_{11} + E_{12}(I + E_{22}^{-1}A_{22})E_{22}^{-1}E_{21} & E_{12}E_{22}^{-1}A_{22} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{41} & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Choosing

$$Q_1^{(3)} = \begin{bmatrix} I & 0 & 0 & 0 \\ -E_{22}^{-1}E_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we determine

$$\begin{aligned} P_1^{(3)} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ E_{22}^{-1}E_{21} & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \\ E_2^{(3)} &= \begin{bmatrix} E_{11} + E_{12}E_{22}^{-1}A_{22}E_{22}^{-1}E_{21} & E_{12} + E_{12}E_{22}^{-1}A_{22} & 0 & -A_{14} \\ E_{21} + A_{22}E_{22}^{-1}E_{21} & E_{22} & 0 & 0 \\ 0 & 0 & -A_{33} & 0 \\ -A_{41} & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

$$(E_2^{(3)})^{-1} = \begin{bmatrix} 0 & 0 & 0 & -A_{41}^{-1} \\ 0 & E_{22}^{-1} & 0 & E_{22}^{-1}(E_{21} + A_{22}E_{22}^{-1}E_{21})A_{41}^{-1} \\ 0 & 0 & -A_{33}^{-1} & 0 \\ -A_{14}^{-1} & A_{14}^{-1}E_{12}(I + E_{22}^{-1}A_{22})E_{22}^{-1} & 0 & -A_{14}^{-1}(\widetilde{E}_{11} - E_{12}E_{22}^{-1}A_{22}(I + E_{22}^{-1}A_{22})E_{22}^{-1}E_{21})A_{41}^{-1} \end{bmatrix}.$$

We verify that $Q_1^{(4)} = -Q_1^{(3)}(E_2^{(3)})^{-1}A_1^{(2)} = Q_1^{(3)}$ and, hence, $P_1^{(4)} = P_1^{(3)}$, $E_2^{(4)} = E_2^{(3)}$ and $(E_2^{(4)})^{-1} = (E_2^{(3)})^{-1}$. The sufficient condition (3.8) of Theorem 3.7 then reads as

$$(E_2^{(4)})^{-1}A_1^{(2)}P_1^{(4)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ E_{22}^{-1}A_{22}E_{22}^{-1}E_{21} & E_{22}^{-1}A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{14}^{-1}E_{12}E_{22}^{-1}A_{22}E_{22}^{-1}A_{22}E_{22}^{-1}E_{21} & A_{14}^{-1}E_{12}E_{22}^{-1}A_{22}E_{22}^{-1}A_{22} & 0 & 0 \end{bmatrix} \geqslant 0.$$

Consider again the eigenvalue problem

$$(\lambda E - A)v = 0.$$

For the given E and A, we obtain

$$\begin{bmatrix} \lambda E_{11} & \lambda E_{12} & 0 & -A_{14} \\ \lambda E_{21} & \lambda E_{22} - A_{22} & 0 & 0 \\ 0 & 0 & -A_{33} & 0 \\ -A_{14} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 0.$$

Since A_{41} and A_{33} are nonsingular, we obtain $v_1 = v_3 = 0$ and the following system of equations:

$$\begin{cases} \lambda E_{12}v_2 - A_{14}v_4 = 0, \\ (\lambda E_{22} - A_{22})v_2 = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} (\lambda I - E_{22}^{-1}A_{22})v_2 = 0, \\ v_4 = \lambda A_{14}^{-1}E_{12}v_2. \end{cases}$$

Condition (3.8) gives $E_{22}^{-1}A_{22} \geqslant 0$ and, hence, we obtain from the first equation that $\rho(E_{22}^{-1}A_{22}) =: \lambda$ is an eigenvalue and there exists a corresponding eigenvector $v_2 \geqslant 0$. By using this, we obtain from the second equation that

$$v_4 = \lambda A_{14}^{-1}E_{12}v_2 = A_{14}^{-1}E_{12}E_{22}^{-1}A_{22}v_2 = -\lambda^{-1}A_{14}^{-1}E_{12}E_{22}^{-1}A_{22}E_{22}^{-1}A_{22}v_2 \geqslant 0,$$

since $A_{14}^{-1}E_{12}E_{22}^{-1}A_{22}E_{22}^{-1}A_{22} \geqslant 0$ by (3.8) and $\lambda \geqslant 0$, $v_2 \geqslant 0$ from the first equation.

The condition in [1], however, is not necessarily applicable, since $E - A$ may not be invertible if $E_{22} - A_{22}$ is not. Also the condition in [11] will not hold in most cases, since we may choose y_1, y_2 in $y = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix}^T$ such that $Ey \geqslant 0$ and choose y_3, y_4 such that $Ay \ngeqslant 0$.

4. Conclusions

We have presented a new generalisation of the Perron–Frobenius theorem to regular matrix pairs (E, A) of arbitrary index. The proof is accomplished via projector-based techniques as introduced for the analysis of DAEs in [13]. Equivalent conditions, also in terms of the original matrices using the Drazin inverse, are given. The new generalisation reduces to the classical

Perron–Frobenius theorem if $E = I$ and is different from previous such generalisations. We have demonstrated the broad applicability of the new generalisation by several examples.

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