The Secant Method for Nondifferentiable Operators

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(Received January 2001; accepted June 2001)

Abstract—In this paper, we use the Secant method to find a solution of a nonlinear operator equation in Banach spaces. A semilocal convergence result is obtained. For that, we consider a condition for divided differences which generalizes the usual once, i.e., Lipschitz continuous or Hölder continuous conditions. Besides, we apply our results to approximate the solution of a nonlinear equation. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—The Secant method, Recurrence relations, Nondifferentiable operator.

1. INTRODUCTION

The Secant method for solving a nonlinear equation in Banach spaces is a well-known iterative process [1]. An important feature of this method is that it does not use derivatives when it is applied. Let $X$, $Y$ be two Banach spaces and $F : \Omega \subseteq X \rightarrow Y$ a nonlinear operator and we consider the equation

$$F(x) = 0.$$  

(1)

Let us denote by $\mathcal{L}(X,Y)$ the space of bounded linear operators from $X$ to $Y$. An operator $[x,y;F] \in \mathcal{L}(X,Y)$ is called a divided difference of first order for the operator $F$ on the points $x$ and $y \ (x \neq y)$ if the following equality holds:

$$[x,y;F](x - y) = F(x) - F(y).$$  

(2)

Using this definition, the Secant method is described by the following algorithm:

$$x_{n+1} = x_n - [x_{n-1},x_n;F]^{-1}F(x_n), \quad x_0, x_1 \text{ given.}$$  

(3)

The convergence of (3) to a solution of (1) has been studied by other authors [1–6]. The basic assumption is that the divided difference of first order for the operator $F$ is Lipschitz or Hölder

Supported in part by a grant from DGES (Ref. PB-98-0198) and a grant from the University of La Rioja (Ref. API-00/B-16).
continuous in some ball around the initial iterate. But these assumptions force to operator \( F \) is differentiable \([1,4]\). In this work, we relax this requirement and just assume the following condition:

\[
\| [x, y; F] - [v, w; F] \| \leq \omega(\|x - v\|, \|y - w\|); \quad x, y, v, w \in \Omega,
\]

where \( \omega : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a continuous nondecreasing function in its two arguments. It is evident that this condition generalizes the conditions previously indicated, by only considering \( \omega(u_1, u_2) = k(u_1 + u_2) \) for the Lipschitz continuous case and \( \omega(u_1, u_2) = k(u_1^p + u_2^p) \) for the \((k, p)\)-Hölder continuous one. Moreover, in general, this condition does not involve \( F \) is differentiable. Then we provide a semilocal convergence result for nondifferentiable operators in general. Finally, we give an example where the last is applied.

2. CONVERGENCE STUDY

**Theorem 2.1.** Assume that, for every pair of distinct points \( x, y \in \Omega \), there exists a first-order divided difference \([x, y; F] \in \mathcal{L}(X, Y)\). Let \( x_0, x_{-1} \in \Omega \) and assume

(a) the linear operator \( L_0 = [x_{-1}, x_0; F] \) is invertible and

\[
\| [x_{-1}, x_0; F]^{-1} \| \leq \beta, \quad \| x_0 - x_{-1} \| = \alpha, \quad \| L_0^{-1}F(x_0) \| \leq \eta;
\]

(b) \( \| [x, y; F] - [v, w; F] \| \leq \omega(\|x - v\|, \|y - w\|); x, y, v, w \in \Omega, \) where \( \omega : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a continuous nondecreasing function in its two arguments;

(c) we denote by \( m = \max(\beta \omega(\alpha, \eta), \beta \omega(\eta, \eta)) \) and assume that the equation

\[
u \left(1 - \frac{m}{1 - \beta \omega(u + \alpha, u)}\right) - \eta = 0
\]

has at least one positive zero, let \( R \) be the minimum positive one.

If \( \beta \omega(R + \alpha, R) < 1, M = m/(1 - \beta \omega(R + \alpha, R)) < 1, \) and \( B(x_0, R) \subseteq \Omega \), then the sequence \( \{x_n\} \) given by (3) is well defined, remains in \( B(x_0, R) \) and converges to a unique solution \( x^* \) of equation (1) in \( B(x_0, R) \).

To simplify the notation, we denote \([x_{n-1}, x_n; F] = L_n\). First, we prove, by mathematical induction, that the sequence given in (3) is well defined, namely iterative procedure (3) makes sense if, at each step, the operator \([x_{n-1}, x_n; F]\) is invertible and the point \( x_{n+1} \) lies in \( \Omega \). From the initial hypotheses, it follows that \( x_1 \) is well defined and \( \| x_1 - x_0 \| \leq \eta < R \). Therefore, \( x_1 \in B(x_0, R) \subseteq \Omega \).

Using (b) and assuming that \( \omega \) is nondecreasing, we obtain

\[
\| I - L_0^{-1}L_1 \| \leq \| L_0^{-1} \| \| L_0 - L_1 \| \leq \| L_0^{-1} \| \omega(\| x_{-1} - x_0 \|, \| x_0 - x_1 \|) \\
\leq \beta \omega(\alpha, \eta) \leq \beta \omega(R + \alpha, R) < 1,
\]

and, by the Banach lemma, \( L_1^{-1} \exists \) exists and

\[
\| L_1^{-1} \| \leq \frac{\beta}{1 - \beta \omega(R + \alpha, R)},
\]

and consequently, the iterate \( x_2 \) is well defined. Moreover, by (2) and (3), we get

\[
F(x_1) = F(x_0) - [x_0, x_1; F](x_0 - x_1) = (L_0 - L_1)(x_0 - x_1).
\]

Then, by (b), we have

\[
\| F(x_1) \| \leq \| L_1 - L_0 \| \| x_1 - x_0 \| \leq \omega(\| x_0 - x_{-1} \|, \| x_1 - x_0 \|) \| x_1 - x_0 \| \leq \omega(\alpha, \eta) \| x_1 - x_0 \|.
\]
So, we obtain
\[ \|x_2 - x_1\| \leq \|L_1^{-1}\| \|F(x_1)\| \leq \frac{m}{1 - \beta \omega(R + \alpha, R)} \|x_1 - x_0\| = M\|x_1 - x_0\| < \eta. \]

On the other hand, if we take into account that \( R \) is a solution of (4), then
\[ \|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq (M + 1)\|x_1 - x_0\| \leq (M + 1)\eta < R, \]
and \( x_2 \in B(x_0, R) \subseteq \Omega. \)

Then, by induction, the following items are shown for \( j \geq 1: \)

(I\(_j\)) \( \exists L_j^{-1} = [x_{j-1}, x_j; F]^{-1} \) such that \( \|L_j^{-1}\| \leq \frac{\beta}{1 - \beta \omega(R + \alpha, R)}; \)

(II\(_j\)) \( \|x_{j+1} - x_j\| \leq M\|x_j - x_{j-1}\| \leq M^j\|x_1 - x_0\| < \eta. \)

Assuming that the linear operators \( L_j \) are invertible and \( x_{j+1} \in B(x_0, R) \subseteq \Omega \) for all \( j = 1, \ldots, n - 1, \) we obtain
\[ \|I - L_0^{-1}L_n\| \leq \|L_0^{-1}\| \|L_0 - L_n\| \leq \beta \omega(\|x_n - x_1\|, \|x_n - x_0\|) \]
\[ \leq \beta \omega(\|x_n - x_0\| + \|x_0 - x_{n-1}\|, \|x_n - x_{n-1}\|) \leq \beta \omega(R + \alpha, R) < 1, \]
and therefore,
\[ \|L_n^{-1}\| \leq \frac{\beta}{1 - \beta \omega(R + \alpha, R)}. \]

From the definition of the first divided difference and the Secant method, we can obtain
\[ F(x_n) = F(x_{n-1}) - [x_{n-1}, x_n; F](x_{n-1} - x_n) = (L_{n-1} - L_n)(x_{n-1} - x_n). \]

Taking norms in the above equality and (b), we obtain
\[ \|F(x_n)\| \leq \|L_n - L_{n-1}\| \|x_n - x_{n-1}\| \leq \omega(\|x_{n-1} - x_{n-2}\|, \|x_n - x_{n-1}\|)\|x_n - x_{n-1}\| \]
\[ \leq \omega(\eta, \eta)\|x_n - x_{n-1}\|. \]

Thus,
\[ \|x_{n+1} - x_n\| \leq \|L_n^{-1}\| \|F(x_n)\| \leq \frac{m}{1 - \beta \omega(R + \alpha, R)} \|x_n - x_{n-1}\| \]
\[ = M\|x_n - x_{n-1}\| \leq M^n\|x_1 - x_0\| < \eta. \]

Consequently, from (4) and (II\(_j\)), it follows:
\[ \|x_{n+1} - x_0\| \leq \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| + \cdots + \|x_2 - x_1\| + \|x_1 - x_0\| \]
\[ \leq \left[ M^n + M^{n-1} + \cdots + 1 \right] \|x_1 - x_0\| \leq \left[ \frac{1 - M^{n+1}}{1 - M} \right] \|x_1 - x_0\| < \frac{1}{1 - M} \eta = R. \]

So, \( x_{n+1} \in B(x_0, R) \subseteq \Omega \) and the induction is complete.

Second, we prove that \( \{x_n\} \) is a Cauchy sequence. For \( k \geq 1, \) we obtain
\[ \|x_{n+k} - x_n\| \leq \|x_{n+k} - x_{n+k-1}\| + \|x_{n+k-1} - x_{n+k-2}\| + \cdots + \|x_{n+1} - x_n\| \]
\[ \leq \left[ M^{k-1} + M^{k-2} + \cdots + 1 \right] \|x_{n+1} - x_n\| \]
\[ \leq \frac{1 - M^k}{1 - M} \|x_{n+1} - x_n\| < \frac{1}{1 - M} M^n\|x_1 - x_0\|. \]

Therefore, \( \{x_n\} \) is a Cauchy sequence and converges to \( x^* \in B(x_0, R). \)
Finally, we see that $x^*$ is a zero of $F$. Since

$$
\|F(x_n)\| \leq \omega(\eta, \eta)\|x_n - x_{n-1}\|
$$

and $\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$, we obtain $F(x^*) = 0$.

To show the uniqueness, we assume that there exists a second solution $y^* \in B(x_0, R)$ and consider the operator $A = [y^*, x^*; F]$. Since $A(y^* - x^*) = F(y^*) - F(x^*)$, if the operator $A$ is invertible then $x^* = y^*$. Indeed,

$$
\|L_0^{-1}A - I\| \leq \|L_0^{-1}\| \|A - L_0\| \leq \|L_0^{-1}\| \|y^*, x^*; F\| - \|y^*, x^*; F\| - [x_{-1}, x_0; F]\|
$$

$$
\leq \beta \omega (\|y^* - x_{-1}\|, \|x^* - x_0\|) \leq \beta \omega (\|y^* - x_0\| + \|x_0 - x_{-1}\|, \|x^* - x_0\|)
$$

$$
\leq \beta \omega (R + \alpha, R) < 1
$$

and the operator $A^{-1}$ exists.

### 3. NUMERICAL EXAMPLE

Now we apply the semilocal convergence result given above to the following system:

$$
x^2 - y + 1 + \frac{1}{9} |x - 1| = 0,
$$

$$
y^2 + x - 7 + \frac{1}{9} |y| = 0.
$$

(5)

We, therefore, have an operator $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F = (F_1, F_2)$. For $x = (x_1, x_2) \in \mathbb{R}^2$, we take $F_1(x_1, x_2) = x_1^2 - x_2 + 1 + (1/9) |x_1 - 1|$, $F_2(x_1, x_2) = x_2^2 + x_1 - 7 + (1/9) |x_2|$.

Let $x = (x_1, x_2) \in \mathbb{R}^2$ and $\|x\| = \|x\|_{\infty} = \max_{1 \leq i \leq 2} |x_i|$. The corresponding norm on $A \in \mathbb{R}^2 \times \mathbb{R}^2$ is

$$
\|A\| = \max_{1 \leq i \leq 2} \sum_{j=1}^{2} |a_{ij}|.
$$

For $v, w \in \mathbb{R}^2$, we take $[v, w; F] \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ as

$$
[v, w; F]_{i1} = \frac{F_1(v_1, w_2) - F_1(w_1, w_2)}{v_1 - w_1}, \quad [v, w; F]_{i2} = \frac{F_1(v_1, v_2) - F_1(v_1, w_2)}{v_2 - w_2}, \quad i = 1, 2.
$$

Therefore,

$$
[v, w; F] = \begin{pmatrix}
\frac{v_2^2 - w_2^2}{v_1 - w_1} -1 & \frac{v_1 - 1 - |w_1 - 1|}{v_1 - w_1} & 0 \\
v_1 - w_1 & \frac{|v_1 - 1| - |w_1 - 1|}{v_1 - w_1} & 0 \\
1 & 0 & \frac{|v_2 - |w_2|}{v_2 - w_2}
\end{pmatrix}
$$

and

$$
\|x, y; F\| - [v, w; F]\| \leq \|x - v\| + \|y - w\| + \frac{2}{9}.
$$

From (b), we consider

$$
\omega(u_1, u_2) = u_1 + u_2 + \frac{2}{9}.
$$

Now, we apply the Secant method to approximate the solution of $F(x) = 0$.

We choose $x_{-1} = (0.9, 1.1)$ and $x_0 = (1, 1)$. Using iteration (3), after three iterations we obtain

$$
x_2 = (1.06867, 2.18207) \quad \text{and} \quad x_3 = (1.14038, 2.34476).
$$

Then, we take $x_{-1} = x_2$ and $x_0 = x_3$. With the notation of Theorem 2.1, we can easily obtain the following:

$$
\alpha = 0.162691, \quad \beta = 0.479385, \quad \eta = 0.0199155, \quad m = 0.194069.
$$

In this case, the solution of equation (4) is $R = 0.0263993$. Besides, $\beta \omega (R + \alpha, R) = 0.209832 < 1$ and $M = 0.0245605 < 1$. Therefore, the hypotheses of Theorem 2.1 are fulfilled, what ensures that a unique solution of equation $F(x) = 0$ exists in $\overline{B(x_0, R)}$.

We obtain the vector $x^*$ as the solution of system (5), after nine iterations

$$
x^* = (1.15936, 2.36182).
$$
REFERENCES