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Shellability of Simplicial Complexes Arising in Representation Theory

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DEDICATED TO H. LENZING ON THE OCCASION OF HIS 60TH BIRTHDAY

Let A be a finite dimensional, connected, associative algebra withunit over an algebraically closed field k. All modules we consider are finitely generated, and mod A will denote the category of (finitely generated)

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over A.

A tilting module T is defined by the following three properties:

(i) the projective dimension pd T of T is finite,

(ii) $\mathrm{Ext}^i_A(T, T) = 0$ for all $i > 0$, and

(iii) there is an exact sequence

$$0 \to {}_AA \to T^1 \to T^2 \to \cdots \to T^{t-1} \to T^t \to 0$$

with T^i in the additive closure add T of T for all $1 \leqslant i \leqslant t$.

Tilting modules play an important role in the representation theory of finite dimensional algebras. Traditionally, one considers tilting modules of projective dimension at most one together with their endomorphism rings. In this case, the category of modules over the endomorphism algebra of a tilting module T over A shares a lot of properties with mod A [BrB, HR]. Moreover, more recent results of Auslander and Reiten [AR] prove that tilting modules are in one–one correspondence to important subcategories of mod A.

Tilting modules are characterized by a singular, i.e., an exceptional behaviour. They occur with a quite small density in the module category of a finite dimensional algebra, and as in other fields of mathematics, one expects that the study of these exceptional objects gives information about the global structure of the object under investigation, here the category mod A. Moreover, it is due to their discrete appearance that the set of tilting modules over A has the tendency to be open to a combinatorial treatment.

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It was observed by C. M. Ringel that the set of tilting modules over A forms a simplicial complex Σ, and in a problems session during a ring theory conference in Antwerp 1987, he proposed to study this complex.

The vertices, i.e., the zero-simplices of Σ, are the isomorphism classes of indecomposable direct sumands of tilting modules, and $\{T_0, ..., T_r\}$ is an r-simplex if $\bigoplus_{i=0}^{r} T_i \oplus Z$ is a tilting module for some Z in mod A.

We will also be interested in the simplicial subcomplex $\Sigma^{\leqslant 1}$ of Σ. Its vertices are the indecomposable direct summands of tilting modules with projective dimension at most one. Note that the chambers of $\Sigma^{\leqslant 1}$ correspond to those tilting modules which were traditionally considered in representation theory.

A tilting module has, up to isomorphism, $n+1$ indecomposable direct summands, where $n+1$ is the number of isomorphism classes of simple A-modules [CHU]. Hence, Σ and $\Sigma^{\leqslant 1}$ are n-dimensional simplicial complexes. Note that a simplicial complex is said to be n-dimensional if it contains an n-simplex but no $(n+1)$-simplex. Moreover, by definition Σ and $\Sigma^{\leqslant 1}$ are of pure dimension n, i.e., every r-simplex is the face of an n-simplex.

In general, Σ and $\Sigma^{\leqslant 1}$ are rather complicated. Usually, they are not finite, not even locally finite, and they are sometimes not connected. Examples may be found in [U2], U3].

In [RiS2] Riedtmann and Schofield proved that the geometric realisation $\|\Sigma^{\leqslant 1}\|$ of $\Sigma^{\leqslant 1}$ is an n-ball provided $\Sigma^{\leqslant 1}$ is finite. Implicitly they used in their proof that $\Sigma^{\leqslant 1}$ is shellable. A simplicial complex Σ of pure dimension n is called shellable, it its n-simplices may be given a linear order $\sigma_1, \sigma_2, \sigma_3, ...$, called a shelling, such that for all $l > 1$, the simplicial subcomplex $(\bigcup_{i=1}^{l-1} \sigma_i) \cap \sigma_l$ is of pure dimension $n-1$. It was proved in [H, RiS1] that $\Sigma^{\leqslant 1}$ is a pseudomanifold, i.e., all $(n-1)$-simplices are the face of at most two n-simplices, and it is easy to see that $\Sigma^{\leqslant 1}$ has a boundary. Recall that the boundary of a purely n-dimensional simplicial complex consists of those $(n-1)$-simplices which are the face of precisely one n-simplex. Under the hypothesis that $\Sigma^{\leqslant 1}$ is finite, shellability together with the fact that $\Sigma^{\leqslant 1}$ is a pseudomanifold with boundary implies that $\|\Sigma^{\leqslant 1}\|$ is an n-ball if it is finite [DK].

In [U1] we proved that also Σ is shellable provided it is finite.

It is one of the aims of this article to give a uniform approach to these results and to prove moreover that, under certain finiteness conditions, also the boundary complexes $\delta\Sigma^{\leqslant 1}$ and $\delta\Sigma$ of $\Sigma^{\leqslant 1}$ respectively Σ are shellable.

In Section 1 we give a purely combinatorial criterion, based on the combinatorics in [RiS2] which implies shellability of a simplicial complex.

The second section is preliminary. We provide some of the representation theoretic background which is needed frequently later.

In the third section we apply the combinatorial criterion to Σ and $\delta\Sigma$ in case Σ is finite.

An important invariant of a shellable simplicial complex is its characteristic. For the definition, which is rather technical, we refer to Section 3 below. A result of Björner [Bj] states that the geometric realisation of a shellable simplicial complex of characteristic h has the homotopy type of the wedge of h n-spheres. In Section 3 we also prove that if Σ is finite, then the characteristic of Σ is zero, whereas the characteristic of $\delta\Sigma$ may be arbitrarily large.

In order to prove shellability of $\delta\Sigma^{\leqslant 1}$ for $\delta\Sigma^{\leqslant 1}$ finite (note that this does not imply that $\Sigma^{\leqslant 1}$ is finite) and to determine its characteristic we have to analyze carefully the structure of tilting modules of projective dimension at most one. This will be done in Section 4. These investigations are closely related to homological questions of subcategories of mod A. Our main result in Section 4 will imply a necessary and sufficient criterion when the subcategory of modules of projective dimension at most one is functorially finite. This answeres a question raised by Auslander and Reiten in [AR].

In Section 5 we prove that $\delta\Sigma^{\leqslant 1}$ is a shellable pseudomanifold if it is finite. Then its geometric realization is a ball or a sphere, and we give examples that both cases occur. In particular, if $\delta\Sigma^{\leqslant 1}$ is finite, then it is shellable of characteristic zero or one.

We saw above that shellability of a simplicial complex has nice topological consequences. The same holds for algebraic combinatorics. Namely, the face ring (for the definition see Section 6) of a shellable simplicial complex is a Cohen–Macaulay ring [Re], sometimes a Gorenstein ring. We briefly recall these concepts in Section 6 and mention some consequences of shellability for the face rings of the simplicial complex of tilting modules respectively the corresponding boundary complexes.

1. A COMBINATORIAL CRITERION IMPLYING SHELLABILITY

In this section we assume that Σ is a countable simplicial complex of pure dimension n.

Let τ be a simplex in Σ, possibly $\tau = \varnothing$. We denote by $\mathscr{C}(\tau)$ the set of chambers in Σ containing τ.

Assume that for all τ in Σ we have a partial order $\overset{\leqslant}{\tau}$ satisfying the following conditions:

(PO1) The chambers which are neighbors in the Hasse diagram of $\overset{\leqslant}{\tau}$ have a common $(n-1)$-simplex.

(PO2) Two minimal elements have a common $(n-1)$-simplex.

(PO3) Every $\sigma \in \mathscr{C}(\tau)$ has a minimal element $\mu \in \mathscr{C}(\tau)$ with $\mu \overset{\leqslant}{\tau} \sigma$.

Assume moreover that the partial orders are compatible with respect to $\overset{\leqslant}{\varnothing}$, that is,

(PO4) $\sigma \overset{\leqslant}{\tau} \sigma'$ implies that $\sigma \overset{\leqslant}{\varnothing} \sigma'$.

PROPOSITION. *Assume that for all simplices $\tau \in \Sigma$ the sets $\mathscr{C}(\tau)$ are partially ordered such that* (PO1) *up to* (PO4) *hold. Then Σ is shellable.*

Proof. We take a linear extension of $\overset{\leqslant}{\varnothing}$, i.e., we order the chambers $\sigma_1, \sigma_2, \sigma_3, ...$ such that $\sigma_i \overset{\leqslant}{\varnothing} \sigma_j$ implies that $i \leqslant j$.

Let $l > 1$ and consider $\Sigma_{l-1} \cap \sigma_l$, where $\Sigma_{l-1} = \bigcup_{i=1}^{l-1} \sigma_i$.

We claim that this intersection is not empty. If σ_l is not minimal with respect to $\overset{\leqslant}{\varnothing}$ then this follows from (PO1). If σ_l is minimal, then, since σ_1 is also minimal with respect to $\overset{\leqslant}{\varnothing}$ we get from (PO2) that $\sigma_1 \cap \sigma_l \neq \varnothing$.

Let $\varnothing \neq \tau$ be a simplex in $\Sigma_{l-1} \cap \sigma_l$. We have to prove that τ is contained in an $(n-1)$-simplex in $\Sigma_{l-1} \cap \sigma_l$. Let $\tau \subset \sigma_s$ for some $s < l$.

If σ_l is minimal with respect to $\overset{\leqslant}{\tau}$ then σ_l and σ_s are not comparable. Otherwise it would follow from (PO4) that $\sigma_l \overset{\leqslant}{\varnothing} \sigma_s$, and the choice of the linear order gives $l \leqslant s$, a contradiction. According to (PO3) there is a chamber $\mu \in \mathscr{C}(\tau)$ with $\mu \overset{\leqslant}{\tau} \sigma_s$. Again using (PO4) we get that $\mu \overset{\leqslant}{\varnothing} \sigma_s$ and conclude that $\mu \in \Sigma_{l-1}$. Now (PO2) states that $\mu \cap \sigma_l$ is $(n-1)$ dimensional and it contains τ.

If σ_l is not minimal with respect to $\overset{\leqslant}{\tau}$ then (PO3) yields a non-trivial path

$$\mu - \cdots - \sigma_m - \sigma_l$$

in the Hasse diagram of $\overset{\leqslant}{\tau}$, and $\sigma_m \overset{\leqslant}{\varnothing} \sigma_l$, which again implies that σ_m lies in Σ_{l-1}. Then $\sigma_m \cap \sigma_l$ is $(n-1)$-dimensional and it contains τ.

2. BASIC RESULTS FROM REPRESENTATION THEORY

We keep the notions of the Introduction and refer for unexplained terminology to [R]. From now on, Σ will always denote the simplicial complex of tilting modules.

2.1. An A-module M is called multiplicity free if in a decomposition $M = \bigoplus_{i=1}^{r} M_i$ of M into indecomposable direct sumands we have that M_i and M_j are not isomorphic for $i \neq j$. Obviously, the r-simplices of Σ correspond bijectively to the isomorphism classes of multiplicity free direct summands of tilting modules with $r+1$ indecomposable direct summands. This allows us to identify simplices with modules.

Let $M = \bigoplus_{i=1}^{r} M_i$ be a multiplicity free module. We denote the module $\bigoplus_{j \neq i} M_j$ by $M[i]$.

A module M in Σ is said to be a partial module. Note that this is a deviation of the common use of this terminology. We also call the module 0 a partial tilting module and identify it with the empty simplex. By definition, the dimension of the empty simplex is -1. Partial tilting modules with n pairwise distinct indecomposable direct summands are called almost complete tilting modules. Let M be a partial tilting module. A multiplicity free module C with the properties that $M \oplus C$ is a tilting module and $\operatorname{add} M \cap \operatorname{add} C = 0$ is said to be a complement to M.

2.2. In [HU2] the notion of a Bongartz complement to a partial tilting module was introduced. Since such a complement will play an essential role in the following sections we recall its definition and basic properties from [HU2].

Following [AR] we denote for an A-module M the full subcategory of A-modules X with $\operatorname{Ext}_A^i(M, X) = 0$ for $i > 0$ by $M^{\perp}$.

Let M be a partial tilting module and C a complement to M. We call C a Bongartz complement to M provided $(M \oplus C)^{\perp} = M^{\perp}$. Obviously, ${}_AA$ is a Bongartz complement to 0. It is easy to see that if M admits a Bongartz complement C, then C is uniquely determined up to isomorphism.

2.3. A subcategory $\mathscr{X}$ of $\operatorname{mod} A$ is called contravariantly finite [AS], if every $C \in \operatorname{mod} A$ has a right $\mathscr{X}$-approximation, i.e., a morphism $F_C \to C$ with $F_C \in \mathscr{X}$ such that the induced morphism $\operatorname{Hom}_A(X, F_C) \to \operatorname{Hom}_A(X, C)$ is surjective for all $X \in \mathscr{X}$. There is an obvious dual notion of left $\mathscr{X}$-approximations, and $\mathscr{X}$ is called covariantly finite if every C in $\operatorname{mod} A$ has a left $\mathscr{X}$-approximation. If $\mathscr{X}$ is both contravariantly and covariantly finite, then $\mathscr{X}$ is said to be functorially finite.

It follows from the results in [AR] that a partial tilting module admits a Bongartz complement if and only if $M^{\perp}$ is covariantly finite.

In [HU2] we gave an example of an algebra A and a partial tilting module M such that $M^{\perp}$ is not covariantly finite, hence Bongartz complements do not always exist.

Let M be a tilting module of projective dimension at most one. Auslander and Reiten proved in [AR] that then $M^{\perp}$ is always covariantly finite. Hence a partial tilting module of projective dimension at most one admits a Bongartz complement.

2.4. Proposition [HU2]. *The following are equivalent for a partial tilting module M with complement C.*

(i) *$C = \bigoplus_{i=1}^{r} C_i$ is the Bongartz complement to M.*

(ii) *None of the modules C_i is generated by $M \oplus C[i]$.*

COROLLARY [HU2]. *If M is an almost complete tilting module, then M admits a Bongartz complement.*

The proposition above is the motivation for the notion "Bongartz complement." There is a well known construction due to Bongartz [Bo] to produce a complement C with pd $C \leqslant 1$ to a partial tilting module M of projective dimension at most one: consider an exact sequence

$$0 \to A \to E \to M^t \to 0$$

such that the connecting homomorphism $\mathrm{Hom}_A(M, M^t) \to \mathrm{Ext}_A^1(M, A)$ is surjective. Then it is easy to check that $E \oplus M$ is a tilting module and pd $E \leqslant 1$. Note that this construction does not produce complements to partial tilting modules of higher projective dimension. When investigating tilting modules of projective dimension at most one in [RiS2], Riedtmann and Schofield called a multiplicity free direct summand C of E such that $M \oplus C$ is a tilting module and add $M \cap$ add $C = 0$ a Bongartz completion to M. They proved an analogous statement to Proposition 2.3 above. In particular we see, that the Bongartz complement to a partial tilting module of projective dimension at most one (compare Subsection 2.2) has always projective dimension at most one. Of course one may also verify our definition of a Bongartz complement directly. In fact, let $X \in M^{\perp}$ and apply $\mathrm{Hom}_A(-, X)$ to

$$0 \to A \to E \to M^t \to 0.$$

This shows that $\mathrm{Ext}_A^1(E, X) = 0$, hence $X \in (M \oplus C)^{\perp}$.

3. SHELLABILITY OF Σ AND $\delta\Sigma$

Let Σ be a shellable simplicial complex. For a given sheling and for $l \geqslant 1$ we denote the union $\bigcup_{i=1}^{l} \sigma_i$ by Σ_l. For $l > 1$ one further defines the restriction $\mathscr{R}(\sigma_l)$ of a chamber σ_l as the set of vertices v of σ_l such that $\sigma_l \backslash \{v\}$ lies in Σ_{l-1}.

Σ is said to be shellable of characteristic h if h is the cardinality of the chambers σ in Σ satisfying that $\mathscr{R}(\sigma) = \sigma$. Note that it follows from [Bj] that the characteristic does not depend on the choice of a shelling.

3.1. Following [RiS2] we associate with Σ an oriented graph $\vec{K}_0$ as follows: The vertices are the elements of $\mathscr{C}(\varnothing)$, i.e., the chambers of Σ, and there is an arrow $T \to T'$ in $\vec{K}_0$ if T and T' have a common $(n-1)$-simplex and if $(T')^{\perp}$ is properly contained in $T^{\perp}$.

The definition immediately implies that $\vec{K}_0$ has no oriented cycle. Hence, as in [RiS2] we obtain a partial order $\overset{\leqslant}{\varnothing}$ on $\mathscr{C}(\varnothing)$ given by the rule that $\sigma \overset{\leqslant}{\varnothing} \sigma'$ if there is a path from σ to σ' in $\vec{K}_0$.

Let M be a partial tilting module which we identify with the simplex τ. By $\vec{K}_M$ we denote the full subgraph of $\vec{K}_0$ whose vertices are the chambers of Σ containing τ. As a subgraph of $\vec{K}_0$, the quiver $\vec{K}_M$ has no oriented cycles, and as above, we obtain a partial order $\overset{\leqslant}{\tau}$ on $\mathscr{C}(\tau)$. Obviously, $\sigma \overset{\leqslant}{\tau} \sigma'$ implies $\sigma \overset{\leqslant}{\varnothing} \sigma'$, hence condition (PO4) holds. Clearly, two neighbors σ and σ' in the Hasse diagram of $\overset{\leqslant}{\tau}$ have a common $(n-1)$-simplex, thus also (PO1) is satisfied for all $\tau \in \Sigma$.

In order to verify the remaining conditions of the partial order we need the following result from [U1]:

LEMMA. *$M \oplus C$ is a source in $\vec{K}_M$ if and only if C is the Bongartz complement to M. In particular, $\vec{K}_M$ has at most one source.*

This lemma implies that the condition (PO2) is empty for the n-simplices of Σ.

By setting $M=0$ in the lemma above we see that ${}_A A$ is the unique source in $\vec{K}_0$. Assume now that Σ is finite. Then $\vec{K}_M$ is connected for all partial tilting modules M (including $M=0$) by the previous lemma. Hence also the Hasse diagrams of $\overset{\leqslant}{\tau}$ are connected for all $\tau \in \Sigma$. This implies the condition (PO3). Applying the criterion of the first section we obtain:

COROLLARY [U3]. *If Σ is finite, then it is shellable.*

3.2. In order to determine the characteristic h of a shelling of Σ we need the following

LEMMA. *Let T be a multiplicity free tilting module. We decompose T into $T^1 \oplus T^2$, where $T^1 = \bigoplus_{i=0}^{r} T_i$ and $T^2 = \bigoplus_{i=r+1}^{n} T_i$ such that every indecomposable direct summand T_i of T^1 is generated by $T[i] = \bigoplus_{j \neq i} T_j$. Then T_i is generated by T^2.*

Proof. The proof is an induction on r.

If $r=0$ there is nothing to show.

Let $r>0$. Since T_r is generated by $T[r]$ we have a non-split surjection

$$T_0^{\alpha_0} \oplus \cdots T_{r-1}^{\alpha_{r-1}} \oplus E \to T_r,$$

where the α_i are non-negative integers and $E \in \operatorname{add} T^2$. By indunction hypothesis, all T_i for $0 \leqslant i \leqslant r-1$ are generated by $T^2 \oplus T_r$. Thus there is a non-split surjection $T_r^{\alpha_r} \oplus E' \to T_r$ with $E' \in \operatorname{add} T^2$. It follows from [AS] that T_r is generated by T^2, the assertion.

PROPOSITION. *Let Σ be finite. The characteristic of a shelling of Σ is zero.*

Proof. As in Section 1 we take as a shelling a linear extension of $\overset{\leqslant}{\emptyset}$. Assume that for $l > 1$ there is an n-simplex σ_l, such that the restriction

$$\mathscr{R}(\sigma_l) = \{v \in \sigma_l \mid \sigma_l \backslash \{v\} \subset \Sigma_{l-1}\}$$

is σ_l. Let $T = \bigoplus_{i=0}^{n} T_i$ be the tilting module associated with σ_l.

Assume that for some $0 \leqslant i \leqslant n$ the module T_i is the Bongartz complement to $T[i]$. Since by assumption, $T[i] = \tau$ is contained in a chamber σ_t, for some $t < l$, and since $\vec{K}_{T[i]}$ is finite and connected there is a non-trivial path from σ_l to σ_t in $\vec{K}_{T[i]}$. This is a contradiction to the choice of the linear order of the chambers of Σ.

Hence none of the T_i is Bongartz complement to the $T[i]$, and it follows from [HU2] that all T_i are generated by $T[i]$. In the terminology of the lemma above, we have $T^2 = 0$, thus T_i is generated by 0, a contradiction.

3.3. It is the aim of this subsection to investigate the boundary of Σ.

LEMMA. *The Bongartz complement to a faithful almost complete tilting module M is cogenerated by M.*

Proof. Let X be the Bongartz complement to M, hence $M^{\perp} = (M \oplus X)^{\perp}$. Since M is faithful, the injective cogenerator $\mathrm{D}(A_A)$ of mod A is generated by M. Here D denotes the standard duality $\mathrm{Hom}_k(-, k)$. Choose an exact sequence

$$\eta : 0 \to K \to E \to \mathrm{D}(A_A) \to 0$$

with $E \in \mathrm{add}\, M$ such that the induced morphism $\mathrm{Hom}_A(M, E) \to \mathrm{Hom}_A(M, \mathrm{D}(A_A))$ is surjective. Then $K \in M^{\perp}$, hence $\mathrm{Ext}^i_A(X, K) = 0$ for all $i > 0$. For some natural number r there is an injective map $f: X \to \mathrm{D}(A_A)^r$. Forming the direct sum of r copies of η and applying $\mathrm{Hom}_A(X, -)$ to

$$0 \longrightarrow K^r \longrightarrow E^r \xrightarrow{\pi} \mathrm{D}(A_A)^r \longrightarrow 0$$

we see that there exists a morphism $h : X \to E^r$ with $f = h\pi$. Obviously, h is injective, hence X is cogenerated by M, the assertion.

COROLLARY. *A faithful almost complete tilting module M has at least two non-isomorphic complements.*

Proof. According to [HU2], the module M admits a Bongartz complement X. The lemma above states that X is cogenerated by M, and it follows from [CHU] that there is a non-split short exact sequence $0 \to X \to E \to Y \to 0$ with $E \in \operatorname{add} M$ and Y a second complement to M.

PROPOSITION. *$\delta\Sigma$ is the set of those partial tilting modules which are direct summands of non-faithful almost complete tilting modules.*

Proof. The inclusion $\subseteq$ is immediate from the corollary above. For the other inclusion assume that M is an almost complete tilting module with non-isomorphic complements X and Y. Since the Bongarz complement to M is unique, we may assume that Y is not the Bongartz complement to M. Thus Y is generated by M according to [HU2]. Since $M \oplus Y$ is a tilting module, it is faithful. Let $g : {}_AA \to F$ be injective with $F \in \operatorname{add} M \oplus Y$. Since Y is generated by M, there is a surjective map $h : E \to F$ with $E \in \operatorname{add} M$. Then there is an injective map $f : {}_AA \to E$ with $fh = g$, in particular, M is faithful. Hence, a non-faithful almost complete tilting module lies in $\delta\Sigma$.

This proposition has been known in several special cases. It was proved for hereditary algebras in [HU1] (partially in [RiS1]), for tilting modules of projective dimension at most one in [H], and for tilting modules of higher (but finite) projective dimension under the additional assumption that the set $\{X \in \operatorname{mod} A \mid \operatorname{Ext}^i_A(M, X) = 0 \text{ for all } i > 0 \text{ and } \operatorname{pd} M < \infty\}$ is contravariantly finite in [CHU].

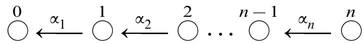
Let $A = k\vec{\Delta}/I$, where $\vec{\Delta}$ is a finite quiver with vertex set $\{0, ..., n\}$ and I an admissable ideal. We denote the (isomorphism classes) of simple A-modules by $S_0, ..., S_n$, the indecomposable projective A-modules with top S_i by P_i and the indecomposable injective A-modules I with $I/\operatorname{soc} I = S_i$ by I_i. Here $\operatorname{soc} I$ denotes the socle of I. Let P_i be an indecomposable projective A-module corresponding to the starting vertex of a path of maximal length in A. It is easy to see that the module $P[i] = \bigoplus_{j \neq i} P_j$ is a non-faithful almost complete tilting module. In particular, Σ is a simplicial complex with boundary. Since $P[i]$ has projective dimension zero, also $\delta\Sigma^{\leqslant 1}$ is not empty.

3.4. Throughout this section we assume that A has at least two simple modules. We will prove that $\delta\Sigma$ is shellable provided Σ is finite.

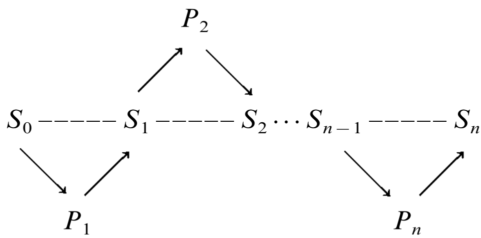
To prove shellability we need to extend the methods used above. We associate with $\delta\Sigma$ an oriented graph $\vec{K}(\delta)_0$ with vertices the chambers of $\delta\Sigma$. There is an arrow $M \to M'$ in $\vec{K}(\delta)_0$ if M and M' have a common $(n-2)$-simplex and if $(M')^\perp$ is propertly contained in $M^\perp$. As before, this induces a partial order $\underset{\varnothing}{\leqslant}$ on $\mathscr{C}(\varnothing)$, the set of chambers in $\delta\Sigma$. Let τ be a simplex in $\delta\Sigma$ which we identify with the partial tilting module M.

Analogously to Subsection 3.1 we denote by $\vec{K}(\delta)_M$ the full subquiver of $\vec{K}(\delta)_0$ with vertices the chambers of $\delta\Sigma$ containing τ. As above, this induces a partial order $\overset{\leqslant}{\tau}$ on $\mathscr{C}(\tau)$. By definition, these partial orders satisfy (PO1) and obviously, also (PO4) holds.

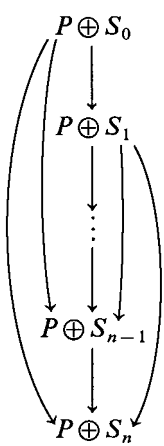
Contrary to $\vec{K}_0$, the quiver $\vec{K}(\delta)_0$ may decompose if it is finite. Consider the following example. Let A_{n+1} be the path algebra of the quiver



bound by the relations $\alpha_{i+1}\alpha_i = 0$ for all $1 \leqslant i < n$. The simplicial complexes Σ and $\delta\Sigma$ are finite. The Auslander–Reiten quiver of A_{n+1} has the form

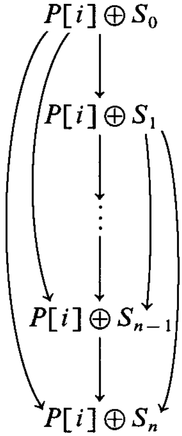


The unique faithful almost complete tilting module is $P = \bigoplus_{i=1}^{n} P_i$, and $\vec{K}_0$ is



with arrows from $P \oplus S_j$ to $P \oplus S_l$ for $0 \leqslant j \leqslant n-1$ and $j < l \leqslant n$.

It is easy to see that $\vec{K}(\delta)_0$ decomposes into n connected components $\vec{C}(\delta)_i$, $1 \leqslant i \leqslant n$, with $n+1$ vertices. A component $\vec{C}(\delta)_i$ has the form



with arrows from $P[i]\oplus S_j$ to $P[i]\oplus S_l$ for $0\leqslant j\leqslant n-1$ and $j<l\leqslant n$.

LEMMA 1. *Let M be an $(n-2)$-simplex in $\delta\Sigma$ and let $U\oplus V$ be the Bongartz complement to M. Then $M\oplus U$ or $M\oplus V$ lie in $\delta\Sigma$.*

Proof. Assume that $M\oplus U$ and $M\oplus V$ are faithful. According to Lemma 3.3, the modules U and V are cogenerated by $M\oplus V$, respectively $M\oplus U$. Thus we obtain two non-split short exact sequences

$$\eta_1 : 0 \longrightarrow U \xrightarrow{\mu_1} E_1\oplus V^r \longrightarrow U' \longrightarrow 0$$

and

$$\eta_2 : 0 \longrightarrow V \xrightarrow{\mu_2} E_2\oplus U^s \longrightarrow V' \longrightarrow 0$$

with $E_1, E_2\in \operatorname{add} M$.

If $r=0$ or $s=0$, then it is easy to see that M is faithful, a contradiction to the assumption that $M\in\delta\Sigma$. Hence $r\neq 0\neq s$. Consider

$$0 \longrightarrow U \xrightarrow{\mu_1} E_1\oplus V^r \xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 0 & \mu_2^r\end{smallmatrix}\right)} E_1\oplus(E_2\oplus U^s)^r = E_3\oplus U^t$$

with $E_3\in\operatorname{add} M$.

Set

$$\mu_2' = \begin{pmatrix}1 & 0\\ 0 & \mu_2^r\end{pmatrix}$$

and $\mu=\mu_1\mu_2'$. Obviously, μ is a monomorphism. If μ does not split, then μ is cogenerated by M, which implies that M is faithful. Hence μ splits and we get $\pi : E_3 \oplus U^t \to U$ such that $\mu\pi = 1_U = \mu_1(\mu_2'\pi)$. This implies that μ_1 is a section. Hence η_1 splits, a contradiction.

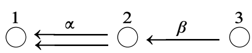
LEMMA 2. *Assume that all partial tilting modules over A admit a Bongartz complement. Let $M \oplus C$ be a chamber in $\delta\Sigma$. Then $M \oplus C$ is a source in $\vec{K}(\delta)_M$ if and only if C is a direct summand of the Bongartz complement to M.*

Proof. Let C be a direct summand of the Bongartz complement $\bar{C}$ of M. Since $(M \oplus \bar{C})^\perp = (M \oplus C)^\perp = M^\perp \supseteq (M \oplus Z)^\perp$ for all partial tilting modules Z such that $M \oplus Z \in \delta\Sigma$, there is no arrow pointing to $M \oplus C$.

For the converse assume that C is not a direct summand of the Bongartz complement to M. Let $M \oplus C \oplus X$ be a tilting module. Since $M \oplus C$ is not faithful, X is the unique complement to $M \oplus C$ by Proposition 3.3, hence the Bongartz complement to $M \oplus C$. In particular, X is not generated by $M \oplus C$. By assumption, $X \oplus C$ is not the Bongartz complement to M, hence we may assume without loss of generality that the indecomposable direct summand C_1 of C is generated by $M \oplus X \oplus C[1]$.

Let $U \oplus V$ be the Bongartz complement to $M \oplus C[1]$. This exists by assumption. According to the previous lemma we may assume that $M \oplus C[1] \oplus U$ is not faithful. Obviously, $(M \oplus C[1] \oplus U)^\perp = (M \oplus C[1])^\perp \supseteq (M \oplus C)^\perp$. We claim that the inclusion is proper. If it is not, then $U \oplus V \in (M \oplus C)^\perp$, implying that $\mathrm{Ext}^i_A(M \oplus C, U \oplus V) = 0 = \mathrm{Ext}^i_A(U \oplus V, M \oplus C)$. Thus $M \oplus C \oplus U \oplus V$ is a tilting module, hence $C_1 = U$ or $C_1 = V$. If $C_1 = U$ then $X = V$ for X is the unique complement to $M \oplus C$. Then U is generated by $M \oplus C[1] \oplus V$, contradicting the assumption that $U \oplus V$ is the Bongartz complement to $M \oplus C[1]$. Analogously, $C_1 = V$ leads to a contradiction. Hence there is an arrow from $M \oplus U \oplus C[1]$ to $M \oplus C$ in $\vec{K}(\delta)_M$.

Note that the assumption that all partial tilting modules admit a Bongartz complement is essential in the lemma. Consider the example in [HU2], where A is the path algebra of the quiver



bound by the relations $\beta\alpha = 0$. The simple injective module I_3 lies in $\delta\Sigma$ and $\vec{K}(\delta)_{I_3}$ is simply the vertex $I_2 \oplus I_3$. In particular, $I_2 \oplus I_3$ is a source in $\vec{K}(\delta)_{I_3}$, but we proved in [HU2] that I_3 does not admit a Bongartz complement.

Note moreover that the hypothesis of the lemma is satisfied provided Σ is finite. Namely then all partial tilting modules have Bongartz complements as was shown in [HU2]. It is actually due to the fact that we do not know whether $\delta\Sigma$ finite implies that all partial tilting modules have a

Bongartz complement that we cannot prove shellability of $\delta\Sigma$ in case it is finite. We need the stronger assumption that Σ is finite.

We assume until the end of this section that Σ is finite.

As in Subsection 3.1, the lemma above implies that a non-faithful almost complete tilting module $M = \bigoplus_{i=1}^{n} M_i$ is a source in $\vec{K}(\delta)_0$ if and only if all M_i are projective. Hence $\vec{K}(\delta)_0$ has at most $n+1$ sources. This gives an upper bound for the number of connected components of $\vec{K}(\delta)_0$, namely:

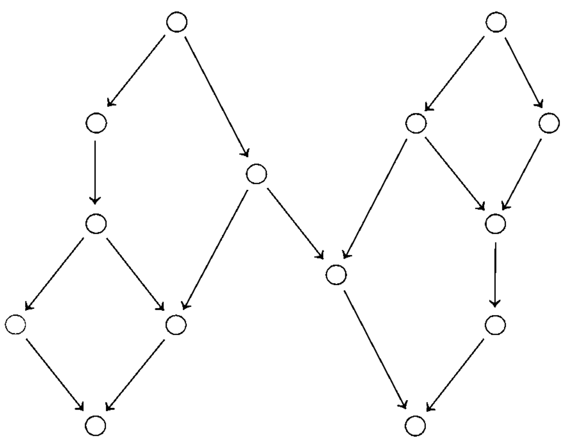
COROLLARY. *If Σ is finite, then $\vec{K}(\delta)_0$ has at most $n+1$ connected components.*

The following example shows that the number of sources in $\vec{K}(\delta)_0$ is not necessarily an optimal upper bound for the number of connected components of $\vec{K}(\delta)_0$.

For the path algebra of the quiver

$$\circ \longleftarrow \circ \longrightarrow \circ \longleftarrow \circ,$$

direct calculation shows that $\vec{K}(\delta)_0$ equals



We now finish our proof that $\delta\Sigma$ is shellable provided Σ is finite. Let $M = \tau$ be a simplex in $\delta\Sigma$. Since the minimal elements in $\mathscr{C}(\tau)$ with respect to the partial order $\overset{\leqslant}{\tau}$ in $\delta\Sigma$ are the sources in $\vec{K}(\delta)_M$, Lemma 2 above implies that $\overset{\leqslant}{\tau}$ satisfies (PO2).

Also, as in Subsection 3.1 it follows that for every chamber $M \oplus Z$ in $\delta\Sigma$ there is a path from $M \oplus C$ to $M \oplus Z$ in $\vec{K}(\delta)_M$, for C a direct summand of the Bongartz complement to M. In particular, all partial order $\overset{\leqslant}{\tau}$ with $\tau \in \delta\Sigma$ satisfy (PO3). With the criterion in Section 1 we conclude:

COROLLARY. *If Σ is finite, then $\delta\Sigma$ is shellable.*

3.5. We finish this section with a series of examples showing that the characteristics of shellings of $\delta\Sigma$ may be arbitrarily large.

For $n \geqslant 2$, let A_{n+1} be the algebras defined in Subsection 3.4.

PROPOSITION. *A shelling of the boundary of the simplicial complex of tilting modules over A_{n+1} has characteristic n.*

Proof. We keep the terminology introduced in Subsection 3.4.

According to the criterion in Section 1 we may choose as a shelling a linear extension of $\overset{\leqslant}{\varnothing}$. In particular, we may order the chambers of $\delta\Sigma$ such that $P[i] \oplus S_j$, the $(j+1)$st vertex of the string $\vec{C}(\delta)_i$, equals $\sigma_{(i-1)(n+1)+j+1}$ for $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n$.

Let $i < n$. The partial tilting module $\bigoplus_{l=1, l \neq i}^{n-1} P_l \oplus S_j$ which is contained in $P[i] \oplus S_j$ is neither contained in a proper predecessor of $P[i] \oplus S_j$ in $\vec{C}(\delta)_i$ nor in a chamber of $\vec{C}(\delta)_r$ for some $r < i$. Thus $\mathscr{R}(\sigma_{(i-1)(n+1)+j+1})$ is properly contained in $\sigma_{(i-1)(n+1)+j+1}$ for all $i < n$. Also the restriction of $P[n] \oplus S_0 = \sigma_{(n-1)(n+1)+1}$ is properly contained in $\sigma_{(n-1)(n+1)+1}$, namely $P[n]$ is not contained in any chamber in $\vec{C}(\delta)_i$ for all $i < n$. This implies that the characteristic of the shelling is at most n.

Now consider for $1 \leqslant j \leqslant n$ the chamber $P[n] \oplus S_j = \sigma_{(n-1)(n+1)+j+1}$. The partial tilting module $P[n]$ is contained in a proper predecessor of $P[n] \oplus S_j$ in $\vec{C}(\delta)_n$, and $\bigoplus_{l=1, l \neq i}^{n-1} P_l \oplus S_j$ is contained in the chamber $\sigma_{(i-1)(n+1)+j+1}$ in $\vec{C}(\delta)_i$. Hence $\mathscr{R}(\sigma_{(n-1)(n+1)+j+1}) = \sigma_{(n-1)(n+1)+j+1}$ for $1 \leqslant j \leqslant n$, implying the assertion.

4. TILTING MODULES OF PROJECTIVE DIMENSION AT MOST ONE

Let $\mathscr{P}^1(A)$ be the full subcategory of mod A of modules of projective dimension at most one.

Tilting modules of projective dimension at most one have been studied in detail in [HR], [Bo], [H], and in connection with $\Sigma^{\leqslant 1}$ in [RiS2].

We recall some result from [H, RiS2].

Let M be an almost complete tilting module of projective dimension at most one. It is proved in [H] that M admits at most two non-isomorphic complements X and Y in $\mathscr{P}^1(A)$, and that M has a unique complement of projective dimension at most one if and only if M is not faithful. Note that this characterizes the boundary of $\Sigma^{\leqslant 1}$. If M is faithful, then there is a short exact sequence η connecting the non-isomorphic complements X and Y, say

$$\eta : 0 \to X \to E \to Y \to 0,$$

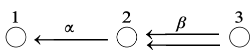
where $E \in \operatorname{add} M$ and $\operatorname{Ext}^1_A(X, Y) = 0$. Such a sequence is unique up to isomorphism of exact sequences, and it will be referred to as the sequence connecting the complements to M.

In this section we first introduce the notion of a Bongartz cocomplement to a partial tilting module of projective dimension at most one, which in some sense is dual to the Bongartz complement. The question of existence of such cocomplements leads to questions whether certain subcategories of mod A are contravariantly finite. This will be discussed in Subsection 4.2 below.

4.1. For an A-module M we denote by ${}^{\perp}M$ the full subcategory of A-modules Z satisfying $\operatorname{Ext}^i_A(Z, M) = 0$ for all $i > 0$.

Let $M \in \mathscr{P}^1(A)$ be a partial tilting module. We call C with $\operatorname{pd} C \leqslant 1$ a Bongartz cocomplement to M if C is a complement to M and if ${}^{\perp}M \cap \mathscr{P}^1(A) = {}^{\perp}(M \oplus C) \cap \mathscr{P}^1(A)$.

Dualizing the example in [HU2], i.e., considering the path algebra of the quiver



bound by the relations $\beta\alpha = 0$, we see that the simple projective A-module P_1 does not admit a Bongartz cocomplement.

Again it is easy to see that if M has a Bongartz cocomplement, then it is uniquely determined up to isomorphism.

LEMMA. *Let $T \in \mathscr{P}^1(A)$ be a tilting module. Then every $Z \in {}^{\perp}T \cap \mathscr{P}^1(A)$ is cogenerated by T.*

Proof. Let

$$0 \longrightarrow P^1 \xrightarrow{\mu} P^0 \xrightarrow{\pi} Z \longrightarrow 0$$

be a minimal projective resolution of Z. Since T is a tilting module, P^0 is cogenerated by T, and there is a short exact sequence

$$0 \longrightarrow P^0 \xrightarrow{\alpha} T^0 \xrightarrow{\beta} T^1 \longrightarrow 0$$

with $T^0, T^1 \in \operatorname{add} T$. This gives the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P^1 & \xrightarrow{\mu} & P^0 & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\
0 & \longrightarrow & P^1 & \xrightarrow{\mu\alpha} & T^0 & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \beta} & & \downarrow & & \\
 & & & & T^1 & = & T^1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

with Q the cokernel coker $\mu\alpha$ of $\mu\alpha$.

An easy calculation shows that $T \oplus Q$ is a tilting module of projective dimension at most one. This implies the assertion.

PROPOSITION. *The following are equivalent for a multiplicity free tilting module $M \oplus C \in \mathscr{P}^1(A)$.*

(i) *$C = \bigoplus_{i=1}^{r} C_i$ is the Bongartz cocomplement to M.*

(ii) *All C_i satisfy that either $M \oplus C_i$ is not faithful or else C_i is generated by $M \oplus C[i]$.*

Proof. Assume that there is an indecomposable direct summand C_i of C such that $M \oplus C[i]$ is faithful and C_i is not generated by $M \oplus C[i]$. Since $M \oplus C[i]$ is faithful, there are two non-isomorphic complements C_i and C_i' of projective dimension at most one to $M \oplus C[i]$. Further, since C_i is not generated by $M \oplus C[i]$, the sequence connecting the complements [H] has the form

$$0 \to C_i \to E_i \to C_i' \to 0,$$

where $E_i \in \operatorname{add} M \oplus C[i]$. Obviously, ${}^{\perp}(M \oplus C) \cap \mathscr{P}^1(A) \subseteq {}^{\perp}(M \oplus C[i] \oplus C_i') \cap \mathscr{P}^1(A) \subseteq {}^{\perp}M \cap \mathscr{P}^1(A)$, and since C_i' does not lie in ${}^{\perp}(M \oplus C) \cap \mathscr{P}^1(A)$, the first inclusion is proper. Hence, C is not the Bongartz cocomplement to M.

For the converse implication assume that C is not the Bongartz complement to M. For $X \in \mathscr{P}^1(A)$ we denote by $t(X)$ the cardinality of the set $\{i \mid \operatorname{Ext}_A^1(X, C_i) \neq 0\}$. If ${}^{\perp}(M \oplus C) \cap \mathscr{P}^1(A)$ is properly contained in ${}^{\perp}M \cap \mathscr{P}^1(A)$ there exists $X \in {}^{\perp}M \cap \mathscr{P}^1(A)$ with $t(X) \neq 0$. Choose such an X

with $t(X)$ minimal. Without loss of generality we may assume that $\mathrm{Ext}^1_A(X, C_i) \neq 0$ for $1 \leqslant i \leqslant s$ and $\mathrm{Ext}^1_A(X, C_j) = 0$ for $s < j \leqslant r$. Let $C = \bar{C} \oplus \bigoplus_{i=1}^{s} C_i$. For all $1 \leqslant i \leqslant s$ we consider exact sequences

$$\eta_i : 0 \to C_i^{t_i} \to E_i \to X \to 0$$

such that the connecting homomorphisms $\mathrm{Hom}_A(C_i^{t_i}, C_i) \to \mathrm{Ext}^1_A(X, C_i)$ are surjective. Obviously, $\mathrm{pd}\, E_i \leqslant 1$ and $E_i \in {}^{\perp}M$, further the minimality of $t(X)$ allows us to deduce that $E_i \in {}^{\perp}(M \oplus C) \cap \mathscr{P}^1(A)$. The previous lemma implies that E_i is cogenerated by $M \oplus C$, and it follows from [CHU] that C_i is cogenerated by $M \oplus C[i]$ for all $1 \leqslant i \leqslant s$. In particular, $M \oplus C[i]$ is faithful for all $1 \leqslant i \leqslant s$, for all of them have at least two non-isomorphic complements.

If one of the C_i's is not generated by $M \oplus C[i]$, then the assertion of the proposition holds.

Hence we may assume that all C_i, for $1 \leqslant i \leqslant s$, are generated by $M \oplus C[i]$. In the terminology of the lemma in Subsection 3.2, we decompose $T = M \oplus C$ into $T^1 = \bigoplus_{i=1}^{s} C_i$ and $T^2 = \bar{C} \oplus M$ and conclude that the C_i for $1 \leqslant i \leqslant s$ are generated by $\bar{C} \oplus M$. Consider an exact sequence

$$0 \to K_i \to F_i \to C_i \to 0$$

with $F_i \in \mathrm{add}\, M \oplus \bar{C}$. Applying $\mathrm{Hom}_A(X, -)$ to this sequence, we obtain that $\mathrm{Ext}^2_A(X, K_i) \neq 0$, a contradiction to the assumption that $\mathrm{pd}\, X \leqslant 1$.

Note that a direct summand of the Bongartz cocomplement to a partial tilting module M in $\mathscr{P}^1(A)$ may be cogenerated by M. This can be seen in the following example. Let A be the path algebra of the quiver $\overset{0}{\circ} \xleftarrow{\alpha} \overset{1}{\circ} \xleftarrow{\beta} \overset{2}{\circ}$ bound by the relations $\beta\alpha = 0$, and $M = P_1 \oplus P_2$. The Bongartz cocomplement to M is the simple module S_1 which is cogenerated by P_2.

Restricting to faithful partial tilting modules M in the proposition we obtain:

COROLLARY 1. *The following are equivalent for a faithful partial tilting module $M \in \mathscr{P}^1(A)$ with complement $C \in \mathscr{P}^1(A)$.*

(i) *$C = \bigoplus_{i=1}^{r} C_i$ is the Bongartz cocomplement to M.*

(ii) *All C_i are generated by M.*

Proof. Let $C = \bigoplus_{i=1}^{r} C_i$ be the Bongartz cocomplement to M. Since $M \oplus C[i]$ is faithful for all $1 \leqslant i \leqslant r$, the proposition implies that all C_i are generated by $M \oplus C[i]$. We set $T^1 = C$ and $T^2 = M$ and conclude from the Lemma in Subsection 3.2 that all C_i are generated by M. Conversely, if all C_i are generated by M, then they are obviously generated by $M \oplus C[i]$. The proposition now implies the assertion.

Combining the proposition above with the characterization of the Bongartz complement and the description of the boundary of $\Sigma^{\leqslant 1}$ (respectively $\delta\Sigma$) we get:

COROLLARY 2. *Let $M \in \mathscr{P}^1(A)$ be a partial tilting module with complement $C = \bigoplus_{i=1}^r C_i$ and* pd $C \leqslant 1$. *Then C is the Bongartz complement and the Bongartz cocomplement to M if and only if $M \oplus C[i]$ is not faithful for all $1 \leqslant i \leqslant r$.*

The phenomenon that the Bongartz complement and the Bongartz cocomplement to M do coincide happens evidently if M is an almost complete tilting module in $\mathscr{P}^1(A)$ which is not faithful. But it can also happen in other cases: take for example the path algebra of the quiver $\overset{1}{\bigcirc} \longrightarrow \overset{0}{\bigcirc} \longleftarrow \overset{2}{\bigcirc}$ and M the simple projective module P_0.

4.2. Let $\mathscr{D}$ be a full subcategory of mod A which is closed under direct sums, direct summands, and isomorphisms. An object $D \in \mathscr{D}$ is called a cocover of $\mathscr{D}$ if for all $X \in \mathscr{D}$ there is an injective map $X \to D'$ with $D' \in \operatorname{add} D$. A cocover D of $\mathscr{D}$ is said to be a minimal cocover if no proper direct summand of D is a cocover of $\mathscr{D}$.

With this notation, the lemma above states that a tilting module T with pd $T \leqslant 1$ is a cocover of ${}^{\perp}T \cap \mathscr{P}^1(A)$.

The next theorem gives a description for the existence of a Bongartz cocomplement in terms of contravariantly finite subcategories and the existence of cocovers. This will be applied later to derive a criterion for $\mathscr{P}^1(A)$ to be contravariantly finite.

THEOREM. *The following are equivalent for a partial tilting module M in $\mathscr{P}^1(A)$.*

(i) *${}^{\perp}M \cap \mathscr{P}^1(A)$ is contravariantly finite.*

(ii) *${}^{\perp}M \cap \mathscr{P}^1(A)$ has a cocover.*

(iii) *M admits a Bongartz cocomplement.*

Proof. The implication (i) $\Rightarrow$ (ii) is just Lemma 3.11(a) in [AS].

We now show that (ii) implies (iii). It is proved in [AS, Proposition 3.6] that for a minimal cocover F of ${}^{\perp}M \cap \mathscr{P}^1(A)$ there is a minimal right ${}^{\perp}M \cap \mathscr{P}^1(A)$ approximation of the injective cogenerator $\mathrm{D}({}_A A)$ of mod A of the form $F' \xrightarrow{\pi} \mathrm{D}({}_A A)$ with $F' \in \operatorname{add} F$. The morphism π is surjective since ${}^{\perp}M \cap \mathscr{P}^1(A)$ contains the projective A-modules. Consider the short exact sequence

$$0 \longrightarrow K \longrightarrow F' \xrightarrow{\pi} \mathrm{D}({}_A A) \longrightarrow 0.$$

According to Wakamatsu's lemma [W] we have $\mathrm{Ext}^1_A(Z, K) = 0$ for all modules Z in ${}^{\perp}M \cap \mathscr{P}^1(A)$. In particular, $\mathrm{Ext}^1_A(M \oplus F, K) = 0 = \mathrm{Ext}^1_A(M \oplus F, F)$. Thus $M \oplus F$ is a partial tilting module, and obviously, $M \oplus F$ is faithful. Consider an exact sequence

$$0 \longrightarrow {}_A A \xrightarrow{\mu} (M \oplus F)^r \longrightarrow Q \longrightarrow 0$$

such that the induced homomorphism $\mathrm{Hom}_A(\mu, M \oplus F)$ is surjective. Clearly, $\mathrm{pd}\, Q \leqslant 1$, $\mathrm{Ext}^1_A(Q, M \oplus F) = 0$ and $\mathrm{Ext}^1_A(M \oplus F, Q) = 0$. Then $\mathrm{Ext}^1_A(Q, Q) = 0$ and $M \oplus F \oplus Q$ is a tilting module of projective dimension at most one.

Let $X \in {}^{\perp}M \cap \mathscr{P}^1(A)$. According to Wakamatsu's lemma, $\mathrm{Ext}^1_A(X, F) = 0$. This implies that $\mathrm{Ext}^1_A(X, Q) = 0$, hence ${}^{\perp}M \cap \mathscr{P}^1(A) = {}^{\perp}(M \oplus F \oplus Q) \cap \mathscr{P}^1(A)$. Thus $F \oplus Q$ has a direct summand C which is a Bongartz cocomplement to M.

To prove the remaining implication (iii) $\Rightarrow$ (i) we want to apply Proposition 1.9 of [AR]. Therefore let C be the Bongartz cocomplement to M, further $\mathscr{X} = {}^{\perp}M \cap \mathscr{P}^1(A)$ and $\mathscr{Y} = (M \oplus C)^{\perp}$. Since $M \oplus C$ is a tilting module, $\mathscr{Y}$ is covariantly finite by Theorem 5.4 of [AR]. In order to establish that $\mathscr{X}$ is contravariantly finite we have to prove that

(a) $X \in \mathscr{X}$ if and only if $\mathrm{Ext}^1_A(X, Y) = 0$ for all $Y \in \mathscr{Y}$ and

(b) $Y \in \mathscr{Y}$ if and only if $\mathrm{Ext}^1_A(X, Y) = 0$ for all $X \in \mathscr{X}$.

To show (a), let $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$. Since $M \oplus C$ is a tilting module, Y is generated by $M \oplus C$. Consider an exact sequence $0 \to K(Y) \to E \to Y \to 0$ with $E \in \mathrm{add}\, M \oplus C$. Applying $\mathrm{Hom}_A(X, -)$ to this sequence, we obtain that $\mathrm{Ext}^1_A(X, Y) = 0$.

For the converse implication assume that $X \in \mathrm{mod}\, A$ satisfies $\mathrm{Ext}^1_A(X, Y) = 0$ for all $Y \in \mathscr{Y}$. We claim that this implies that $\mathrm{Ext}^i_A(X, Y) = 0$ for all $i > 0$ and $Y \in \mathscr{Y}$. The proof is an induction on i, the case $i = 1$ is just the assumption. Let $\mathrm{Ext}^i_A(X, Y) = 0$ for $i > 0$ and all $Y \in \mathscr{Y}$. We denote by $I(Y)$ the injective hull of Y. Applying $\mathrm{Hom}_A(M \oplus C, -)$ to the exact sequence

$$0 \to Y \to I(Y) \to Q(Y) \to 0,$$

we deduce from $\mathrm{pd}\, M \oplus C \leqslant 1$ that $Q(Y) \in \mathscr{Y}$. Now applying $\mathrm{Hom}_A(X, -)$ we see that $\mathrm{Ext}^{i+1}_A(X, Y) = \mathrm{Ext}^i_A(X, Q(Y)) = 0$. This proves the claim.

In particular, since $M \oplus C \in \mathscr{Y}$ we get that $X \in {}^{\perp}(M \oplus C)$.

Let R be some A-module. Since $\mathcal{Y}$ is covariantly finite, R admits a minimal left $\mathcal{Y}$-approximation $R \to Y_R$ which is injective. Consider the exact sequence

$$0 \to R \to Y_R \to Q_R \to 0.$$

Applying $\mathrm{Hom}_A(M \oplus C, -)$ we get that Q_R belongs to $\mathcal{Y}$. Now applying $\mathrm{Hom}_A(X, -)$ and using the claim above, we obtain that $\mathrm{Ext}^i_A(X, R) = 0$ for all $i > 1$. Hence, $X \in \mathcal{X}$, and (a) is established.

The fact that $Y \in \mathcal{Y}$ implies that $\mathrm{Ext}^1_A(X, Y) = 0$ for all $X \in \mathcal{X}$ has already been proved before. For the converse let $Y \in \mathrm{mod}\, A$ be such that $\mathrm{Ext}^1_A(X, Y) = 0$ for all $X \in \mathcal{X}$. In particular, $\mathrm{Ext}^1_A(M \oplus C, Y) = 0$, and since $\mathrm{pd}\, M \oplus C \leqslant 1$, also $\mathrm{Ext}^i(M \oplus C, Y) = 0$ for all $i > 0$. Hence $Y \in \mathcal{Y}$ and (b) holds.

Now Proposition 1.9 in [AR] states that ${}^{\perp}M \cap \mathcal{P}^1(A)$ is contravariantly finite.

We mention some consequences of the theorem.

Specializing to $M = 0$ we obtain:

Corollary 1. *$\mathcal{P}^1(A)$ is contravariantly finite if and only if $\mathcal{P}^1(A)$ has a cocover.*

This result was, with a different proof, recently obtained in [HU3].

Auslander and Reiten proved in [AR] that $\mathcal{P}^1(A)$ is always covariantly finite. Combining this result with the theorem above we get:

Corollary 2. *$\mathcal{P}^1(A)$ is functorially finite if and only if it admits a cocover.*

Assume that the injective envelope $I(A)$ of the module ${}_AA$ has projective dimension at most one. As in the proof of the theorem, $I(A)$ may be extended to a tilting module which is a Bongartz cocomplement to $M = 0$. Then the theorem implies:

Corollary 3 [IST]. *If the projective dimension of the injective envelope of ${}_AA$ is at most one, then $\mathcal{P}^1(A)$ is contravariantly finite.*

In [RiS2], Riedtmann and Schofield associated with $\Sigma^{\leqslant 1}$ an oriented graph $\vec{K}_0^{\leqslant 1}$ with vertices the chambers of $\Sigma^{\leqslant 1}$ and an arrow $T \to T'$ if T and T' have a common $(n-1)$-simplex and if $(T')^{\perp}$ is properly contained in $T^{\perp}$. The latter condition is equivalent to the fact that ${}^{\perp}T \cap \mathcal{P}^1(A)$ is properly contained in ${}^{\perp}T' \cap \mathscr{P}^1(A)$. Using the proposition in Subsection 4.1 it is easy to see that T is a sink in $\vec{K}_0^{\leqslant 1}$ if and only if T is the Bongartz cocomplement to 0. In particular, such a sink is unique.

Since $\vec{K}_0^{\leqslant 1}$ has no oriented cycles, it has a sink provided it is finite. Hence we obtain:

COROLLARY 4. *If A has only finitely many tilting modules of projective dimension at most one the $\mathscr{P}^1(A)$ is contravariantly (even functorially) finite.*

If A is hereditary, then every partial tilting module admits a Bongartz cocomplement. This can be proved by a construction dual to the one in [Bo]. Then it follows:

COROLLARY 5. *A partial tilting module M over a hereditary algebra satisfies that ${}^{\perp}M$ is contravariantly finite.*

5. THE BOUNDARY OF $\Sigma^{\leqslant 1}$

In this section we prove that $\delta\Sigma^{\leqslant 1}$ is a pseudomanifold in case it is finite, and we determine the geometric realization $\|\delta\Sigma^{\leqslant 1}\|$ of $\delta\Sigma^{\leqslant 1}$.

5.1. As in Subsection 3.4 we associate with $\delta\Sigma^{\leqslant 1}$ an oriented graph $\vec{K}(\delta)_0^{\leqslant 1}$ and consider its corresponding subgraphs $\vec{K}(\delta)_M^{\leqslant 1}$. Since the Bongartz complement to a partial tilting module in $\mathscr{P}^1(A)$ has projective dimension at most one, Lemma 1 of Subsection 3.4 holds for $\delta\Sigma^{\leqslant 1}$, and since a partial tilting module in $\mathscr{P}^1(A)$ always admits a Bongartz complement, Lemma 2 of Subsection 3.4 states that $M \oplus C$ is a source in $\vec{K}(\delta)_M^{\leqslant 1}$ if and only if C is a direct summand of the Bongartz complement to M. As before, these facts imply that the partial order induced by paths in $\vec{K}(\delta)_M^{\leqslant 1}$ satisfies (PO1), (PO2), and (PO4). Moreover, if $\delta\Sigma^{\leqslant 1}$ is finite, then also (PO3) holds. Hence we obtain:

COROLLARY. *$\delta\Sigma^{\leqslant 1}$ is shellable provided it is finite.*

5.2. LEMMA. *Let M be an $(n-2)$-simplex in $\delta\Sigma^{\leqslant 1}$, and let $M \oplus U$ be a chamber in $\delta\Sigma^{\leqslant 1}$. Then U is a direct summand of the Bongartz complement to M or a direct summand of the Bongartz cocomplement to M.*

Proof. Let C_U be the unique complement to $M \oplus U$. If $M \oplus C_U$ is not faithful, then $U \oplus C_U$ is the Bongartz complement and the Bongartz cocomplement to M by Corollary 2 in Subsection 4.2, implying the assertion.

If $M \oplus C_U$ is faithful, then $M \oplus C_U$ admits a second complement V with pd $V \leqslant 1$, and the sequence connecting the complements to $M \oplus C_U$ has the form

$$0 \to U \to E_1 \to V \to 0$$

with E_1 in add $M \oplus C_U$ or

$$0 \to V \to E_2 \to U \to 0$$

with $E_2 \in \operatorname{add} M \oplus C_U$.

In the first case, U is a direct summand of the Bongartz complement to $M \oplus C_U$. Since C_U is not generated by $M \oplus U$ (otherwise $M \oplus U$ would be faithful), $C_U \oplus U$ is the Bongartz complement to M.

In the second case, U is a direct summand of the Bongartz cocomplement to $M \oplus C_U$, and since $M \oplus U$ is not faithful, $C_U \oplus U$ is the Bongartz cocomplement to M.

PROPOSITION. *Let M be an $(n-2)$-simplex in $\delta\Sigma^{\leqslant 1}$. Then M is contained in at most two chambers in $\delta\Sigma^{\leqslant 1}$.*

Proof. Assume that $M \oplus X$, $M \oplus Y$, and $M \oplus Z$ are pairwise distinct chambers in $\delta\Sigma^{\leqslant 1}$. Since the Bongartz complement (respectively the Bongartz cocomplement) to M has two indecomposable direct summands, the above lemma implies that we may assume without loss of generality that $M \oplus X \oplus Y$ is a tilting module. Then Corollary 2 in Subsection 4.2 implies that $X \oplus Y$ is the Bongartz complement and the Bongartz cocomplement to M. Using again the lemma above we conclude that X and Z or Y and Z are isomorphic, a contradiction to our assumption.

Now assume that $\delta\Sigma^{\leqslant 1}$ is finite. We saw in Subsection 5.1 that $\delta\Sigma^{\leqslant 1}$ is shellable, and with the proposition above be conclude:

COROLLARY 1. *If $\delta\Sigma^{\leqslant 1}$ is finite, then it is a pseudomanifold.*

Using [DK] this implies:

COROLLARY 2. *If $\delta\Sigma^{\leqslant 1}$ is finite, then $\|\delta\Sigma^{\leqslant}\|$ is an $(n-1)$-ball or an $(n-1)$-sphere.*

The fact that $\|\Sigma^{\leqslant 1}\|$ is an n-ball provided $\Sigma^{\leqslant 1}$ is finite [RiS2] immediately implies that in this situation $\|\delta\Sigma^{\leqslant 1}\|$ is an $(n-1)$-sphere. Note that it is possible that $\Sigma^{\leqslant 1}$ is infinite and $\delta\Sigma^{\leqslant 1}$ is finite. The tame hereditary algebras furnish a class of examples with this phenomenon.

5.3. PROPOSITION. *Let $\delta\Sigma^{\leqslant 1}$ be finite. The following are equivalent.*

(i) *$\|\delta\Sigma^{\leqslant 1}\|$ is an $(n-1)$-sphere.*

(ii) *Every $(n-2)$-simplex $M \in \delta\Sigma^{\leqslant 1}$ admits a Bongartz cocomplement.*

Proof. $\|\delta\Sigma^{\leqslant 1}\|$ is an $(n-1)$-sphere if and only if every $(n-2)$-simplex M in $\delta\Sigma^{\leqslant 1}$ is contained in precisely two chambers $M \oplus X$ and $M \oplus Y$ in $\delta\Sigma^{\leqslant 1}$.

Let $M \oplus X$ and $M \oplus Y$ be two chambers in $\delta\Sigma^{\leqslant 1}$. If $M \oplus X \oplus Y$ is a tilting module, then Corollary 2 in Subsection 4.2 states that $X \oplus Y$ is the Bongartz cocomplement to M. Otherwise, Lemma 1 in Subsection 3.4 implies that we may assume that Y is a direct summand of the Bongartz cocomplement to M, in particular, M admits one.

Conversely assume that $C_1 \oplus C_2$ is the Bongartz cocomplement to M. A proof dual to the one of Lemma 1 in Subsection 3.4 shows that we may assume without loss of generality that $C_1 \oplus M \in \delta\Sigma^{\leqslant 1}$. If also $C_2 \oplus M \in \delta\Sigma^{\leqslant 1}$, then M is contained in two chambers in $\delta\Sigma^{\leqslant 1}$, the assertion. Hence assume that $C_2 \oplus M$ is faithful. Then C_1 is generated by $M \oplus C_2$. In particular, C_1 is not a direct summand of the Bongartz complement to M. Since M has a Bongartz complement, Lemma 1 in Subsection 3.4 states that there is a direct summand U of the Bongartz complement to M such that $M \oplus U \in \delta\Sigma^{\leqslant 1}$. Then M is contained in the chambers $M \oplus U$ and $M \oplus C_1$ in $\delta\Sigma^{\leqslant 1}$, implying the assertion.

COROLLARY 1. *If A is hereditary and $\delta\Sigma^{\leqslant 1}$ is finite, the $\|\delta\Sigma^{\leqslant 1}\|$ is an $(n-1)$-sphere.*

Consider the example of Igusa *et al.* [IST], where A is the path algebra of the quiver

$$1\bigcirc \begin{array}{c} \xleftarrow{\gamma} \\ \xleftarrow{\beta} \\ \xrightarrow{\alpha} \end{array} \bigcirc 2$$

bound by the relations $0 = \alpha\beta = \alpha\gamma = \beta\alpha$. Obviously, the partial tilting module 0 is an $n-1 = -1$ simplex in $\delta\Sigma^{\leqslant 1}$. Since $\mathscr{P}^1(A) = {}^{\perp}0 \cap \mathscr{P}^1(A)$ is not contravariantly finite [IST], the module 0 does not admit a Bongartz cocomplement by Theorem 4.2. Moreover, $\delta\Sigma^{\leqslant 1}$ is finite. Now the proposition above implies that $\|\delta\Sigma^{\leqslant 1}\|$ is a zero-ball. Indeed, it is easy to see that $\delta\Sigma^{\leqslant 1}$ is the simple projective module P_1.

Summarizing the results of this section, we obtain:

COROLLARY 2. *If $\delta\Sigma^{\leqslant 1}$ is finite, then it is shellable of characteristic zero or one, and both cases occur.*

6. COHEN–MACAULAY COMPLEXES, GORENSTEIN COMPLEXES, AND THE FACE RING OF A SIMPLICIAL COMPLEX

Assume that Σ is a finite simplicial complex of pure dimension n. Let $\tau \in \Sigma$ be a simplex. The link $\operatorname{lk} \tau$ of τ is the simplicial subcomplex $\{\sigma \in \Sigma \mid \tau \cup \sigma \in \Sigma \text{ and } \tau \cap \sigma = \varnothing\}$.

Let R be the ring of integers or a field. By $H_*(\Sigma, R)$ we denote the reduced simplicial homology of Σ with coefficients in R. For the definition and further details we refer to [Sp].

Σ is said to be Cohen–Macaulay over R if $\tilde{H}_i(\operatorname{lk} \tau, R) = 0$ for all τ in Σ, including $\tau = \varnothing$, and all $i < \dim \operatorname{lk} \tau$.

The motivation for this terminology comes from another object studied in algebraic combinatorics, the so-called face ring or Stanley–Reisner ring of Σ.

Let $V = \{x_0, ..., x_r\}$ be the vertex set of Σ. Consider the polynomial ring $R[x_0, ..., x_r]$ and the ideal I_Σ generated by all square free monomials $x_{i_1} \dots x_{i_s}$ such that $\{x_{i_1}, ..., x_{i_s}\} \notin \Sigma$. The quotient $R[\Sigma] = R[x_0, ..., x_r]/I_\Sigma$ is called the face ring of Σ.

A theorem of Reisner [Re] states that Σ is Cohen–Macaulay over R if and only if $R[\Sigma]$ is a Cohen–Macaulay ring. For a definition we refer to [Ma].

As a consequence of the result above, Reisner proves [Re] that if $\|\Sigma\|$ is a closed manifold, then $R[\Sigma]$ is a Cohen–Macaulay ring if and only if $H_i(\Sigma, R) = 0$ for all $0 \leqslant i < \dim \Sigma$.

It was first observed by Hochster [Ho1], that a shellable simplicial complex Σ is Cohen–Macaulay over the integers. For a direct algebraic proof we refer to [St1].

A simplicial complex Σ is called Gorenstein over R if its face ring is a Gorenstein ring over R. Again we refer to [Ma] for a definition. In [Ho2], Hochster gave a very nice combinatorial criterion for a simplicial complex to be Gorenstein, namely

THEOREM [Ho2]. *Let Σ be a simplicial complex of dimension at least one. The following conditions are equivalent*:

(i) *Σ is Gorenstein over $\mathbb{Z}$*

(ii) *Σ is Cohen–Macaulay over $\mathbb{Z}$ and all one-dimensional links in Σ are cycles or lines with at most three vertices.*

From this result he deduces that Σ is Gorenstein over the integers proved the geometric realization of Σ is a sphere.

For further details about Cohen–Macaulay and Gorenstein complexes we refer to the survey articles of Hochster [Ho2] and Stanley [St2].

Applying these concepts and results to the simplicial complex of tilting modules we obtain:

COROLLARY 1. *If the simplicial complex of tilting modules is finite, then the face rings of Σ and $\delta\Sigma$ are Cohen–Macaulay rings over the integers.*

Using the theorem of Hochster, it is easy to construct examples showing that the face rings of Σ and $\delta\Sigma$ are in general not Gorenstein.

COROLLARY 2. *Assume that $\delta\Sigma^{\leqslant 1}$ is finite. If every $(n-2)$ simplex in $\delta\Sigma^{\leqslant 1}$ admits a Bongartz cocomplement, then the face ring of $\delta\Sigma^{\leqslant 1}$ is Gorenstein over the integers.*

COROLLARY 3. *Let A be hereditary. If A is representation finite or tame, then the face ring of $\delta\Sigma^{\leqslant 1}$ is Gorenstein over $\mathbb{Z}$.*

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