Relating edge-coverings to the classification of $\mathbb{Z}_2^k$-magic graphs

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Let $G = (V, E)$ be a finite graph and let $(A, +)$ be an abelian group with identity 0. Then $G$ is $A$-magic if and only if there exists a function $\phi$ from $E$ into $A - \{0\}$ such that for some $c \in A$, $\sum_{e \in E} \phi(e) = c$ for every $v \in V$, where $E(v)$ is the set of edges incident to $v$. Additionally, $G$ is zero-sum $A$-magic if and only if $\phi$ exists such that $c = 0$. In this paper, we explore $\mathbb{Z}_2^k$-magic graphs in terms of even edge-coverings, graph parity, factorability, and nowhere-zero 4-flows. We prove that the minimum $k$ such that bridgeless $G$ is zero-sum $\mathbb{Z}_2^k$-magic is equal to the minimum number of even subgraphs that cover the edges of $G$, known to be at most 3. We also show that bridgeless $G$ is zero-sum $\mathbb{Z}_2^k$-magic for all $k \geq 2$ if and only if $G$ has a nowhere-zero 4-flow, and that $G$ is zero-sum $\mathbb{Z}_2^k$-magic for all $k \geq 2$ if $G$ is Hamiltonian, bridgeless planar, or isomorphic to a bridgeless complete multipartite graph. Finally, we establish equivalent conditions for graphs of even order with bridges to be $\mathbb{Z}_2^k$-magic for all $k \geq 4$.

1. Introduction

Throughout this paper, graphs shall be finite and loopless with at least one edge, and not necessarily simple. Abelian groups shall have identity element 0 and binary operator $+$.

Let $G = (V, E)$ be a graph and let $(A, +)$ be an abelian group. Then an $A$-labeling of $G$ is a function $\phi$ from $E$ into $A - \{0\}$. For edge $e \in E$, the label of $e$ under $\phi$ shall refer to $\phi(e)$, and for vertex $v \in V$, the weight of $v$ under $\phi$, denoted $w_{\phi}(v)$, shall refer to the sum of the labels (under $\phi$) of the edges incident to $v$. It is clear that $w_{\phi}$ is a function induced by $\phi$ from $V$ to $A$. We call $\phi$ an $A$-magic labeling of $G$ with weight $c$ if and only if all vertices have the same weight $c$ under $\phi$, and we call $\phi$ a zero-sum $A$-magic labeling of $G$ if and only if all vertices have weight 0 under $\phi$. Accordingly, we will say that $G$ is $A$-magic (respectively zero-sum $A$-magic) if and only if there exists an $A$-magic (respectively zero-sum $A$-magic) labeling of $G$. For illustration, we give a zero-sum $\mathbb{Z}_2^k$-magic labeling of the wheel $W_5$ in Fig. 1.1.

We observe that if $(A', +)$ is a subgroup of $(A, +)$ and $G$ is $A'$-magic (resp. zero-sum $A'$-magic), then $G$ is $A$-magic (resp. zero-sum $A$-magic). Thus, if $G$ is $\mathbb{Z}_{2^{k_0}}$-magic or zero-sum $\mathbb{Z}_{2^{k_0}}$-magic, then $G$ is $\mathbb{Z}_{2^k}$-magic or zero-sum $\mathbb{Z}_{2^k}$-magic for all $k \geq k_0$.

The study of group-magic labelings of graphs was motivated in the 1960s by the work of Sedláček [12] and Stewart [15] on integer-magic labelings. Surveys of the field have been written by Gallian [3] and Wallis [16].

In recent years, particular attention has been given to the $A$-magic labelings of graphs in various classes where $A$ is some cyclic group (see [1,9–11,13] for examples), leading to the notion of the integer-magic spectrum of a graph $G$ (the set of all $k$ such that $G$ is $\mathbb{Z}_k$-magic). Additionally, some attention has been paid to the $V_4$-magic labelings of graphs, where $V_4$ denotes the Klein group $\mathbb{Z}_2^2$ [5,8]. The particular comparisons of $V_4$-magic graphs and $\mathbb{Z}_4$-magic graphs led to the question: Are $V_4$-magic graphs necessarily $\mathbb{Z}_4$-magic? This question was settled in the negative in [5] via the investigation of zero-sum...
In this paper, we consider the application of even edge-coverings, graph parity, odd factors, cubic extensions, and 4-flows to the study of $A$-magic labels of graphs for $A$ in the set $\mathbb{Z}_2^k$, $1 \leq k < \infty$. Notation and preliminary results are given in Section 2. In Section 3, we consider the zero-sum $\mathbb{Z}_2^k$-magicness of bridgeless graphs, showing that the smallest $k$ for which $G$ is zero-sum $\mathbb{Z}_2^k$-magic is equal to the minimum cardinality of an even edge-covering of $G$, a number known to be at most 3. We conclude the equivalence between the existence of a nowhere-zero 4-flow in $G$ and the existence of a zero-sum $\mathbb{Z}_4$-magic labeling of $G$. We also develop a simple cubic derivative of $G$, called the cubic extension of $G$, that is zero-sum $\mathbb{Z}_4$-magic if and only if $G$ is zero-sum $\mathbb{Z}_4$-magic. The section closes with a consideration of complete multipartite graphs, continuing the work of Low and Shiu in [14]. In Section 4, we determine conditions under which graphs with at least one bridge (necessarily not zero-sum $\mathbb{Z}_2^k$-magic for each $k$) are $\mathbb{Z}_2^k$-magic for $k \geq 4$. Finally, Section 5 summarizes the results of the preceding sections.

2. Definitions and preliminary results

The vertex set and edge set of a graph $G$ will be denoted $V(G)$ and $E(G)$, respectively. A graph $G$ is odd (resp. even) if and only if the degree of each vertex in $V(G)$ is odd (resp. even). If $F$ is a spanning subgraph of $G$, then $F$ is an odd factor of $G$ if and only if $F$ is odd. We note that each even graph is zero-sum $\mathbb{Z}_2$-magic, and that each odd graph is $\mathbb{Z}_2$-magic but not zero-sum $\mathbb{Z}_2$-magic. It is clear that no other graph is $\mathbb{Z}_2$-magic.

Let $G$ be a graph and let $C$ be a non-empty set of subgraphs of $G$. Then $C$ is an edge-covering of $G$ if and only if $E(G)$ equals $\bigcup_{H \in C} E(H)$. Moreover, $C$ is an even edge-covering of $G$ if and only if $C$ is an edge-covering of $G$ such that every subgraph in $C$ is even. The following theorem from [2] shall be applied in Section 3.

**Theorem 2.1.** Every bridgeless graph $G$ has an even edge-covering of cardinality at most 3. □

Let $G$ be a bridgeless graph. Then $s(G)$ shall denote the minimum cardinality over the set of even edge-coverings of $G$. By Theorem 2.1, $s(G)$ exists and is at most 3. Moreover, $s(G) = 1$ if and only if $G$ is an even graph.

In Section 3, we will make use of the following definitions and theorems, all of which can be found in [17].

**Definition 2.2.** Let $G$ be a graph and let $k$ be a positive integer. Then a nowhere-zero $k$-flow is a pair $(D, f)$ such that $D$ is an orientation of $G$, and $f$ is a function from $E(G)$ into the set of non-zero integers strictly between $-k$ and $k$, and for each $v \in V(G)$, $\sum_{e \in E^+_D(v)} f(e) = -\sum_{e \in E^-_D(v)} f(e)$, where $E^+_D(v)$ (resp. $E^-_D(v)$) is the set of edges that are incident to $v$ and pointed away from (resp. toward) $v$. □

**Theorem 2.3.** A planar bridgeless graph $G$ is $k$-face-colorable if and only if $G$ has a nowhere-zero $k$-flow. □

**Theorem 2.4.** A graph $G$ has a nowhere-zero 4-flow if and only if $G$ has an even edge-covering of cardinality at most two. □

**Theorem 2.5.** If $G$ is Hamiltonian, then $G$ has a nowhere-zero 4-flow. □

Let $G$ be a graph, let $F$ be a spanning subgraph of $G$, and let $g$ be a function from $V(G)$ into $\{1, 3, 5, \ldots\}$. Then $F$ is a $(1, g)$-odd factor of $G$ if and only if for each $v \in V(G)$, the degree of $v$ in $F$ is in $\{1, 3, 5, \ldots, g(v)\}$. The next two results, the first of which appears in [7], will be applied in Section 5.

**Theorem 2.6.** Let $T$ be a tree of even order and let $g$ be a function from $V(T)$ into $\{1, 3, 5, \ldots\}$. Then $T$ has a $(1, g)$-odd factor if and only if for every $v \in V(T)$, $o(T - v) \leq g(v)$, where $o(T - v)$ is the number of components of odd order of $T - v$. □

**Theorem 2.7.** Let $T$ be a tree of even order and let $g$ be a function on $V(T)$ such that for each $v \in V(T)$, $g(v)$ is the largest odd integer less than or equal to $d(v)$ (the degree of $v$). Then $T$ has a $(1, g)$-odd factor. Consequently, if $G$ is a connected graph with even order, then $G$ has an odd factor, since $G$ has a spanning tree $T$ of even order.

**Proof.** Select $v \in V(T)$. If $d(v)$ is odd, then $g(v) = d(v)$, implying $o(T - v) \leq g(v)$. If $d(v)$ is even, then $g(v) = d(v) - 1$. Since the order of $T$ is even, then $o(T - v)$ cannot be $d(v)$. Thus, $o(T - v) \leq g(v)$. The result now follows from Theorem 2.6. □
3. On zero-sum $\mathbb{Z}_2^k$-magic graphs

In [5], it was proved that if $G$ is a graph with a bridge, then for each positive integer $k$, $G$ is not zero-sum $\mathbb{Z}_2^k$-magic. Thus, in this section, we focus on bridgeless graphs.

**Theorem 3.1.** Let $G$ be a bridgeless graph. Then $G$ is zero-sum $\mathbb{Z}_2^{s(G)}$-magic.

**Proof.** By Theorem 2.1, let $C = \{H_i | 1 \leq i \leq s(G) \leq 3\}$ be an even edge-covering of minimum cardinality. We produce a zero-sum $\mathbb{Z}_2^{s(G)}$-magic labeling $\phi(G)$ as follows: for each $e \in E(G)$ and each $j$, $1 \leq j \leq s(G)$, let the $j$th coordinate of $\phi(e)$ be the scaler 1 if $e$ is in $E(H_j)$, and the scaler 0 otherwise. Noting that each assigned label has at least one coordinate equal to the scalar 1, we have that no assigned label is equal to 0. To show that $w_\phi(v) = 0$ for each vertex $v$ of $G$, fix $v_0 \in V(G)$ and fix $j_0$, $1 \leq j_0 \leq s(G)$. Then there is an even number of edges in $E(H_{j_0})$ that are incident to $v_0$, implying that the $j_0$th coordinate of the weight of $v_0$ is the scaler 0. \hfill \square

Let $G$ be a bridgeless graph. Then $(t)$ shall denote the minimum positive integer $k$ such that $G$ is zero-sum $\mathbb{Z}_2^k$-magic. By Theorem 3.1, $t(G)$ is well-defined and $1 \leq t(G) \leq s(G)$.

**Theorem 3.2.** For bridgeless graph $G$, $s(G) = t(G)$.

**Proof.** It suffices to show that $s(G) \leq t(G)$. Let $\phi$ denote a zero-sum $\mathbb{Z}_2^{t(G)}$-magic labeling of $G$. By the minimality of $t(G)$, for each $j$, $1 \leq j \leq t(G)$, there exists an edge $e$ such that $\phi(e)$ is the scaler 1 in the $j$th component. We may thus produce an edge-covering $C$ of $G$ with cardinality $t(G)$ as follows: $C = \{H_1, H_2, \ldots, H_{t(G)}\}$ where, for $1 \leq j \leq t(G)$, $e \in E(H_j)$ if and only if the $j$th coordinate of $\phi(e)$ is scaler 1. Since the weight of each vertex under $\phi$ is 0, it is now easily seen that every element of $C$ is even. Thus $s(G) \leq t(G)$. \hfill \square

From Theorem 3.2 and the previous note that $s(G) \leq 3$, we have.

**Corollary 3.3.** Let $G$ be a bridgeless graph. Then $t(G) \leq 3$. Moreover, $t(G) = 1$ if and only if $G$ is even. \hfill \square

The above results imply that each non-even bridgeless graph $G$ can be classified into one of two categories: $t(G) = 2$ (so $G$ is zero-sum $\mathbb{V}_4$-magic) or $t(G) = 3$ (and thus $G$ is not zero-sum $\mathbb{V}_4$-magic). Our investigation of this classification problem begins with the consideration of cubic bridgeless graphs. The following is a result from [5].

**Theorem 3.4.** Let $G$ be a cubic graph. Then $G$ is zero-sum $\mathbb{V}_4$-magic if and only if the chromatic index of $G$ is 3. \hfill \square

It thus follows that every bridgeless cubic graph $G$ has $t(G) = 2$ if $\chi'(G) = 3$, and $t(G) = 3$ if $\chi'(G) = 4$. Hence the Petersen graph $P$ has $t(P) = 3$, as do all snarks. Such graphs are necessarily non-Hamiltonian; however, we point out the existence of non-Hamiltonian cubic bridgeless graphs with chromatic index 3. (See [4]). Moreover, since the determination of the chromatic index of an arbitrary cubic graph is known to be NP-complete (see [6]), it follows that the determination of whether or not a cubic graph is zero-sum $\mathbb{V}_4$-magic is also NP-complete.

We point out that the classification problem over bridgeless graphs in general is not clearly linked to chromatic index. For example, $t(K_5 - e) = 2$ and $\chi'(K_5 - e) = 5$. On the other hand, we may link the classification of a bridgeless graph $G$ to the chromatic index of a certain cubic graph generated from $G$, as described below.

Let $G$ be a graph (not necessarily bridgeless) with $\delta(G) \geq 2$. From $G$, we form a cubic graph of order $2|E(G)|$ by executing the following pseudo-code:

Let the vertices of $G$ be $v_1, v_2, v_3, \ldots, v_n$.

1. Set $G := G_0$.
2. Set $i := 1$.
3. While $i \leq n$, do.
   - Form graph $G_i^1$ by subdividing each edge of $G_{i-1}$ that is incident to $v_i$;
   - Form graph $G_i^2$ by inserting $d(v_i)$ edges in $G_i^1$ so that the subgraph induced by the set of subdividing vertices is a cycle of length $d(v_i)$ (there may be more than one way to select the incidence structure of the subdividing vertices);
   - Form graph $G_i^3$ by deleting from $G_i^2$ the vertex $v_i$ and its incident edges;
   - Set $G_i := G_i^3$.
   - Set $i := i + 1$.
4. The output graph $G_n$ is a simple cubic graph.

Any graph $G_n$ that is output by this code will be called a cubic extension of $G$, and will be denoted $ce(G)$. The cycle that is created by the code when $i = n$ shall be called the cycle in $ce(G)$ induced by $v_n$ and denoted $C_{v_n}$. Since there is more than one way to form the cycle $C_e$ for $d(v)$ sufficiently large, it follows that a graph $G$ may have non-isomorphic cubic extensions. Each edge in $ce(G)$ that is incident to some $C_e$ but does not lie along $C_e$ shall be called a spoke. There is a natural bijection $f$ from $E(G)$ to the set of spokes of $ce(G)$; in particular, if an edge $e \in E(G)$ is incident to distinct $x, y \in V(G)$, then the corresponding spoke in $ce(G)$ shall be an edge that connects $C_x$ and $C_y$. The spoke will be unique if $G$ is simple. On the other
hand, if there are precisely \( k \) distinct edges incident to \( x \) and \( y \) in \( V(G) \), then there will be precisely \( k \) distinct spokes incident to \( C_x \) and \( C_y \) in \( ce(G) \). We illustrate a graph \( G \) and two non-isomorphic cubic extensions of \( G \) in Fig. 3.1, alluding to the bijection \( f \).

We observe that for every cubic extension \( ce(G) \), \( e \) is a bridge of \( G \) if and only if \( f(e) \) is a bridge of \( ce(G) \).

We now turn to the relationship between \( G \) and at least one of its cubic extensions, preceded by a supporting lemma.

**Lemma 3.5.** For fixed integer \( k \), let \( \phi \) be a zero-sum \( \mathbb{Z}_2^k \)-magic labeling of a graph \( G \) (so \( G \) is necessarily bridgeless) and let \( S \) be a subset of \( V(G) \). Let \( P_S \) be the set of edges of \( G \) that are incident to precisely one vertex in \( S \), and let \( Q_S \) be the set of edges of \( G \) that are incident to precisely two vertices in \( S \). Then \( \sum_{e \in P_S} \phi(e) = 0 \).

**Proof.** We have \( 0 = \sum_{v \in S} \mu_\phi(v) = \sum_{e \in P_S} \phi(e) + 2 \sum_{e \in Q_S} \phi(e) \). Since \( 2 \sum_{e \in Q_S} \phi(e) = 0 \), the result follows. \( \square \)

**Theorem 3.6.** Let \( G \) be a bridgeless graph. Then \( G \) is zero-sum \( V_4 \)-magic if and only if there exists a cubic extension \( ce(G) \) that is zero-sum \( V_4 \)-magic.

**Proof.** Since this theorem deals only with \( V_4 \), we will denote its zero element by \((0,0)\) throughout the proof.

Let \( \phi^* \) be a zero-sum \( V_4 \)-magic labeling of \( ce(G) \) and let \( v_0 \) be an element of \( V(G) \). Then by Lemma 3.5, the sum of the labels assigned by \( \phi^* \) to the spokes incident to \( C_{v_0} \) is \((0,0)\). We now form a zero-sum \( V_4 \)-magic labeling \( \phi \) of \( G \) as follows: \( \phi(e) = \phi^*(f(e)) \), where \( f \) is the natural bijection from the edges of \( G \) to the spokes of \( ce(G) \).

Now let \( \phi \) be a zero-sum \( V_4 \)-magic labeling of \( G \). For each \( v \in V(G) \) and \( i \in V_4 \), let \( X_i(v) \) denote the set of edges incident to \( v \) with label \( i \) under \( \phi \). We shall construct a cubic extension \( ce(G) \) of \( G \) and a zero-sum \( V_4 \)-magic labeling \( \phi^* \) of \( ce(G) \).

If \( e \) is a spoke of our constructed cubic extension, we will let \( \phi^*(e) \) equal \( \phi(f^{-1}(e)) \). If \( e \) is not a spoke, then \( \phi^*(e) \) will depend on the length \( d(v) \) of \( C_v \) and the labels of the two spokes incident to \( e \). We observe that it suffices to form \( C_v \) and the labeling of the edges of \( C_v \) for arbitrary fixed vertex \( v \in V(G) \) in each of two cases.

**Case 1.** \( d(v) \) is odd. Noting that the weight of \( v \) under \( \phi \) is \((0,0)\), it follows that the cardinalities of \( X_{(1,1)}(v) \), \( X_{(0,1)}(v) \), and \( X_{(1,0)}(v) \) are each odd with respective cardinalities \( 2j_1 + 1 \), \( 2j_2 - 1 \), and \( 2j_3 + 1 \), summing to \( d(v) \). Let \( C_v = (v_0, v_1, v_2, \ldots, v_{d(v)-1}) \) denote a cycle induced by \( v \) such that:
- for \( 0 \leq i < 2j_1 \), vertex \( v_i \) is incident to a spoke with label \((1,1)\) under \( \phi^* \);
- for \( 2j_1 + 1 \leq i < 2j_1 + 2j_2 + 1 \), vertex \( v_i \) is incident to a spoke with label \((0,1)\) under \( \phi^* \);
- for \( 2j_1 + 2j_2 + 2 \leq i < d(v) - 1 \), vertex \( v_i \) is incident to a spoke with label \((1,0)\) under \( \phi^* \).

For \( 0 \leq i \leq 2j_1 \), we define
\[
\phi^*(v_i, v_{i+1}) = \begin{cases} (1,0) & \text{if } i = 0 \text{ mod } 2 \\ (0,1) & \text{if } i = 1 \text{ mod } 2. \end{cases}
\]

For \( 2j_1 + 1 \leq i \leq 2j_1 + 2j_2 + 1 \), we define
\[
\phi^*(v_i, v_{i+1}) = \begin{cases} (1,1) & \text{if } i = 1 \text{ mod } 2 \\ (0,0) & \text{if } i = 0 \text{ mod } 2. \end{cases}
\]

For \( 2j_1 + 2j_2 + 2 \leq i < d(v) - 2 \), we define
\[
\phi^*(v_i, v_{i+1}) = \begin{cases} (1,1) & \text{if } i = 1 \text{ mod } 2 \\ (0,1) & \text{if } i = 0 \text{ mod } 2. \end{cases}
\]

Finally, we let \( \phi^*(v_{d(v)-1}, v_0) = (0,1) \).

It can be verified that every vertex along \( C_v \) has weight \((0,0)\)
Case 2. $d(v)$ is even. Let $a$, $b$, $c$ be the distinct elements of $\{(1, 1), (0, 1), (1, 0)\}$. Noting that the weight of $v$ under $\phi$ is $(0, 0)$, it follows that the cardinalities of $X_a(v)$, $X_b(v)$, and $X_c(v)$ are each even with respective cardinalities $2j_1$, $2j_2$, and $2j_3$, summing to $d(v)$. Without loss of generality, suppose $v$ is a vertex such that $j_3 \leq j_2 \leq j_1$. We form $C_v = \{v_0, v_1, v_2, \ldots, v_{d(v) - 1}\}$ as follows (with the understanding that any reference to $v_i$ for $i \geq d(v)$ is vacuous):

: for $0 \leq i \leq 6j_3 - 1$, vertex $v_i$ is incident to a spoke with respective label $a$, $b$, $c$ under $\phi^*$ if $i = 0 \mod 3$, $i = 1 \mod 3$, $i = 2 \mod 3$.

: vertex $v_{6j_3 + 1}$ is incident to a spoke with label $a$ under $\phi^*$;

: for $6j_3 + 1 \leq i \leq 4j_3 + 2j_2$, vertex $v_i$ is incident to a spoke with label $b$ under $\phi^*$;

: for $4j_3 + 2j_2 + 1 \leq i \leq d(v) - 1$, vertex $v_i$ is incident to a spoke with label $a$ under $\phi^*$.

For $0 \leq i \leq 6j_3 - 1$, we define

$$\phi^*(v_i, v_{i+1}) = \begin{cases} c & \text{if } i = 0 \mod 3 \\ a & \text{if } i = 1 \mod 3 \\ b & \text{if } i = 2 \mod 3. \end{cases}$$

We define $\phi^*(v_{6j_3}, v_{6j_3 + 1}) = c$.

For $6j_3 + 1 \leq i \leq 4j_3 + 2j_2$, we define

$$\phi^*(v_i, v_{i+1}) = \begin{cases} c & \text{if } i = 0 \mod 2 \\ a & \text{if } i = 1 \mod 2. \end{cases}$$

For $4j_3 + 2j_2 + 1 \leq i \leq d(v) - 2$, we define

$$\phi^*(v_i, v_{i+1}) = \begin{cases} c & \text{if } i = 0 \mod 2 \\ b & \text{if } i = 1 \mod 2. \end{cases}$$

Finally, we let $\phi^*(v_{d(v) - 1}, v_0) = b$.

It can be verified that every vertex along $C_v$ has weight $(0, 0)$. □

We observe that if $G$ is bridgeless planar, then $G$ has a cubic extension $ce(G)$ that is also bridgeless planar. By the Four-Color Theorem and Tait’s Theorem [see 17], $ce(G)$ thus has chromatic index 3, from which it follows by Theorem 3.4 that $ce(G)$ is zero-sum $V_4$-magic. So, by Theorem 3.6, $G$ is zero-sum $V_4$-magic as well. We also observe that if $G$ is Hamiltonian (and thus bridgeless), it is easy to construct a Hamiltonian cubic extension $ce(G)$ as well. Thus $ce(G)$ has chromatic index 3, again implying that $G$ is zero-sum $V_4$-magic. We therefore have.

**Theorem 3.7.** If $G$ is bridgeless planar or Hamiltonian, then $G$ is zero-sum $\mathbb{Z}_2^k$-magic for $k \geq 2$. □

Turning to the relationship between zero-sum magicness and nowhere-zero 4-flows, we observe by Theorem 2.4 that if $G$ is a bridgeless graph, then $s(G) \leq 2$ if and only if $G$ has a nowhere-zero 4-flow. We therefore have.

**Theorem 3.8.** Let $G$ be a bridgeless graph. Then the following are equivalent.

1. some cubic extension $ce(G)$ has chromatic index 3
2. some cubic extension $ce(G)$ is zero-sum $\mathbb{Z}_2^k$-magic for $k \geq 2$
3. $G$ is zero-sum $\mathbb{Z}_2^k$-magic for $k \geq 2$
4. $t(G) \leq 2$
5. $s(G) \leq 2$
6. $G$ has a nowhere-zero 4-flow. □

We observe that Theorem 3.7 can also be shown in the context of nowhere-zero 4-flows. Particularly, if $G$ is bridgeless planar or Hamiltonian, then by Theorems 2.3 and 2.5, $s(G) \leq 2$. Hence, by Theorem 3.1, $G$ is zero-sum $\mathbb{Z}_2^k$-magic for $k \geq 2$.

Although the hypotheses of Theorem 3.7 include bridgeless planarity or Hamiltonicity, we note that zero-sum $V_4$-magic graphs exist which satisfy neither condition. If $G$ is a graph and $\mathcal{P}$ is a partition $\{E_1, E_2, \ldots, E_k\}$ of $E(G)$ such that the subgraph of $G$ induced by $E_1$ has a zero-sum $A$-magic labeling $\phi_1$, then it is clear that $G$ is zero-sum $A$-magic. (Particularly, let $\phi$ be an $A$-labeling of $G$ such that $\phi(e) = \phi_1(e)$ if and only if $e \in E_1$. Then the weight of each vertex $v \in V(G)$ is 0.) It therefore follows that if each $E_i$ induces a subgraph that is either bridgeless planar or Hamiltonian, then $G$ is zero-sum $V_4$-magic. For example, since the complete multipartite graph $K_{n_1, n_2, \ldots, n_m}$, $n_i \geq 2$, admits a partitioning $\mathcal{P}$ of its edge set such that each element of $\mathcal{P}$ induces the bridgeless planar or Hamiltonian subgraph $K_2, 2$, $K_2, 3$, or $K_3, 3$, then $K_{n_1, n_2, \ldots, n_m}$ is zero-sum $\mathbb{Z}_2^k$-magic for $k \geq 2$, a result shown in [14].

We also observe that if $G$ has a 2-factor $F = \{C_1, C_2, \ldots, C_{m/2}\}$ such that each cycle in $F$ is incident to vertices with degrees in $G$ that sum to an even number, then $G$ is zero-sum $V_4$-magic. To see this, note that $G$ has a cubic extension $ce(G)$ with a 2-factor $F' = \{C_{i_1}, C_{i_2}, \ldots, C_{i_{m/2}}\}$ such that each cycle in $F'$ has even length. (Each $C_{i_1}$ will be incident to precisely the vertices of the cycles in $ce(G)$ induced by the vertices along $C_{i_1}$, from which the evenness of the length of $C_{i_1}$ follows.)

We then form a zero-sum $V_4$-magic labeling of $ce(G)$ by alternating the labels $(0, 1)$ and $(1, 0)$ about each cycle in $F'$, and assigning $(1, 1)$ to each of the other edges of $ce(G)$. The result follows by Theorem 3.6.

We state these results below.
Theorem 3.9. If $G$ is a graph and $\mathcal{P}$ is a partition $\{E_1, E_2, \ldots, E_k\}$ of $E(G)$ such that the subgraph of $G$ induced by $E_i$ has a zero-sum $V_4$-magic labeling $\phi_i$, then $G$ is zero-sum $Z_2^k$-magic for $k \geq 2$. □

Theorem 3.10. If $G$ has a 2-factor $F$ such that each cycle in $F$ is incident to vertices with degrees in $G$ that sum to an even number, then $G$ is zero-sum $Z_2^k$-magic for $k \geq 2$. □

Corollary 3.11. If $G$ and $H$ are zero-sum $V_4$-magic, then by Theorem 3.9, the Cartesian product $G \square H$ is zero-sum $Z_2^k$-magic for $k \geq 2$. □

Corollary 3.12. Let $G$ be a $2m + 1$-regular graph with chromatic index $2m + 1$. Then $G$ is zero-sum $Z_2^k$-magic for $k \geq 2$. Hence $G$ is bridgeless with a nowhere-zero 4-flow.

Proof. Let $C$ denote a $2m + 1$-coloring of $G$. Then for distinct colors $c_1$ and $c_2$ in the range of $C$, the set of edges with colors $c_1$ and $c_2$ induce a 2-factor of $G$ in which each cycle is even. By Theorems 3.9 and 3.10, the results follow. □

The Petersen graph $P$ and the Hoffman-Singleton graph $HS$ are the only known odd-regular Moore graphs with diameter 2. We have already observed that $P$ is not zero-sum $V_4$-magic. However, since $HS$ is 7-regular with chromatic index 7, it follows from Corollary 3.12 that $HS$ is zero-sum $V_4$-magic and has a nowhere-zero 4-flow.

Corollary 3.13. Let $G$ be an odd graph. If $G$ has a 2-factor in which each cycle is of even order, then by Theorem 3.10, $G$ is zero-sum $Z_2^k$-magic for $k \geq 2$. Hence $G$ is bridgeless with a nowhere-zero 4-flow. □

We close this section with a characterization of complete multipartite graphs $K_{n_1, n_2, \ldots, n_m}$, $n_1 \leq n_2 \leq \cdots \leq n_m$, that are zero-sum $V_4$-magic, and hence zero-sum $Z_2^k$-magic for $k \geq 2$. For convenience, we will use the term magical to describe any partition $\mathcal{P}$ of an edge set such that each cycle is even. By Theorems 3.6 and 3.8, we observe that if $G$ is triangle-free with an even number of edges, then $G$ is $Z_2^k$-magic if and only if $G$ is $Z_2^k$-magic.

Case 1. $m < 3$. If $m = 3$, then $K_{n_1, n_2, n_3}$ has a magic partition $\{E_1, E_2, E_3\}$ such that $E_i$ induces the complete graph $K_{n_i}$ and $E_1$ induces $K_{n_1, n_2, n_3}$. If $m < n_i$, then $G$ is not zero-sum $V_4$-magic by Theorem 3.7.

Case 2. $m \geq 3$. If $n_1 = 1$, then $G$ is $Z_2^k$-magic by Theorem 3.7. If $n_1 = 2$, then $G$ is $Z_2^k$-magic by Theorem 3.8.

Case 3. $m > 3$. If $m > 3$, then $G$ is $Z_2^k$-magic by Theorem 3.8. Therefore, $G$ is $Z_2^k$-magic if and only if $G$ has no bridge. □

4. On $Z_2^k$-magic graphs

In this section we consider the conditions under which a connected graph is $Z_2^k$-magic (not necessarily zero-sum) for some $k$. We note that since $G$ is $Z_2$-magic if and only if $G$ is $Z_2^k$-magic for all $k \geq 1$, then $G$ is $Z_2^k$-magic if and only if $G$ is odd or $G$ is even.

Theorem 4.1. Let $G$ be a graph of odd order (connected or not) and for some $k$ let $\phi$ be a $Z_2^k$-magic labeling of $G$ with weight $a$. Then $a = 0$.

Proof. Since $|V(G)|$ is odd, $a = \sum_{v \in V(G)} w_\phi(v) = 2 \sum_{e \in E(G)} \phi(e) = 0$. □

Let $G$ be a graph (connected or not) and let $\tau(G)$ denote the smallest $k$ such that $G$ is $Z_2^k$-magic if such a $k$ exists. By Corollary 3.3 and Theorem 4.1, if $G$ has no bridges, then $\tau(G) \leq t(G) \leq 3$, with $\tau(G) = t(G)$ if $G$ has odd order. Furthermore, by Theorem 4.1 and the opening remark of Section 3, we observe that if $G$ (connected or not) has odd order and a bridge, then for all $k$, $G$ is not $Z_2^k$-magic. Thus, $Z_2^k$-magic graphs with a bridge have even order, and we will show that for such graphs, $\tau(G) \leq 4$. 
Lemma 4.2. Let $G$ be a connected graph with non-empty bridge set. Let $\phi$ be a $\mathbb{Z}_2^k$-magic labeling with weight $a$. Then:

(i) $G$ has even order and $a \neq 0$, and
(ii) for any bridge $e^*$, each component of $G - e^*$ has odd order and $\phi(e^*) = a$.

Proof. Part (i) follows from the remark at the beginning of Section 3 and the immediately preceding remark.

To show (ii), let $\phi$ be a $\mathbb{Z}_2^k$-magic labeling of $G$ and let $e^*$ be a bridge. Let $G_1$ and $G_2$ denote the components of $G - e^*$. If $G_i$ has even order, then

$$0 = \sum_{v \in V(G_i)} w_{\phi}(v) = \phi(e^*) + 2 \sum_{e \in E(G_i)} \phi(e),$$

implying the contradiction $\phi(e^*) = 0$. Since $G_i$ thus has odd order, then

$$a = \sum_{v \in V(G_i)} w_{\phi}(v) = \phi(e^*) + 2 \sum_{e \in E(G_i)} \phi(e) = \phi(e^*).$$

\[\square\]

Theorem 4.3. Let $\phi$ denote a $\mathbb{Z}_2^k$-magic labeling of connected $G$ with weight $a \neq 0$. Then $G$ has an odd factor. Moreover, if $G$ has a non-empty bridge set, then $G$ has an odd factor containing every bridge.

Proof. Since $a \neq 0$, some coordinate of $a$ (with no loss of generality, the first coordinate) is equal to scaler 1. Consider the set $E_\phi$ of edges with labels under $\phi$ that have 1 in the first coordinate. Then for each vertex $v$, the number of such edges incident to $v$ is necessarily odd. Thus $E_\phi$ is an odd factor.

If $e^*$ is a bridge of $G$, then by Lemma 4.2, $\phi(e^*) = a$, and hence $e^* \in E_\phi$. \[\square\]

Theorem 4.4. Let $G$ be a connected graph with non-empty bridge set $B$. Then $G$ is $\mathbb{Z}_2^k$-magic for some $k \leq 4$ if and only if $G$ has an odd factor containing every bridge.

Proof. Let $\phi$ be a $\mathbb{Z}_2^k$-magic labeling of $G$. Then by Lemma 4.2, $\phi$ has non-zero weight $a$, and the result follows from Theorem 4.3.

Let $H$ be an odd factor of $G$ that contains every bridge. By Theorem 2.1, $G - B$ has an even edge-covering $\{G_1, G_2, \ldots, G_m\}$ for some $m$, $m \leq 3$. We construct a $\mathbb{Z}_2^{m+1}$-labeling $\phi$ of $G$. For $1 \leq i \leq m$, let the $i$th coordinate of $\phi(e)$ be the scaler 1 if $e \in E(G_i)$; the scaler 0 otherwise. Similarly, let the $(m + 1)$st coordinate of $\phi(e)$ be scaler 1 if $e \in E(H)$; scaler 0 otherwise. It is easily checked that $\phi$ is a $\mathbb{Z}_2^{m+1}$-labeling with weight $a$, where the first $m$ coordinates of $a$ are scaler 0 and the last coordinate of $a$ is scaler 1. \[\square\]

Theorem 4.5. Let $G$ be a connected graph with non-empty bridge set. Then $G$ is $\mathbb{Z}_2^k$-magic for some $k \leq 4$ if and only if for every bridge $e^*$, $G - e^*$ has two components each of odd order.

Proof. If $G$ is $\mathbb{Z}_2^k$-magic for some $k \leq 4$, the result follows from Lemma 4.2.

Assume that for every bridge $e^*$, $G - e^*$ has two components, each of odd order. We show that $G$ has an odd factor that contains every bridge of $G$, from which the result will follow by Theorem 4.4.

Since $G$ is necessarily of even order, then $G$ has an odd factor $F$ by Theorem 2.7. To see that $F$ contains every bridge of $G$, suppose to the contrary that $e'$ is a bridge of $G$ not in $E(F)$. Noting that $F$ is therefore an odd factor of $G - e'$ and that $G - e'$ has two components $G_1$ and $G_2$ each of odd order, we have the contradiction that the restriction of $F$ to $G_i$ is an odd factor on a graph of odd order. \[\square\]

Now suppose that for some $k$, $G$ is a connected $\mathbb{Z}_2^k$-magic graph with non-empty bridge set. By Lemma 4.2, the weight of the labeling is not 0. Hence, by Theorem 4.3. and 4.4, $G$ is $\mathbb{Z}_2^k$-magic for some $k \leq 4$. Thus by Corollary 3.3 we have.

Corollary 4.6. If $G$ is a connected (and either bridgeless or not) $\mathbb{Z}_2^k$-magic graph for some $k$, then $\tau(G) \leq 4$. \[\square\]

5. Closing remarks

The collection of connected graphs that are $\mathbb{Z}_2^k$-magic for some $k$ has a partitioning into three types:

Type 1: bridgeless graphs of even order.
Type 2: bridgeless graphs of odd order.
Type 3: graphs of even order having at least one bridge such that for any bridge $e$, $G - e$ has two components each of odd order.

By Theorems 2.1 and 3.1, all graphs of Type 1 are zero-sum $\mathbb{Z}_2^3$-magic. Additionally, since these graphs contain an odd factor by Theorem 2.7, then by the method of label construction in the proof of Theorem 4.4, $G$ has a $\mathbb{Z}_2^4$-magic labeling $\phi$ with weight $(0, 0, 0, 1)$.

By Theorem 4.1 and Corollary 3.3, each graph $G$ of Type 2 has the properties that for all $k \geq \tau(G)$, $G$ is $\mathbb{Z}_2^k$-magic and all $\mathbb{Z}_2^k$-magic labelings of $G$ have weight 0. Since $\tau(G) \leq 3$, then $G$ is $\mathbb{Z}_2^4$-magic and all $\mathbb{Z}_2^4$-magic labelings of $G$ have weight 0.
By the opening comments of Section 3, each graph $G$ of Type 3 is not zero-sum $\mathbb{Z}_2^k$-magic for any $k$. Moreover, by the method of label construction in the proof of Theorem 4.4, there exists a $\mathbb{Z}_2^4$-magic labeling of $G$ with weight $(0, 0, 0, 1)$. We thus have the following.

**Theorem 5.1.** Let $G$ be a (not necessarily connected) graph. Then $G$ is $\mathbb{Z}_2^k$-magic for some $k$ only when

(i) each component of $G$ is of Type 1 or Type 2, or
(ii) each component of $G$ is of Type 1 or Type 3.

Moreover, if (i) holds, $G$ is zero-sum $\mathbb{Z}_2^3$-magic. And if (ii) holds, then there is a $\mathbb{Z}_2^4$-magic labeling $\phi$ with weight $(0, 0, 0, 1)$. □

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**References**