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Endomorphisms, Derivations, and Polynomial Rings

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Given a ring A with a ring endomorphism σ and a σ-derivation δ (i.e., $\delta(ab) = \delta(a)b + \sigma(a)\,\delta(b)$), one can form a twisted polynomial ring in the noncommuting variable x, $A[x; \sigma, \delta]$, subject to the relation

$$xa = \sigma(a)x + \delta(a).$$

Such rings were studied by Ore [10], Jacobson [7], Amitsur [1], Jategaonkar [8], Carcanague [2] and others, usually in the case when A was a division ring, although Jategaonkar in particular considers more general coefficient rings but with $\delta = 0$.

In this paper we study the ring $A[x; \sigma, \delta]$ in the case when A is a semisimple Artinian ring and σ is any injective endomorphism. Our results may be summarized as establishing close connections with the case when A is a division ring.

We start by considering the nature of the pair σ, δ under the circumstances of the ring A being a finite product of rings or a matrix ring. This enables us to concentrate our attention on rings of the form $A[x; \sigma]$, i.e., $\delta = 0$, with A a finite product of division rings, say $A = \prod_{i=1}^{n} D_i$ where $\sigma(D_i) \subseteq D_{i+1}$ (letting $D_{n+1} = D_1$). This is a class of rings studied by Jategaonkar [8]. Such a ring too can be described in an alternative fashion; namely as a multiple idealizer subring of the $n \times n$ matrix ring over $D_n[x^n; \sigma^n]$. This provides a route for obtaining its properties. We illustrate this by describing its ideals.

Some of the results of this paper are used in [3] where some further properties of twisted polynomial rings are studied.

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1. Endomorphisms and Derivations of Products

Throughout this section we consider a triple $(A; \sigma, \delta)$ where A is a ring with 1, σ is an injective ring endomorphism of A, and δ is a σ-derivation of A (i.e., δ is an additive map from A to A such that $\delta(ab) = \delta(a)b + \sigma(a)\,\delta(b)$ for all $a, b \in A$). We will assume that there is a finite bound on the cardinality of sets of orthogonal idempotents of A. Thus $A = \prod_{i=1}^{m} A_i$, a finite product of indecomposable rings. We will let e_i denote the identity of A_i.

Lemma 1.1. *For each i there is a unique j such that $\sigma(A_i) \subseteq A_j$.*

Proof. Choose a set, X say, of orthogonal idempotents of A, having maximum cardinality. Clearly $\sum \{x \mid x \in X\} = 1$. Also, each $x \in X$ belongs to one of the A_i; for otherwise x would decompose as the sum of its projections, producing a larger set of orthogonal idempotents. We set $X_i = X \cap A_i$. It follows that $e_i = \sum \{x \mid x \in X_i\}$.

Note that σX is also a set of orthogonal idempotents, and has maximum cardinality. Therefore, as before, $\sigma 1 = \sum \{\sigma(x) \mid x \in X\} = 1$, and each member of σX belongs to one of the A_j. Suppose that

$$\sigma(X_i) \cap A_j \neq \varnothing.$$

Let

$$f = \sum \{x \mid x \in X_i,\ \sigma(x) \in A_j\},$$

$$g = \sum \{x \mid x \in X_i,\ \sigma(x) \notin A_j\}.$$

Then $f + g = e_i$. Also

$$\sigma(f)\,\sigma(A_i)\,\sigma(g) \subseteq \sigma(f)\,A\sigma(g) \subseteq A_j \cap \prod_{k \neq j} A_k = 0.$$

Since σ is injective, $fA_ig = 0$. Similarly $gA_if = 0$. It follows that f, g are central idempotents and $A_i = fA_i \oplus gA_i$. However A_i is indecomposable and $f \neq 0$. Therefore $g = 0$, and so $\sigma(X_i) \subseteq A_j$. Finally, if $a \in A_i$, then $a = e_ia$ and thus $\sigma(a) = \sigma(e_i)\,\sigma(a) \in A_j$. Hence $\sigma(A_i) \subseteq A_j$. ∎

Notation. This result shows that σ induces a permutation of the index set $\{1, \ldots, m\}$. We denote this permutation by ρ. Thus $\sigma(A_i) \subseteq A_{\rho(i)}$.

Lemma 1.2. $\delta(A_i) \subseteq A_i + A_{\rho(i)}$.

Proof. If $a \in A_i$, then $a = e_ia$. Therefore

$$\delta(a) = \delta(e_i)a + \sigma(e_i)\,\delta(a) \in A_i + A_{\rho(i)}. \quad ∎$$

These results combine to give

THEOREM 1.3. *Suppose that $\gamma_1, \ldots, \gamma_k$ are the orbits of ρ. Let*

$$B_j = \prod \{A_i \mid i \in \gamma_j\}$$

and let σ_j, δ_j be the restriction to B_j of σ, δ. Then σ_j is an injective endomorphism of B_j, and δ_j is a σ_j-derivation and

$$(A; \sigma, \delta) = \prod_{j=1}^{k} (B_j\, ; \sigma_j\, , \delta_j). \quad \blacksquare$$

We restrict our attention now to the indecomposable case. Thus, after re-ordering if necessary, we can suppose that ρ is the cycle $(1\ 2 \cdots m)$. Then $\sigma(e_i) = e_{i+1}$, with the convention that $e_{m+1} = e_1$.

We recall that, if $b \in A$, then there is a derivation δ_b given by

$$\delta_b(a) = ba - \sigma(a)b.$$

This is called an *inner σ-derivation* of A.

LEMMA 1.4. *If $m > 1$, then δ is an inner σ-derivation.*

Proof. Let $a \in A$. Then $ae_i = e_ia = e_iae_i$, and so

$$\begin{aligned} \delta(e_ia) &= \delta(e_iae_i) \\ &= \delta(e_i)\, ae_i + \sigma(e_i)\, \delta(a)e_i + \sigma(e_i)\, \sigma(a)\, \delta(e_i) \\ &= \delta(e_i)\, e_ia + \delta(a)\, e_{i+1}e_i + \sigma(a)\, e_{i+1}\delta(e_i). \\ \therefore\ \delta(a) &= \sum_i \delta(e_ia) \\ &= \left(\sum_i \delta(e_i)e_i\right) a + \sigma(a) \left(\sum_i e_{i+1}\delta(e_i)\right). \end{aligned}$$

If we let $a = 1$, we see that

$$0 = \delta(1) = \sum \delta(e_i)e_i + \sum e_{i+1}\delta(e_i).$$

Thus, letting $b = \sum \delta(e_i)\, e_i = -\sum e_{i+1}\delta(e_i)$ we have that

$$\delta(a) = ba - \sigma(a)b$$

and so δ is an inner σ-derivation. $\blacksquare$

This result has also been proved independently by H. Wexler in a paper to appear in the seminaire d'algebre (Aribaud, Dubreil, M. P. Malliavin) (Paris 1977).

2. Endomorphisms and Derivations of Matrices

We continue to study the triple (A, σ, δ). We suppose that the conditions on A and the A_i are as in Section 1, and that ρ is the cycle $(1\ 2 \cdots m)$. We will also impose an extra condition. We will be considering the case when A_1 is an $n \times n$ matrix ring; say $A_1 = M_n(D_1)$. If E_1 is a D_1-module such that $E_1^{(n)} \simeq D_1^{(n)}$, we will require that $E_1 \simeq D_1$. This condition holds, for example, if D_1 is Artinian, or if D_1 is a semifir [4].

Lemma 2.1. *Let A_1 be an $n \times n$ matrix ring. Then so too are $A_2, \ldots, A_m$.*

Proof. Let $\{e_{ij}\}$ be a complete set of matrix units for A_1. Then $\{\sigma^{k-1}(e_{ij})\}$ is a complete set of matrix units for A_k. ▮

Lemma 2.2. *Let $\{e_{ij}\}$ and $\{e'_{ij}\}$ be two complete sets of $n \times n$ matrix units for A_1. Then there is an inner automorphism α_1 of A_1 such that $e_{ij} = \alpha_1(e'_{ij})$.*

Proof. We write $A = M_n(D_1) \simeq \operatorname{End}(D_1^{(n)})$ with respect to $\{e_{ij}\}$. We choose a basis $v_1, \ldots, v_n$ for $D_1^{(n)}$. Now consider the set $\{e'_{ij}\}$. It is clear that

$$e'_{ii}D_1^{(n)} \simeq e'_{jj}D_1^{(n)} \simeq E_1 \quad \text{say;}$$

and then $D_1^{(n)} = \sum_i e'_{ii}D_1^{(n)} \simeq E_1^{(n)}$.

By assumption, $E_1 \simeq D_1$. Thus we obtain a second basis for $D_1^{(n)}$; say $v'_1, \ldots, v'_n$. Note that

$$e_{ij}(v_k) = \delta_{jk}v_i \qquad \text{and} \qquad e'_{ij}v'_k = \delta_{jk}v'_i$$

where δ_{jk} is the Kronecker symbol. Let u be the automorphism of $D_1^{(n)}$ given by $u(v'_i) = v_i$, $i = 1, \ldots, n$. Then $e_{ij} = ue'_{ij}u^{-1}$ for all i, j. We set α_1 to be the inner automorphism of A_1 given by u. ▮

Lemma 2.3. *Let $\{e_{ij}\}$ be a complete set of $n \times n$ matrix units for A_1. Then there is an inner automorphism α of A such that, for all i, j,*

$$\alpha(\sigma^k(e_{ij})) = \sigma^k(e_{ij}) \qquad \textit{for} \quad k = 1, 2, \ldots, m-1,$$
$$\alpha(\sigma^m(e_{ij})) = e_{ij}.$$

Proof. Define α to be 1 on $A_2, \ldots, A_m$ and to be α_1 on A_1, where α_1 is obtained, as described in Lemma 2.2, taking $e'_{ij} = \sigma^m(e_{ij})$. ▮

Theorem 2.4. *Let A_1 be an $n \times n$ matrix ring. Then $A \simeq M_n(D)$ with $D = \prod_{i=1}^m D_i$ and $M_n(D_i) \simeq A_i$. Moreover there is an inner automorphism σ_2 of A and an injective endomorphism σ_1 of D such that $\sigma = \sigma_2\hat{\sigma}_1$ where $\hat{\sigma}_1$ denotes the natural extension of σ_1 to $M_n(D)$.*

Proof. If we write

$$f_{ij} = e_{ij} + \sigma(e_{ij}) + \cdots + \sigma^{m-1}(e_{ij}),$$

then $\{f_{ij}\}$ is a complete set of $n \times n$ matrix units for A. Then $A \simeq M_n(D)$, and $A_i \simeq M_n(D_i)$, and $D = \prod D_i$. Let α be the inner automorphism described in Lemma 2.3. Then, for all i, j,

$$\alpha\sigma(\sigma^k(e_{ij})) = \sigma^{k+1}(e_{ij}) \qquad \text{for} \quad k \neq m-1,$$
$$\alpha\sigma(\sigma^{m-1}(e_{ij})) = \alpha\sigma^m(e_{ij}) = e_{ij}.$$

Thus $\alpha\sigma(f_{ij}) = f_{ij}$. It follows that $\alpha\sigma$ is the extension to $M_n(D)$ of an injective endomorphism, σ_1 say, of D. Then $\alpha\sigma = \hat{\sigma}_1$ and so $\sigma = \sigma_2\hat{\sigma}_1$ where $\sigma_2 = \alpha^{-1}$ is inner. ▮

It remains to consider the σ-derivation δ in these circumstances. If $m > 1$ then, as shown in Lemma 1.3, δ is inner. So we will consider only the case when $m = 1$. In the light of Theorem 2.4, we will suppose also that $\sigma = \hat{\sigma}_1$, the extension of an injective endomorphism σ_1 of D.

THEOREM 2.5. *Let δ be a $\hat{\sigma}_1$-derivation of $M_n(D)$. Then $\delta = \hat{\delta}_1 + \delta_2$ where δ_2 is an inner $\hat{\sigma}_1$-derivation and δ_1 is a σ_1-derivation of D.*

Proof. $1 = \sum_i e_{i1}e_{1i}$ and therefore

$$0 = \delta(1) = \sum \delta(e_{i1})\, e_{1i} + \sum \hat{\sigma}_1(e_{i1})\, \delta(e_{1i})$$
$$= \sum \delta(e_{i1})\, e_{1i} + \sum e_{i1}\delta(e_{1i}).$$

Hence

$$0 = \left(\sum \delta(e_{i1})e_{1i} + \sum e_{i1}\delta(e_{1i})\right) e_{hk}$$

$$= \delta(e_{h1})e_{1k} + \left(\sum e_{i1}\delta(e_{1i})\right) e_{hk}.$$

Similarly

$$0 = e_{hk}\left(\sum \delta(e_{i1})e_{1i}\right) + e_{h1}\delta(e_{1k}).$$

Now $\delta(e_{hk}) = \delta(e_{h1}e_{1k}) = \delta(e_{h1})\, e_{1k} + e_{h1}\delta(e_{1k})$ and so

$$\delta(e_{hk}) = -\left(\sum e_{i1}\delta(e_{1i})\right) e_{hk} - e_{hk}\left(\sum \delta(e_{i1})e_{1i}\right).$$

Let $b = \sum \delta(e_{i1})\, e_{1i} = -\sum e_{i1}\delta(e_{1i})$. Then we see that

$$\delta(e_{hk}) = be_{hk} - e_{hk}b \qquad \text{for all } h, k.$$

Thus, if we define $\delta_2(a) = ba - \hat{\sigma}_1(a)b$, then

$$\delta_2(e_{hk}) = \delta(e_{hk}),$$

and so $\delta - \delta_2$ is the extension to $M_n(D)$ of a σ_1-derivation, δ_1 say, of D. ∎

Note. This proof is based on an argument of Kawada [9] pointed out to us by D. Jordan.

3. Twisted Polynomial Rings

Given a triple (A, σ, δ) as before, one can construct a twisted polynomial ring $A[x; \sigma, \delta]$ in which the commutation law is

$$xa = \sigma(a)x + \delta(a).$$

We will consider the structure of this ring in the case when A is semisimple Artinian, σ is an injective endomorphism and δ is a σ-derivation. We let ρ denote the permutation induced by σ on the simple Artinian factors of A.

THEOREM 3.1. *The ring $A[x; \sigma, \delta]$ decomposes as a direct product $B_j[x; \sigma, \delta]$ where each B_j comprises the product of the simple Artinian factors belonging to an orbit of ρ.*

Proof. This is clear from Theorem 1.3. ∎

We can therefore, without serious loss, restrict our attention to the case when ρ is a cycle on the simple Artinian factors $A_1, \ldots, A_m$ of A. By Lemma 2.1 these will all be $n \times n$ matrix rings, for some n, over division rings $D_1, \ldots, D_m$ say. We let $D = \prod_{i=1}^m D_i$.

THEOREM 3.2. *Suppose that $A_i \simeq M_n(D_i)$ for $i = 1, \ldots, m$. Then there is an injective endomorphism σ of $D = \prod_{i=1}^m D_i$ and a σ_1-derivation δ_1 of D, and elements $u, v \in A$, u being a unit, such that*

$$A[x; \sigma, \delta] = A[y; \hat{\sigma}_1, \hat{\delta}_1] = M_n(D[y; \sigma_1, \delta_1])$$

where $y = ux + v$. Moreover, if $m > 1$, we can choose δ_1 to be trivial.

Proof. We see, by Theorem 2.4 that there is an inner automorphism σ_2 of A and an injective endomorphism σ_1 of D such that $\sigma = \sigma_2\hat{\sigma}_1$. Say $\sigma_2(a) = u^{-1}au$. We set $x_1 = ux$. Then we have

$$A[x, \sigma, \delta] = A[x_1; \hat{\sigma}_1, \delta']$$

where $\delta' = u\delta$ and δ' is a $\hat{\sigma}_1$-derivation of A. But then, by Theorem 2.5, $\delta' = \hat{\delta}_1 + \delta_2$, where δ_2 is an inner $\hat{\sigma}_1$-derivation (say $\delta_2(a) = va - \sigma_1(a)v$) and δ_1 is a σ_1-derivation of D. Therefore

$$A[x; \sigma, \delta] = A[x_1 ; \hat{\sigma}_1, \delta'] = A[y, \hat{\sigma}_1, \hat{\delta}_1]$$
$$= M_n(D[y; \sigma_1, \delta_1]),$$

as claimed.

Finally, if $m > 1$ then, by Lemma 1.4, δ' is an inner $\hat{\sigma}_1$-derivation, and so we can arrange that δ_1 is trivial. ∎

This effectively reduces the study of $A[x; \sigma, \delta]$ to the case when A is the product of m division rings which are cycled by σ. The case when $m = 1$, or $m \neq 1$, are rather different to each other so we will discuss them separately.

4. One Division Ring

In this section we will describe briefly the ideals of the ring $R = D[x; \sigma, \delta]$ with D a division ring, σ an endomorphism and δ a derivation. Our main interest, for the following section, is in the case when $\delta = 0$. This case has been discussed by Jacobson [7], and the general case by Carcanague [2]. However, we will sketch proofs of some of the facts we will be needing.

By Euclid's algorithm one sees that $R = D[x; \sigma, \delta]$ is a principal left ideal domain (and it is not hard to deduce that R is also a principal right ideal domain if and only if σ is an automorphism). Of course, each non zero left ideal I is generated by the monic polynomial of least degree belonging to it. That makes it plain that R/I is a finite D-vector space and hence, as an R-module, has finite composition length. Thus, each proper factor ring of R is a left Artinian principal left ideal ring. The theory of such rings [7, pp. 75–76] shows that each ideal is a commutative product of maximal ideals. Hence

Proposition 4.1. *Each nonzero ideal of R is a unique commutative product of maximal ideals.*

Suppose for the moment that $\delta = 0$. Then it is clear that Rx is a maximal ideal. Let Rp be another maximal ideal, with p a polynomial of degree n say. If we choose p of the form $p = 1 + qx$ it is an elementary calculation that p is central. Moreover, if the leading coefficient of p is $u \in D$, then $\sigma^n(a) = u^{-1}au$ for $a \in D$. This demonstrates

Proposition 4.2 (Jacobson). *If $\delta = 0$, each ideal of R has the form Rpx^m with p central. Moreover, unless some power of σ is an inner automorphism, the nonzero ideals all have the form Rx^m.* ∎

We note that Carcanague proves that, in the case when $\delta \neq 0$, if q is a monic polynomial of minimal degree such that Rq is a proper ideal, then the ideals of R have the form Rpq^m with p central. And again, if R has more than one maximal ideal, then σ must be an automorphism some power of which is inner.

5. A Cycle of Division Rings

In this section we consider the type of ring described in Theorem 3.2. Thus $D = \prod_{i=1}^{n} D_i$ is the direct product of division rings $D_1, \ldots, D_n$, and σ is an injective endomorphism of D such that $\sigma(D_i) \subseteq D_{i+1}$, $\sigma(D_n) \subseteq D_1$. As shown in Theorem 3.2, any σ-derivation would be inner. So we consider only the ring $R = D[x; \sigma]$. Rings of this type have been studied before by Jategaonkar [8]. We wish to describe some further facts and an alternative viewpoint. We let e_i denote the identity element of D_i.

It is well known that R is a left Noetherian left hereditary prime ring (and R is right Noetherian and right hereditary if and only if σ is an automorphism). Thus R has a left quotient ring $\underset{\sim}{Q}$ which is simple Artinian. We recall [7, Chap. 6] that if I is an ideal of R then

$$O_r(I) = \{q \in Q \mid Iq \subseteq I\} \simeq \operatorname{End}({}_R I).$$

Lemma 5.1. *Let $I = Re_nR$ and $S = O_r(I)$. Then*

(i) $I = D_n + (D_n + D_1)x + (D_n + D_1 + D_2)x^2 + \cdots + (D_n + D_1 + \cdots + D_{n-1})x^{n-2} + Rx^{n-1}$.

(ii) $S = x^{-(n-1)}(D_n + (D_n + \sigma D_n)x + \cdots + (D_n + \sigma D_n + \cdots + \sigma^{n-1} D_n)x^{n-1} + (D_n + \sigma D_n + \cdots + \sigma^{n-1} D_n)x^n + \cdots)$.

Proof. (i) This is an easy computation.

(ii) From (i), $I \supseteq Rx^{n-1}$, and so

$$q \in S \Rightarrow Iq \subseteq R \Rightarrow Rx^{n-1}q \subseteq R \Rightarrow x^{n-1}q \in R \Rightarrow q \in x^{-(n-1)}R.$$

Therefore $S \subseteq x^{-(n-1)}R$. Bearing in mind the fact that I is the sum of homogeneous subsets, it is enough to check homogeneous subsets of $x^{-(n-1)}R$. The result follows after an easy, but lengthy, computation. ▮

Proposition 5.2. $S = O_r(Re_nR) \simeq M_n(D_n[x^n; \sigma^n])$.

Proof. First we note that S contains the elements displayed in the following $n \times n$ array:

$$\begin{array}{ccccc} e_1 & e_1x^{-1}e_2 & e_1x^{-2}e_3 & \cdots & e_1x^{-(n-1)}e_n \\ e_2xe_1 & e_2 & e_2x^{-1}e_3 & \cdots & e_2x^{-(n-2)}e_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ e_nx^{n-1}e & e_nx^{n-2}e_2 & e_nx^{n-3}e_3 & \cdots & e_n \end{array}$$

These form a complete set of $n \times n$ matrix units. Moreover $e_nSe_n = e_nRe_n = D_n[x^n; \sigma^n]$. Therefore

$$S \simeq M_n(e_nSe_n) = M_n(D_n[x^n; \sigma^n]). \quad \blacksquare$$

We note that $I = Re_nR = Re_n \oplus Re_nx \oplus \cdots \oplus Re_nx^{n-1}$. Now this is a decomposition of I as a direct sum of n isomorphic left ideals. Therefore

$$\mathrm{End}({}_RI) \simeq M_n(\mathrm{End}\, Re_n) \simeq M_n(e_nRe_n).$$

Moreover $S = O_r(I) \simeq \mathrm{End}({}_RI)$, with the elements of S acting via right multiplication. This provides an alternative route to the description of S.

There is a converse result to Proposition 5.2 as follows.

PROPOSITION 5.3. *Let D_n be a division ring with an endomorphism τ and let $S = M_n(D_n[y; \tau])$. Then there is a ring R of the form $D[x; \sigma]$ with D the direct sum of n division rings and σ an injective endomorphism, such that $S \simeq O_r(Re_nR)$.*

Proof. Let $D = D_1 \oplus \cdots \oplus D_n$ with $D_i \simeq D_n$ and define $\sigma: D \to D$ by $\sigma(d_1, d_2, \ldots, d_n) = (\sigma d_n, d_1, d_2, \ldots, d_{n-1})$. Let $R = D[x; \sigma]$. Then, by Proposition 5.2,

$$O_r(Re_nR) \simeq M_n(D_n[x^n; \sigma^n]).$$

Now $\sigma^n = \tau$ on D_n. Therefore, setting $y = x^n$, we have

$$O_r(Re_nR) \simeq S. \quad \blacksquare$$

We now consider the ideals of $R = D[x; \sigma]$ in the special case when σ is not an automorphism.

PROPOSITION 5.4. *If σ is not an automorphism and B is a nonzero ideal of R then, for some m,*

$$Rx^m \subseteq B \subseteq Rx^{m-n+1}.$$

Proof. It is clear that e_nBe_n is a nonzero ideal of e_nRe_n. Now $e_nRe_n \simeq D_n[x^n; \sigma^n]$ and, of course, σ^n is not an automorphism of D_n. Thus e_nBe_n contains a power of $e_nx^ne_n$. The same is true of e_iBe_i for $i = 1, \ldots, n-1$ and thus B contains a power of x^n.

Now we choose m minimal such that $Rx^m \subseteq B$. Let $b \in B$; $b \equiv a_ix^i + \cdots + a_{m-1}x^{m-1} \pmod{Rx^m}$, with $a_i \neq 0$. By multiplying by an element of D, we can arrange that $a_i = e_j$ for some j. However,

$$x^ke_jx^ix^{n-k-1} = e_{j+k}x^{i+n-1}.$$

Hence there is a monic polynomial of degree $i + n - 1$ belonging to B. If $i + n - 1 < m$, one could deduce that $x^{m-1} \in B$, contradicting the minimality of m. Hence $i + n - 1 \geqslant m$ and $B \subseteq Rx^{m-n+1}$. $\blacksquare$

Next we aim to describe in more detail the relationship between R and S. We will use the notion of a ***multiple idealizer subring.*** If U is a ring and $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_k$ is a chain of right ideals of U then

$$V = \mathbf{I}(A_1, A_2, ..., A_k) = \{u \in U \mid uA_i \subseteq A_i,\ i = 1, ..., k\}$$

is called the multiple idealizer of U at that chain (see [6, 13]). The rings U and V are particularly closely linked when A_k is semimaximal (i.e., when U/A_k is a semisimple module). Similar remarks apply to left ideals B_j.

Returning now to the rings R and S, we let

$$f_i = e_{i+1} + e_{i+2} + \cdots + e_n, \qquad g_i = e_1 + e_2 + \cdots + e_i,$$

and $A_i = f_iS + Re_nR$, and $B_{n-i} = Sg_i + Re_1R$. Note that $A_{n-1} = Re_nR$, $B_{n-1} = Re_1R$.

THEOREM 5.5. (i) $R = \mathbf{I}(A_1, A_2, ..., A_{n-1})$.

(ii) *If σ is an automorphism, then A_{n-1} and B_{n-1} are semimaximal in S, and* $R = \mathbf{I}(B_1, B_2, ..., B_{n-1})$.

(iii) *R is a multiple idealizer from S at a chain of left ideals if and only if $\sigma^{n-1}D_1 = D_n$; and in that case $R = \mathbf{I}(B_1, ..., B_{n-1})$.*

Proof. (i) Let $A_0 = S$. Then we will prove by induction on k that $\mathbf{I}(A_0, A_1, ..., A_k) = A_k + R$. This is obvious if $k = 0$, and is what we wish to establish if $k = n - 1$. We suppose it holds for $k - 1$. We note that one can compute that $\mathbf{I}(A_0, A_1, ..., A_k) \supseteq A_k + R$. Now $A_{k-1} = e_kS + A_k$. Thus we need only prove that, if $e_ksA_k \subseteq A_k$, then $e_ks \in A_k + R$. However

$$e_ksA_k \subseteq A_k \Rightarrow e_ksA_k \subseteq e_kA_k \Rightarrow e_ksA_k \subseteq R.$$

Consider the homogeneous components of e_kS; namely

$$x^{-(n-1)}\sigma^{k-1}D_nx^{k-1},\ x^{-(n-1)}\sigma^{k-1}D_nx^k,\ x^{-(n-1)}\sigma^{k-1}D_nx^{k+1}, ...$$

Multiply each, on the right, respectively by the elements

$$e_n,\ e_{n-1}, ..., e_{k+1},\ e_kx^n,\ e_{k-1}x^n, ..., e_1x^n,\ e_nx^n, ..., e_1x^n,\ e_nx^n, ...$$

of A_k. The fact that the first $n - k$ products have negative degrees shows that these homogeneous components contain no elements of $\mathbf{I}(A_k)$. And the other products show that the elements of the remaining components belong to $\mathbf{I}(A_k)$ only if they already belong to R.

(ii) If σ is an automorphism, then $Sx^n = x^nS$ is a maximal ideal of S, the factor ring being simple Artinian. Moreover, $A_{n-1} \supseteq x^nS$ and $B_{n-1} \supseteq x^nS$ and so both are semimaximal. By (i), $R = \mathbf{I}(A_1, ..., A_{n-1})$ and so, using [6] or [13] it follows immediately that $R = \mathbf{I}(B_1, ..., B_{n-1})$.

(iii) We start by considering under what circumstances $Sx^l \subseteq R$. By Lemma 5.1,

$$Sx^l = x^{-(n-1)}(D_n + (D_n + \sigma D_n)x + \cdots + (D_n + \sigma D_n + \cdots + \sigma^{n-1}D_n)\, x^{n-1} + \cdots)\, x^l.$$

Checking the homogeneous components one by one, we see that $Sx^l \subseteq R$ if and only if $l \geqslant n - 1$ and $D_n \subseteq \sigma^{n-1}D$, $D_n + \sigma D_n \subseteq \sigma^{n-1}D, \ldots, D_n + \sigma D_n + \cdots + \sigma^{n-1}D_n \subseteq \sigma^{n-1}D$. These latter conditions are equivalent to the condition that $l \geqslant n - 1$ and $D_n = \sigma^{n-1}D_1$.

Suppose now that R is a multiple idealizer from S. If σ is an automorphism then, of course, $\sigma^{n-1}D_1 = D_n$ and the result follows from (ii). If σ is not an automorphism then, by Proposition 5.4, we know that each ideal of R contains a power of x. However, if $R = \mathbf{I}(C_1, \ldots, C_t)$ then C_t is a left ideal of S and an ideal of R. Thus $Sx^l \subseteq R$ for some l. Hence $\sigma^{n-1}D_1 = D_n$ by the preceding paragraph.

Conversely, suppose $\sigma^{n-1}D_1 = D_n$. Then $Sx^n \subseteq R$. However Sx^n is a maximal ideal of S, with S/Sx^n being simple Artinian. Moreover $Sx^n \subseteq Re_1R$. Thus $B_1, \ldots, B_{n-1}$ are semimaximal left ideals of S and, calculating modulo Sx^n, it is easily verified that $R = \mathbf{I}(B_1, \ldots, B_{n-1})$. ▌

Next we consider the ideal structure of R, starting with the collection of maximal ideals.

THEOREM 5.6. *Each maximal ideal M of R is either of the form $M = R(1 - e_i) + Rx$ or of the form $M = X \cap R$ where X is a maximal ideal of S other than Sx^n.*

Proof. First suppose σ is not an automorphism. By Proposition 5.4, each ideal of R contains a power of x. Thus each maximal ideal M contains Rx. Therefore M has the form $R(1 - e_i) + Rx$.

Second, suppose σ is an automorphism. By Theorem 5.5(ii), R is a multiple idealizer from S at semimaximal left ideals containing Sx^n. Hence, by [12, Proposition 2.6], the simple left R-modules are of one of the following types;

(i) simple left S-modules A not annihilated by Sx^n,

(ii) subfactors B of the left R-module $S/Sx^n = S/x^nS$.

Now each maximal ideal M of R arises as the annihilator of some unfaithful simple module. If A is unfaithful over R, and hence over S, then $M = \operatorname{ann}_R A = \operatorname{ann}_S A \cap R = X \cap R$. As for B, note that $x^nB = 0$. Thus $\operatorname{ann}_R B \supseteq Rx$. Hence $\operatorname{ann}_R B = R(1 - e_i) + Rx$ for some i. ▌

THEOREM 5.7. *Each ideal I of R can be written uniquely in the form $I = AM_1^{n_1} \cdots M_t^{n_t}$ where $M_i = X_i \cap R$, with X_i a maximal ideal of S other than Sx^n, and where A is an ideal such that $Rx^l \supseteq A \supseteq Rx^{l+n-1}$ for some l.*

Proof. If σ is not an automorphism, this is clear from Proposition 5.4. If σ is an automorphism, then R is a hereditary Noetherian prime ring and, by Theorem 5.6, has only finitely many idempotent maximal ideals (for the ideal $M = X \cap R$ cannot be idempotent since X is an ideal of the principal ideal ring S and so $\bigcap X^m = 0$). By [5, Theorems 2.9, 4.2] each ideal of R is the unique product of an eventual idempotent ideal and some maximal invertible ideals. The latter all have either the form $M_i = X_i \cap R$, or else equal Rx. The former all contain Rx^{n-1} by [5, Proposition 4.3]. The result now follows. ∎

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