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A pathway to matrix-variate gamma and normal densities

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Abstract

A general real matrix-variate probability model is introduced here, which covers almost all real matrix-variate densities used in multivariate statistical analysis. Through the new density introduced here, a pathway is created to go from matrix-variate type-1 beta to matrix-variate type-2 beta to matrix-variate gamma to matrix-variate Gaussian or normal densities. Other densities such as extended matrix-variate Student t, F, Cauchy density will also come in as particular cases. Connections to the distributions of quadratic forms and generalized quadratic forms in the new matrix are established. The present day analysis of these problems is mainly confined to Gaussian random variables. Thus, through the new distribution, all these theories are extended. Connections to certain geometrical probability problems, such as the distribution of the volume of a random parallelotope in Euclidean space, is also established.

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1. Introduction

Let $X = (x_{ij}),\ i = 1, \ldots p,\ j = 1, \ldots, r,\ r \geqslant p$, of rank p and of real scalar variables x_{ij}'s for all i and j, subject to the condition that the rank of X is p, having the density $f(X)$, where $f(X)$ is a scalar function of X given by

$$f(X) = c|A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\alpha}|I - a(1-q)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\frac{\beta}{1-q}} \tag{1.1}$$

for $A = A' > 0$ and $p \times p$, $B = B' > 0$ and $r \times r$, a, β, q scalars, $a > 0$, $\beta > 0$, $I - a(1-q)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}} > 0$, where A and B are free of the elements in X and c is the normalizing constant. For convenience let $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ denote the real symmetric positive definite square roots of A and B respectively. A prime denotes the transpose, $|(\cdot)|$ denotes the determinant of $(\cdot)$, I is the identity matrix, $(\cdot) > 0$ means that the real symmetric matrix $(\cdot)$ is positive definite. Also $\mathrm{tr}(\cdot)$ will denote the trace of $(\cdot)$ and $\Re(\cdot)$ will denote the real part of $(\cdot)$. The normalizing constant c can be evaluated by using the following transformations. Let

$$Y = A^{\frac{1}{2}}XB^{\frac{1}{2}} \Rightarrow \mathrm{d}Y = |A|^{\frac{r}{2}}|B|^{\frac{p}{2}}\,\mathrm{d}X$$

by using Theorem 1.18 of [3]. Let

$$U = YY' \Rightarrow \mathrm{d}Y = \frac{\pi^{\frac{rp}{2}}}{\Gamma_p\left(\frac{r}{2}\right)}|U|^{\frac{r}{2}-\frac{p+1}{2}}\,\mathrm{d}U$$

by using Theorem 2.16 of [3], where for example,

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}}\Gamma(\alpha)\Gamma\left(\alpha - \frac{1}{2}\right)\cdots\Gamma\left(\alpha - \frac{p-1}{2}\right), \quad \alpha > \frac{p-1}{2}, \tag{1.2}$$

taking α as real, and if complex the condition is $\Re(\alpha) > \frac{p-1}{2}$. Let

$$V = a(1-q)U \Rightarrow \mathrm{d}V = [a(1-q)]^{\frac{p(p+1)}{2}}\,\mathrm{d}U$$

by using Theorem 1.20 of [3]. Then

$$\begin{aligned} 1 = \int_X f(X)\,\mathrm{d}X &= \frac{c}{|A|^{\frac{r}{2}}|B|^{\frac{p}{2}}}\int_Y |YY'|^{\alpha}|I - a(1-q)YY'|^{\frac{\beta}{1-q}}\,\mathrm{d}Y \\ &= \frac{\pi^{\frac{rp}{2}}}{\Gamma_p\left(\frac{r}{2}\right)|A|^{\frac{r}{2}}|B|^{\frac{p}{2}}}\int_u |U|^{\alpha+\frac{r}{2}-\frac{p+1}{2}}|I - a(1-q)U|^{\frac{\beta}{1-q}}\,\mathrm{d}U. \end{aligned} \tag{1.3}$$

At this stage we can consider three possibilities: (i) $q < 1$, (ii) $q > 1$, (iii) $q = 1$. Let us consider these one by one.

Case (i): $q < 1$.

Then $a(1-q) > 0$ and then by making the transformation $V = a(1-q)U$ we have

$$c^{-1} = \frac{\pi^{\frac{rp}{2}}}{\Gamma_p\left(\frac{r}{2}\right)|A|^{\frac{r}{2}}|B|^{\frac{p}{2}}[a(1-q)]^{p\left(\alpha+\frac{r}{2}\right)}} \int_V |V|^{\alpha+\frac{r}{2}-\frac{p+1}{2}}|I-V|^{\frac{\beta}{1-q}}\,\mathrm{d}V. \tag{1.4}$$

Now, evaluating the integral in (1.4) by using a matrix-variate type-1 beta, see Section 5.1.4 of [3], we have

$$c^{-1} = \frac{\pi^{\frac{rp}{2}}}{\Gamma_p\left(\frac{r}{2}\right)|A|^{\frac{r}{2}}|B|^{\frac{p}{2}}[a(1-q)]^{p\left(\alpha+\frac{r}{2}\right)}} \frac{\Gamma_p\left(\alpha+\frac{r}{2}\right)\Gamma_p\left(\frac{\beta}{1-q}+\frac{p+1}{2}\right)}{\Gamma_p\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+\frac{p+1}{2}\right)} \tag{1.5}$$

for $\alpha+\frac{r}{2}>\frac{p-1}{2}$. We will assume the parameters to be real for convenience.

Case (ii): $q>1$.

In this case write $1-q=-(q-1)$ so that $q-1>0$. Then in (1.3)

$$|I-a(1-q)U|^{\frac{\beta}{1-q}} = |I+a(q-1)U|^{-\frac{\beta}{q-1}} \tag{1.6}$$

and then make the transformation $V=a(q-1)U$. Then

$$c^{-1} = \frac{\pi^{\frac{rp}{2}}}{\Gamma_p\left(\frac{r}{2}\right)|A|^{\frac{r}{2}}|B|^{\frac{p}{2}}[a(q-1)]^{p\left(\alpha+\frac{r}{2}\right)}} \int_V |V|^{\alpha+\frac{r}{2}-\frac{p+1}{2}}|I+V|^{-\frac{\beta}{q-1}}\,\mathrm{d}V.$$

Evaluating the integral by using a matrix-variate type-2 beta integral, see Section 5.1.4 of [3], we have the following:

$$c^{-1} = \frac{\pi^{\frac{rp}{2}}}{\Gamma_p\left(\frac{r}{2}\right)|A|^{\frac{r}{2}}|B|^{\frac{p}{2}}[a(q-1)]^{p\left(\alpha+\frac{r}{2}\right)}} \frac{\Gamma_p\left(\alpha+\frac{r}{2}\right)\Gamma_p\left(\frac{\beta}{q-1}-\alpha-\frac{r}{2}\right)}{\Gamma_p\left(\frac{\beta}{q-1}\right)} \tag{1.7}$$

for $\alpha+\frac{r}{2}>\frac{p-1}{2}$, $\frac{\beta}{q-1}-\alpha-\frac{r}{2}>\frac{p-1}{2}$.

Case (iii): $q=1$.

Irrespective of whether q approaches 1 from the left or from the right it can be shown that the determinant containing q in (1.3) and (1.6) has the following form, which will be stated as a lemma:

Lemma 1.1

$$\lim_{q\to 1}|I-a(1-q)U|^{\frac{\beta}{1-q}} = \mathrm{e}^{-a\beta\,\mathrm{tr}(U)}.$$

This result can be seen by observing the following: For a real symmetric positive definite matrix U there exists a matrix Q such that

$$QQ' = I, \quad Q'Q = I, \quad Q'UQ = \mathrm{diag}(\lambda_1, \ldots, \lambda_p), \quad \lambda_j > 0, \quad j = 1, \ldots, p \tag{1.8}$$

where $\mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ denotes a diagonal matrix with the diagonal elements $\lambda_1, \ldots, \lambda_p$. Then

$$\begin{aligned} |I - a(1-q)U| &= |I - a(1-q)QQ'UQQ'| \\ &= |I - a(1-q)Q'UQ| = |I - a(1-q)\mathrm{diag}(\lambda_1, \ldots, \lambda_p)| \\ &= \prod_{j=1}^{p}(1 - a(1-q)\lambda_j). \end{aligned}$$

But

$$\lim_{q \to 1}(1 - a(1-q)\lambda_j)^{\frac{\beta}{1-q}} = \mathrm{e}^{-a\beta\lambda_j}.$$

Then

$$\lim_{q \to 1}|I - a(1-q)U|^{\frac{\beta}{1-q}} = \mathrm{e}^{-a\beta\left(\sum_{j=1}^{p}\lambda_j\right)} = \mathrm{e}^{-a\beta\mathrm{tr}(U)}$$

which establishes the result. Hence in case (iii)

$$\begin{aligned} c^{-1} &= \frac{\pi^{\frac{rp}{2}}}{\Gamma_p\left(\frac{r}{2}\right)|A|^{\frac{r}{2}}|B|^{\frac{p}{2}}}\int_U |U|^{\alpha + \frac{r}{2} - \frac{p+1}{2}}\mathrm{e}^{-a\beta\mathrm{tr}(U)}\,\mathrm{d}U \\ &= \frac{\pi^{\frac{rp}{2}}}{\Gamma_p\left(\frac{r}{2}\right)|A|^{\frac{r}{2}}|B|^{\frac{p}{2}}}\frac{\Gamma_p(\alpha + \frac{r}{2})}{(a\beta)^{p(\alpha + \frac{r}{2})}}, \quad \alpha + \frac{r}{2} > \frac{p-1}{2} \end{aligned} \tag{1.9}$$

by using Section 5.1.1 of [3].

2. A general density

For X, A, B, a, β, q as defined in (1.1) let

$$f(X) = c|A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\alpha}|I - a(1-q)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\frac{\beta}{1-q}} \tag{2.1}$$

for $q \neq 1$, and for $q = 1$

$$= c|A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\alpha}\mathrm{e}^{-a\beta\mathrm{tr}\left[A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}\right]} \tag{2.2}$$

where c in (2.1) is given by (1.5) for $q < 1$ and by (1.7) for $q > 1$. From (1.9) we have the c in (2.2). In (2.1) a necessary condition to be met is that $I - a(1 - q)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}} > 0$. Note that when q moves from $-\infty$ to 1, that is, $-\infty < q < 1$ then (2.1) maintains a matrix-variate type-1 beta form and when q becomes greater than 1 then the type-1 beta form switches to a type-2 beta form. That is, to the left of 1 for q a type-1 beta form is available and to the right of 1 for q a type-2 beta form is available. Both these type-1 and type-2 beta forms go to a matrix-variate gamma form at $q = 1$. Thus the pathway for q describes a wide range of statistical densities covering type-1 and type-2 beta forms and gamma forms. It may also be noted from (1.1) that one need not go for the symmetric square roots $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ of A and B, one needs to obtain only a representation $A = A_1 A_1'$ and $B = B_1 B_1'$. Then one $A^{\frac{1}{2}}$ could be replaced by A_1' and one $B^{\frac{1}{2}}$ by B_1'.

2.1. Arbitrary moments

Arbitrary hth moment for the determinant $|A^{\frac{1}{2}} X B X' A^{\frac{1}{2}}|$ or that for $|XBX'|$ can be obtained from c^{-1} in (1.5), (1.7), (1.9) for the cases $q < 1, q > 1, q = 1$ respectively, by changing α to $\alpha + h$ and then taking the ratio of the normalizing constants. Thus we have the following, where E denotes the expected value.

Theorem 2.1

$$E|A^{\frac{1}{2}} X B X' A^{\frac{1}{2}}|^h = \frac{1}{[a(1-q)]^{ph}} \frac{\Gamma_p\left(\alpha + h + \frac{r}{2}\right)}{\Gamma_p\left(\alpha + \frac{r}{2}\right)} \frac{\Gamma_p\left(\alpha + \frac{r}{2} + \frac{\beta}{1-q} + \frac{p+1}{2}\right)}{\Gamma_p\left(\alpha + h + \frac{r}{2} + \frac{\beta}{1-q} + \frac{p+1}{2}\right)}$$

$$\text{for} \quad q < 1, \alpha + h + \frac{r}{2} > \frac{p-1}{2} \tag{2.3}$$

$$= \frac{1}{[a(q-1)]^{ph}} \frac{\Gamma_p\left(\alpha + h + \frac{r}{2}\right)}{\Gamma_p\left(\alpha + \frac{r}{2}\right)} \frac{\Gamma_p\left(\frac{\beta}{q-1} - \alpha - h - \frac{r}{2}\right)}{\Gamma_p\left(\frac{\beta}{q-1} - \alpha - \frac{r}{2}\right)} \tag{2.4}$$

$$\text{for} \quad q > 1, \frac{\beta}{q-1} - \alpha - h - \frac{r}{2} > \frac{p-1}{2}, \alpha + h + \frac{r}{2} > \frac{p-1}{2}$$

$$= \frac{1}{(a\beta)^{ph}} \frac{\Gamma_p(\alpha + h + \frac{r}{2})}{\Gamma_p(\alpha + \frac{r}{2})} \quad \text{for } q = 1, \quad \alpha + h + \frac{r}{2} > \frac{p-1}{2}. \tag{2.5}$$

One may wonder whether (2.3) and (2.4) go to (2.5) when $q \to 1$ from the left and right respectively. This can be seen from an asymptotic expansion for gamma functions or from Stirling's approximation. These will be stated as lemmas.

Lemma 2.1. *For $|z| \to \infty$ and a a bounded quantity,*

$$\Gamma(z + a) \approx \sqrt{2\pi} z^{z + a - \frac{1}{2}} e^{-z}, \tag{2.6}$$

where $\approx$ means "approximately equal to".

Then by applying lemma 2.1 and writing $\Gamma_p(\cdot)$ in explicit forms one has the following results.

Lemma 2.2

$$\lim_{q\to 1}\left\{\frac{1}{[a(1-q)]^{ph}}\frac{\Gamma_p\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+\frac{p+1}{2}\right)}{\Gamma_p\left(\alpha+h+\frac{r}{2}+\frac{\beta}{1-q}+\frac{p+1}{2}\right)}\right\}=\frac{1}{[a\beta]^{ph}}. \tag{2.7}$$

This can be seen by observing the following:

$$\frac{\Gamma_p\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+\frac{p+1}{2}\right)}{\Gamma_p\left(\alpha+h+\frac{r}{2}+\frac{\beta}{1-q}+\frac{p+1}{2}\right)}=\prod_{j=1}^{p}\left[\frac{\Gamma\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+\frac{p+1}{2}-\frac{j-1}{2}\right)}{\Gamma\left(\alpha+h+\frac{r}{2}+\frac{\beta}{1-q}+\frac{p+1}{2}-\frac{j-1}{2}\right)}\right].$$

When q goes to 1 from the left $\frac{\beta}{1-q}\to\infty$. Then, for example,

$$\begin{aligned}&\prod_{j=1}^{p}\Gamma\left(\alpha+h+\frac{r}{2}+\frac{\beta}{1-q}+\frac{p+1}{2}-\frac{j-1}{2}\right)\\ &\quad=\prod_{j=1}^{p}\sqrt{2\pi}\left(\frac{\beta}{1-q}\right)^{\alpha+h+\frac{r}{2}+\frac{\beta}{1-q}+\frac{p+1}{2}-\frac{j-1}{2}-\frac{1}{2}}\mathrm{e}^{-\frac{\beta}{1-q}}\\ &\quad=(\sqrt{2\pi})^{p}\left(\frac{\beta}{1-q}\right)^{p\left(\alpha+h+\frac{r}{2}+\frac{\beta}{1-q}\right)+\frac{p(p+1)}{4}}\mathrm{e}^{-\frac{p\beta}{1-q}}.\end{aligned}$$

Hence,

$$\frac{1}{[a(1-q)]^{ph}}\prod_{j=1}^{p}\frac{\Gamma\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+\frac{p+1}{2}-\frac{j-1}{2}\right)}{\Gamma\left(\alpha+h+\frac{r}{2}+\frac{\beta}{1-q}+\frac{p+1}{2}-\frac{j-1}{2}\right)}=\frac{1}{(a\beta)^{ph}}.$$

This establishes that (2.3) goes to (2.5) when $q\to 1$ from the left. In a similar way one can see that (2.4) also goes to (2.5). Thus q is a pathway from moments in (2.3) and (2.4) to go to the moments in (2.5).

One can make some interesting observations from (2.3)–(2.5). From (2.3) we have,

$$\begin{aligned}&E|a(1-q)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{h}\\ &\quad=\prod_{j=1}^{p}\frac{\Gamma\left(\alpha+\frac{r}{2}+h-\frac{j-1}{2}\right)}{\Gamma\left(\alpha+\frac{r}{2}-\frac{j-1}{2}\right)}\frac{\Gamma_p\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+\frac{p+1}{2}-\frac{j-1}{2}\right)}{\Gamma\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+\frac{p+1}{2}+h-\frac{j-1}{2}\right)}\\ &\quad=\prod_{j=1}^{p}E\left(x_j^{h}\right),\end{aligned} \tag{2.8}$$

where x_j is a real scalar type-1 beta random variable with the parameters

$$\left(\alpha+\frac{r}{2}-\frac{j-1}{2},\ \frac{\beta}{1-q}+\frac{p+1}{2}\right),\quad j=1,\ldots,p.$$

Thus, structurally, $|a(1-q)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|$, for $q<1$, is a product p of statistically independently distributed real type-1 beta random variables with the parameters as mentioned above. Similarly for $q>1$, $|a(q-1)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|$ is a product of p statistically independently distributed type-2 real scalar beta random variables, and from (2.5), $|a\beta A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|$ is a product of p independently distributed gamma random variables. These products of independent real scalar type-1 beta and type-2 beta random variables go to a product of independent real scalar gamma random variables when $q\to 1$. Thus, through q a pathway is achieved to go to product of independent gamma variables from products of independent type-1 beta and type-2 beta variables.

2.2. Some special cases

In (2.2) when $\alpha=0$ one has the famous matrix-variate Gaussian or normal density. For $q=1$ and α replaced by the degrees of freedom and with appropriate change in $A=\frac{1}{2}V^{-1}$, $V=V'>0$ and $B=I$ and expected value of X null, we have the extended Wishart density. The standard Wishart density is the central density in multivariate statistical analysis. We have extended type-1 beta, extended type-2 beta, F, Student t, Cauchy and other distributions coming as special cases. Note that all these are defined on rectangular matrices and hence we call them the extended versions. The following is a list of some particular cases and the transformations are listed to go from the extended versions to the regular cases. If a location matrix is to be introduced then one may replace X by $X-M$ where M is a $p\times r$ constant matrix.

$q<1, a(1-q)=1$	Extended type-1 beta density
$q<1, a(1-q)=1, Y=XBX'$	Non-standard type-1 beta density
$q<1, a(1-q)=1, Y=A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}$	Standard type-1 beta density
$q<1, a(1-q)=1, \alpha=0, \beta=0$	Extended uniform density
$q<1, a(1-q)=1, Y=XBX'$	Non-standard uniform density
$q<1, a(1-q)=1, \alpha+\frac{r}{2}=\frac{p+1}{2}, \beta=0, Y=A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}$	Standard uniform density
$q<1, a(1-q)=1, \alpha=0, \frac{\beta}{1-q}=\frac{1}{2}(m-p-r-1)$	Inverted T density of Dickey
$q<1, a=1, \alpha=0, \beta=1$	A q-binomial density
$q>1, a(q-1)=1$	Extended type-2 beta density
$q>1, a(q-1)=1, Y=XBX'$	Non-standard type-2 beta density
$q>1, a(q-1)=1, Y=A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}$	Standard type-2 beta density
$q>1, a(q-1)=1, \frac{\beta}{q-1}=\frac{m}{2}, \alpha=0$	T density of Dickey
$q>1, a(q-1)=\frac{1}{n}, \frac{\beta}{q-1}=\frac{n+1}{2}, Y=A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}$	Standard T density
$q>1, a(q-1)=1, \alpha+\frac{r}{2}=\frac{p+1}{2}, \frac{\beta}{q-1}=1, Y=A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}$	Standard Cauchy density
$q>1, a(q-1)=\frac{m}{n}, \alpha+\frac{r}{2}=\frac{m}{2}, \frac{\beta}{q-1}=\frac{m+n}{2}, Y=BXB'$	Non-standard F density
$q>1, a(q-1)=\frac{m}{n}, \alpha+\frac{r}{2}=\frac{m}{2}, \frac{\beta}{q-1}=\frac{m+n}{2}, Y=A^{\frac{1}{2}}BXB'A^{\frac{1}{2}}$	Standard F density
$q=1, a=1, \beta=1$	Extended gamma density
$q=1, a=1, \beta=1, Y=XBX'$	Non-standard gamma density
$q=1, a=1, \beta=1, Y=A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}$	Standard gamma density
$q=1, a=1, \beta=1, \alpha=0$	Gaussian density
$q=1, a=1, \beta=1, \alpha+\frac{r}{2}=\frac{n}{2}, Y=XBX', A=\frac{1}{2}V^{-1}$	Wishart density

For $p = 1, r = 1$, that is, in the scalar case,

$$|I - a(1-q)Z|^{\frac{\beta}{1-q}} = (1 - a(1-q)z)^{\frac{\beta}{1-q}}.$$

Let us see what we obtain if we expand this by Taylor series.

$$\begin{aligned}\left[1 + a(q-1)z\right]^{-\frac{\beta}{q-1}} &= 1 - \beta\frac{az}{1!} + \beta[\beta + (q-1)]\frac{(az)^2}{2!} \\ &\quad -\beta[\beta+(q-1)][\beta+2(q-1)]\frac{(az)^3}{3!} \\ &\quad -\beta[\beta+(q-1)][\beta+2(q-1)][\beta+3(q-1)]\frac{(az)^4}{4!} - \cdots\end{aligned} \tag{2.9}$$

This is a type of q-binomial series. Hence one can also look upon (2.1) for $\alpha = 0$ as a matrix-variate analogue of a q-binomial series.

2.3. Special cases as quadratic forms

One interesting special case is when $p = 1, r > p$. Then the constant matrix A is a scalar and without any loss of generality we may take it as 1.

$$A^{\frac{1}{2}}XBX'A^{\frac{1}{2}} = (x_1, \ldots, x_r)B\begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} = u \text{ (say)} \tag{2.10}$$

which is a real positive definite quadratic form in the first row of X, denoted by $(x_1, \ldots, x_r)$. The density of this quadratic form is available from (1.3) for the case $q < 1$ and from (1.6) for the case $q > 1$. Denoting the density of u by $g(u)$ we have the following:

Theorem 2.2. *The density of u in (2.10) is given by*

$$g(u) = c_1 u^{\alpha + \frac{r}{2} - 1}[1 - a(1-q)u]^{\frac{\beta}{1-q}} \tag{2.11}$$

with $1 - a(1-q)u > 0$, where, for $q < 1$

$$c_1 = \frac{\left[a(1-q)\right]^{\alpha+\frac{r}{2}}\Gamma\left(\alpha + \frac{r}{2} + \frac{\beta}{1-q} + 1\right)}{\Gamma\left(\alpha + \frac{r}{2}\right)\Gamma\left(\frac{\beta}{1-q} + 1\right)}, \quad \alpha + \frac{r}{2} > 0, \tag{2.12}$$

for $q > 1$

$$c_1 = \frac{[a(q-1)]^{\alpha+\frac{r}{2}}\Gamma\left(\frac{\beta}{q-1}\right)}{\Gamma\left(\alpha+\frac{r}{2}\right)\Gamma\left(\frac{\beta}{q-1} - \alpha - \frac{r}{2}\right)}, \quad \alpha + \frac{r}{2} > 0, \quad \frac{\beta}{q-1} - \alpha - \frac{r}{2} > 0, \tag{2.13}$$

and for $q = 1$

$$c_1 = \frac{(a\beta)^{\alpha+\frac{r}{2}}}{\Gamma\left(\alpha+\frac{r}{2}\right)}, \quad \alpha + \frac{r}{2} > 0. \tag{2.14}$$

Distributions of quadratic forms in real Gaussian random variables are discussed in [6] and the distributions of generalized quadratic forms with Gaussian vector random variables are considered in [7]. But if the $p \times r$, $r \geqslant p$ real random matrix X has a matrix-variate distribution as in (1.1), which covers rectangular matrix-variate type-1 beta, type-2 beta, gamma type and Gaussian type distributions, then the density of the generalized quadratic form follows trivially from (1.1). This will be given as the next theorem.

Theorem 2.3. *When the $p \times r$, $r \geqslant p$ real random matrix X has the matrix-variate distribution as given in (1.1) then the generalized quadratic form $Y = A^{\frac{1}{2}} B X B' A^{\frac{1}{2}}$ has the following density, denoted by*

$$f_1(Y) = c_2 |Y|^{\alpha+\frac{r}{2}-\frac{p+1}{2}} |I - a(1-q)Y|^{\frac{\beta}{1-q}}, \tag{2.15}$$

where, for $q > 1$

$$c_2 = \frac{[a(1-q)]^{p(\alpha+\frac{r}{2})} \Gamma_p\left(\alpha + \frac{r}{2} + \frac{\beta}{1-q} + \frac{p+1}{2}\right)}{\Gamma_p\left(\alpha+\frac{r}{2}\right) \Gamma_p\left(\frac{\beta}{1-q} + \frac{p+1}{2}\right)}, \quad \alpha + \frac{r}{2} > \frac{p-1}{2}, \tag{2.16}$$

for $q > 1$

$$c_2 = \frac{[a(q-1)]^{p(\alpha+\frac{r}{2})} \Gamma_p\left(\frac{\beta}{q-1}\right)}{\Gamma_p\left(\alpha+\frac{r}{2}\right) \Gamma_p\left(\frac{\beta}{q-1} - \alpha - \frac{r}{2}\right)}, \quad \alpha + \frac{r}{2} > \frac{p-1}{2},$$
$$\frac{\beta}{q-1} - \alpha - \frac{r}{2} > \frac{p-1}{2} \tag{2.17}$$

and for $q = 1$

$$c_2 = \frac{(a\beta)^{p(\alpha+\frac{r}{2})}}{\Gamma_p(\alpha+\frac{r}{2})}, \quad \alpha + \frac{r}{2} > \frac{p-1}{2}. \tag{2.18}$$

3. Connection to geometrical probability problems

While considering the distributional aspects of the volume content of a r-parallelotope generated by the convex hull of linearly independent random points in Euclidean n-space many authors had considered the problem when the points are isotropic and are distributed according to a beta type-1, type-2 and Gaussian situations, see for example [8,9,10]. The distributions of the random points that they considered were particular cases of (2.11) with $B = I$. More general situations in this category of problems are considered in [4]. Since the determinant of the type $|A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|$, appearing in (1.1), can be considered to be volume of an appropriately defined parallelotope a more general model in this category of problems is available from (1.1). Note that the $p \times r$ matrix X of full rank can also be looked upon as p linearly independent points in a r-dimensional Euclidean space. Then $|XX'|$ is the square of the volume of the parallelotope generated by the convex hull of these p points in r-space, $r \geqslant p$. Hence $|a(1-q)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|$ is the square of the volume of the parallelotope generated by p points in a transformed space. Also from (2.3)–(2.5) it is seen that $|A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|$ is structurally a product of p independent type-1 beta, type-2 beta and gamma random variables corresponding to $q < 1, q > 1$ and $q = 1$ respectively. The same structure is appearing in geometrical probability problems also. For such structures, approximations and asymptotic results are derived in [2] and in Chapter 4 of [5]. Hence approximations and asymptotic results will not be discussed here.

4. Remarks

In (1.1) we dealt with a general model when the elements in X are real scalar random variables. If a relocation parameter matrix is to be introduced then we may replace X by $X - M$ where M is a location parameter matrix. If the model in (1.1) is to be extended to the complex domain then the Jacobians and the integrals will be slightly different. The necessary tools are available in [3] and the procedure is parallel to the real case. Hence we will not deal with the case here when X is in the complex domain.

When $p = 1, r = 1, A = I, B = I, \beta = 1$ and $\alpha = 0$ we obtain Tsallis' statistics as a special case from (1.1). The q-binomial function $[1-(1-q)t]^{\frac{1}{1-q}}$ is also a solution of the power law

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y^q \tag{4.1}$$

which is associated with the generalized entropy

$$k\frac{\int_{-\infty}^{\infty} f^{1-q}(t)\mathrm{d}t - 1}{q-1}, \tag{4.2}$$

where $f(t)$ is a density function and k is a constant. When $q \to 1$, (4.2) goes to Shannon's entropy. These considerations are very relevant in physics problems. Nowadays Tsallis' statistics is a hot subject, applicable in a wide range of problems in astrophysics, extending the theories in various topics in astrophysics areas. For a window into the vast areas of research activities one may start with [11].

Extensions of the ideas in (1.1) to densities involving many matrix variables in the real or complex domain are also straightforward. As an example let us look into the matrix-variate Dirichlet type family of distributions. For a discussion of matrix-variate Dirichlet and Liouville distributions see [1]. Let $X_j,\ p \times r_j,\ r_j \geqslant p,\ j = 1, \ldots, k$ be real matrix random variables having a joint density of the following type:

$$f(X_1, \ldots, X_k) = C_k \left\{ \prod_{j=1}^{k} \left| A_j^{\frac{1}{2}}(X_j - M_j)B_j(X_j - M_j)' A_j^{\frac{1}{2}} \right|^{\alpha_j} \right\} \times \left| I - (1-q) \sum_{j=1}^{k} \left(A_j^{\frac{1}{2}}(X_j - M_j)B_j(X_j - M_j)' A_j^{\frac{1}{2}} \right) \right|, \tag{4.3}$$

where C_k is the normalizing constant, $A_j,\ p \times p,\ B_j,\ r_j \times r_j,\ j = 1, \ldots, k$ are real symmetric positive definite constant matrices and $M_j,\ p \times r_j,\ j = 1, \ldots, k$ are constant matrices. The normalizing constant C_k can be evaluated by using the steps described in this paper. Many interesting properties can be seen from the model in (4.3). For $q < 1$ and the last factor in (4.3) remaining positive, the density is an extended Dirichlet type-1 type, then when $q > 1$ the model switches to an extended type-2 type. But when $q = 1$ the random matrices are independently distributed and of the extended gamma types. This statistical independence property is a surprising result. There are various generalizations of the Dirichlet model available in the literature. Such generalizations can also be extended to the rectangular matrix-variate cases, real or complex, and those can then be extended to their q-versions by using the procedure discussed in this paper.

Another observation that one can make is the following: In the real scalar case our model in (1.1) becomes

$$f(x) = cy^{\alpha}[1 - (1-q)y]^{\frac{\beta}{1-q}}, \quad y = xx' = x^2 \tag{4.4}$$

with $1 - (1-q)y > 0$, taking $A = I,\ B = I$. In this case, we can replace y by z^{δ}, $\delta > 0$. Then when $q \to 1$ one can go to generalized gamma, Weibull and other distributions. But in the matrix case, powers such as δ are not feasible, even though we are dealing with real symmetric positive definite or hermitian positive definite matrices, because when transformations are needed the Jacobians do not go into nice forms. Even for $\delta = 2$ see the complicated form of the Jacobian from [3]. The special cases available from (4.4), which itself is a special case of (1.1), are the following:

$q = 1, \alpha = 0, a = 1$	Gaussian or normal density for $-\infty < x < \infty$
$q = 1, \alpha = \frac{3}{4}, a = 1$	Maxwell–Boltzmann density in physics
$q = 1, \alpha = \frac{1}{2}, a = 1$	Rayleigh density
$q = 1, \alpha = \frac{n}{2} - 1, a = 1$	Hermert density
$q = 0, \alpha = 0, \beta = 1$	U-shaped density
$q = 2, a = \frac{1}{v}, \beta = \frac{v+1}{2}, \alpha = 0$	Student-t for v degrees of freedom, $-\infty < x < \infty$

$q = 2, a = 1, \beta = 1$	Cauchy density for $-\infty < x < \infty$
$q < 1, a(1-q) = 1, x^2 = y$	Standard type-1 beta density
$q > 1, a(q-1) = 1, x^2 = y$	Standard type-2 beta density
$\alpha = \frac{1}{2}, \beta = 1, a = 1, x^2 = y$	Tsallis statistics in astrophysics, power law, q-binomial density
$\alpha = \frac{1}{2}, q = 0, \beta = 1, x^2 = y$	Triangular density
$q = 2, \alpha + \frac{1}{2} = \frac{m}{2}, a = \frac{m}{n}, \beta = \frac{m+1}{2}, x^2 = y$	F-density
$q = 1, \alpha = \frac{1}{2}, a = 1, \beta = \frac{mg}{KT}, x^2 = y$	Helley's density in physics
$q = 1, a = 1, x^2 = y$	Gamma density
$q = 1, a = 1, \beta = \frac{1}{2}, \alpha + \frac{1}{2} = \frac{v}{2}, x^2 = y$	Chisquare density for v degrees of freedom
$q = 1, a = 1, \alpha = \frac{1}{2}, x^2 = y$	Exponential density (Laplace density with y=$\|z\|$,$-\infty < z < \infty$)
$q = 1, a = 1, x^2 = z^\delta, \delta > 0$	Generalized gamma density
$q = 1, a = 1, \alpha = \frac{1}{2}, x^2 = z^\delta, \delta > 0$	Weibull density
$q = 2, a = 1, \beta = 2, \alpha = \frac{1}{2}, x^2 = e^y$	Logistic density for $-\infty < y < \infty$
$q = 2, a = e^\delta, \beta = 1, \alpha = -\frac{1}{2}, x^2 = e^{\gamma y}, \gamma > 0$	Fermi–Dirac density in physics

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