Extension to maximal semidefinite invariant subspaces for hyponormal matrices in indefinite inner products

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Abstract

It is proved that under certain essential additional hypotheses, a nonpositive invariant subspace of a hyponormal matrix admits an extension to a maximal nonpositive subspace which is invariant for both the matrix and its adjoint. Nonpositivity of subspaces and the hyponormal property of the matrix are understood in the sense of a nondegenerate inner product in a finite dimensional complex vector space. The obtained theorem combines and extends several previously known results. A Pontryagin space formulation, with essentially the same proof, is offered as well.

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1. Introduction

On the vector space \( \mathbb{C}^n \), equipped with the standard inner product, we fix an indefinite inner product \( [\cdot, \cdot] \) determined by an invertible Hermitian \( n \times n \) matrix \( H \) via the formula

\[
[\cdot, \cdot] = \cdot^* H \cdot
\]
[x, y] = \langle Hx, y \rangle, \quad x, y \in \mathbb{C}^n.

Here, \( \langle \cdot, \cdot \rangle \) denotes the standard inner product.

A subspace \( \mathcal{M} \subseteq \mathbb{C}^n \) is said to be \( H \)-nonnegative if \( [x, x] \geq 0 \) for every \( x \in \mathcal{M} \), \( H \)-positive if \( [x, x] > 0 \) for every nonzero \( x \in \mathcal{M} \), \( H \)-nonpositive if \( [x, x] \leq 0 \) for every \( x \in \mathcal{M} \), \( H \)-negative if \( [x, x] < 0 \) for every nonzero \( x \in \mathcal{M} \), and \( H \)-neutral if \( [x, x] = 0 \) for every \( x \in \mathcal{M} \). Note that by default the zero subspace is \( H \)-positive as well as \( H \)-negative. An \( H \)-nonnegative subspace is said to be \textit{maximal \( H \)-nonnegative} if it is not properly contained in any larger \( H \)-nonnegative subspace.

It is easy to see that an \( H \)-nonnegative subspace is maximal if and only if its dimension is equal to the number \( i_+(H) \) of positive eigenvalues of \( H \) (counted with multiplicities). Analogously, an \( H \)-nonpositive subspace is maximal if and only its dimension is equal to the number of negative eigenvalues of \( H \).

Let \( X^{[*]} \) denote the adjoint of a matrix \( X \in \mathbb{C}^{n \times n} \) with respect to the indefinite inner product, i.e., \( X^{[*]} \) is the unique matrix satisfying \( [x, Xy] = (X^{[*]}x, y) \) for all \( x, y \in \mathbb{C}^n \). One easily sees that \( X^{[*]} = H^{-1}X^*H \). We recall that a matrix \( X \in \mathbb{C}^{n \times n} \) is called \textit{\( H \)-normal} if \( X^{[*]}X = XX^{[*]} \), and \textit{\( H \)-hyponormal} if \( H(X^{[*]}X - XX^{[*]}) \geq 0 \) (positive semidefinite). We note that it is easy to check that if \( X \) is \( H \)-normal, resp., \( H \)-hyponormal, then \( P^{-1}XP \) is \( P^{[*]}HP \)-normal, resp., \( P^{[*]}HP \)-hyponormal, provided that \( P \in \mathbb{C}^{n \times n} \) is nonsingular.

It is well known that several classes of matrices in indefinite inner product spaces allow extensions of invariant \( H \)-nonnegative subspaces to invariant maximal \( H \)-nonnegative subspaces. Those classes are for example the ones of \( H \)-expansive matrices (including \( H \)-unitary matrices), \( H \)-dissipative matrices (including \( H \)-selfadjoints), and \( H \)-skew-adjoint matrices, see, e.g., [5] for a proof. The natural question arises if this extension problem still has a solution for arbitrary \( H \)-normal matrices. A partial answer to this question is contained in the following result.

**Theorem 1.** Let \( X \in \mathbb{C}^{n \times n} \) be \( H \)-normal, and let \( \mathcal{M}_0 \) be an \( H \)-neutral \( X \)-invariant subspace. Then there exists an \( X \)-invariant subspace \( \mathcal{M} \) which is also maximal \( H \)-nonnegative, i.e., \( H \)-nonnegative of dimension \( i_+(H) \), and such that \( \mathcal{M}_0 \subseteq \mathcal{M} \). Also, there exists an \( X \)-invariant maximal \( H \)-nonpositive subspace containing \( \mathcal{M}_0 \).

Theorem 1 can be obtained from results of [2,3], and it holds also for Pontryagin spaces; see [6] for details. A more general theorem is proved in [5]. The proof of Theorem 1 given in [5] depends essentially on the \( H \)-neutrality of the given invariant subspace \( \mathcal{M}_0 \).

Moreover, it was proven in [6] that if \( \mathcal{M} \) is a maximal \( H \)-nonnegative subspace invariant under an \( H \)-normal \( X \), then it is also invariant under \( X^{[*]} \). Also, the authors proved an extension result in the framework of \( H \)-hyponormal matrices. For sake of convenience, we recall the two main results from that paper.

**Theorem 2.** Let \( X \in \mathbb{C}^{n \times n} \) be \( H \)-hyponormal. If the spectrum of \( X + X^{[*]} \) is real or if the spectrum of \( X - X^{[*]} \) is purely imaginary (including zero), then there exists an \( X \)-invariant maximal \( H \)-nonnegative subspace that is also invariant for \( X^{[*]} \). Also, there exists an \( X \)-invariant maximal \( H \)-nonpositive subspace that is also invariant for \( X^{[*]} \).

The assumption that either the spectrum of \( X + X^{[*]} \) is real or the spectrum of \( X - X^{[*]} \) is purely imaginary in Theorem 2 was shown in [6] to be essential even for the case of \( H \)-normal matrices.
For a subspace \( \mathcal{M}_0 \subseteq \mathbb{C}^n \), we denote by
\[
\mathcal{M}_0^{\perp} = \{ \mathbf{x} \in \mathbb{C}^n | [\mathbf{x}, \mathbf{y}] = 0 \text{ for every } \mathbf{y} \in \mathcal{M}_0 \}
\]
the \( H \)-orthogonal companion of \( \mathcal{M}_0 \).

**Theorem 3.** Let \( X \in \mathbb{C}^{n \times n} \) be \( H \)-hyponormal and let \( \mathcal{M}_0 \) be an \( X \)-invariant \( H \)-negative subspace. Define \( X_{22} = X^{[\ast]}|_{\mathcal{M}_0^{\perp}} : \mathcal{M}_0^{\perp} \rightarrow \mathcal{M}_0^{\perp} \). Equip \( \mathcal{M}_0^{\perp} \) with the indefinite inner product induced by \( H \). Assume that at least one of the two inclusions \( \sigma(X_{22}^{[\ast]} + X_{22}) \subseteq \mathbb{R} \) and \( \sigma(X_{22}^{[\ast]} - X_{22}) \subseteq i\mathbb{R} \) holds true. Then there exists an \( X \)-invariant maximal \( H \)-nonpositive subspace that contains \( \mathcal{M}_0 \).

The aim of this note is to unify and complete the theory of extensions of semidefinite subspaces for \( H \)-normal and \( H \)-hyponormal subspaces. In particular, we prove a generalization of Theorem 3, where we start with an \( H \)-nonpositive \( X \)-invariant subspace \( \mathcal{M}_0 \) instead of an \( H \)-negative one. The extension result is then not true without further conditions, as it was already shown in [6].

**2. Extension of nonpositive invariant subspaces**

We start by generalizing the fact that, for \( H \)-normal matrices \( X \), invariant maximal \( H \)-semidefinite subspaces are also invariant under the adjoint \( X^{[\ast]} \). Indeed, it turns out that this result holds true even for \( H \)-hyponormal matrices if the subspace under consideration is assumed to be \( H \)-nonpositive.

**Proposition 4.** Let \( X \in \mathbb{C}^{n \times n} \) be \( H \)-hyponormal and let \( \mathcal{M} \) be an \( X \)-invariant maximal \( H \)-nonpositive subspace. Then \( \mathcal{M} \) is invariant also for \( X^{[\ast]} \).

**Proof.** The proof is essentially the same as the corresponding proof for the case that \( X \) is \( H \)-normal (see [6]). Nevertheless we provide the proof here to keep the paper self-contained. Applying otherwise a suitable transformation \( X \mapsto P^{-1}XP, H \mapsto P^{\ast}HP \), where \( P \) is invertible, we may assume that \( \mathcal{M} \) is spanned by the first (say) \( m \) unit vectors and that \( X \) and \( H \) have the forms
\[
X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ 0 & 0 & X_{33} & X_{34} \\ 0 & 0 & X_{43} & X_{44} \end{bmatrix}, \quad H = \begin{bmatrix} -I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.
\]

Indeed, this follows easily by decomposing \( \mathcal{M} = \mathcal{M}_p \oplus \mathcal{M}_0 \) into an \( H \)-neutral subspace \( \mathcal{M}_0 \) and its orthogonal complement \( \mathcal{M}_p \) (in \( \mathcal{M} \)), and choosing an \( H \)-neutral subspace \( \mathcal{M}_sl \) that is skewly linked to \( \mathcal{M}_0 \) (see [1,4], for the definition and properties of skewly linked subspaces). Note that the \( H \)-orthogonal complement to \( \mathcal{M} + \mathcal{M}_sl \) is necessarily an \( H \)-positive subspace due to the maximality of \( \mathcal{M} \). Then, selecting appropriate bases in all subspaces constructed above, and putting the bases as the consecutive columns of a matrix \( P \), we get a transformation that yields the desired result. From (1), we then obtain that
\[
X^{[\ast]} = \begin{bmatrix} X_{11}^{[\ast]} & 0 & -X_{21}^{[\ast]} & 0 \\ -X_{13}^{[\ast]} & X_{33}^{[\ast]} & X_{23}^{[\ast]} & X_{43}^{[\ast]} \\ -X_{12}^{[\ast]} & 0 & X_{22}^{[\ast]} & 0 \\ -X_{14}^{[\ast]} & X_{34}^{[\ast]} & X_{24}^{[\ast]} & X_{44}^{[\ast]} \end{bmatrix}
\]
and
\[ H\left(X^{[*]}X - XX^{[*]}\right) = \begin{bmatrix}
* & * & * & * \\
* - X_1^{[*]}X_1 - X_{34}X_{34}^{[*]} & * & * \\
* & * & * \\
* & * & X_{44}^{[*]}X_{44} - X_{14}^{[*]}X_{14} + X_{24}^{[*]}X_{34} + X_{34}^{[*]}X_{24} - X_{44}X_{44}^{[*]} & *
\end{bmatrix}. \] (3)

Since \(X\) is \(H\)-hyponormal, i.e., \(H\left(X^{[*]}X - XX^{[*]}\right) \succeq 0\), we obtain from the block \((2, 2)\)-entry in (3) that \(X_{12} = 0\) and \(X_{34} = 0\). But then the inequality for the block \((4, 4)\)-entry of (3) becomes
\[ X_{44}^{[*]}X_{44} - X_{14}^{[*]}X_{14} \succeq 0 \] (4)
which is easily seen to imply (by taking traces of both sides in (4)) that \(X_{44}\) is normal and that \(X_{14} = 0\). Thus, we obtain from (2) that \(\mathcal{M}\) is also invariant for \(X^{[*]}\). \(\Box\)

The following example illustrates Proposition 4 and shows that we cannot replace \(H\)-nonpositivity in the hypothesis of the proposition by \(H\)-nonnegativity.

**Example 5.** Let
\[ X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \] (5)

Then one easily computes
\[ X^{[*]} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad H\left(X^{[*]}X - XX^{[*]}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X + X^{[*]} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & 0 & 2 \end{bmatrix} \]
that is, \(X\) is \(H\)-hyponormal and the spectrum of \(\sigma(X + X^{[*]}) = \{2\}\) is real. Then the only \(X\)-invariant subspace that is maximal \(H\)-nonpositive is given by \(\mathcal{M}_- = \text{Span}(e_2, e_3)\). Obviously, \(\mathcal{M}_-\) is also invariant under \(X^{[*]}\). On the other hand, \(\mathcal{M}_+ = \text{Span}(e_1)\) is a maximal \(H\)-nonnegative subspace that is invariant under \(X\), but \(\mathcal{M}_+\) is not invariant under \(X^{[*]}\). However, Theorem 2 implies that \(X\) has a maximal \(H\)-nonnegative subspace that is also invariant under \(X^{[*]}\). Such a subspace is given by \(\tilde{\mathcal{M}}_+ = \text{Span}(e_2)\).

The main results of this note is the following. It combines elements of Theorems 1–3.

**Theorem 6.** Let \(X\) be \(H\)-hyponormal, and let \(\mathcal{M}\) be an \(H\)-nonpositive subspace that is invariant under \(X\). Let \(\mathcal{M}_0\) be the isotropic part of \(\mathcal{M}\) and decompose \(\mathcal{M}^{(\bot)}\) as
\[ \mathcal{M}^{(\bot)} = \mathcal{M}_0 \oplus \mathcal{M}_{\text{nd}} \] (6)
for an \(H\)-nondegenerate subspace \(\mathcal{M}_{\text{nd}}\). Denote by \(X_{44}\) and \(H_4\) the compressions of \(X\) and \(H\) to \(\mathcal{M}_{\text{nd}}\), respectively. Assume that \(\mathcal{M}_0\) is invariant under \(X^{[*]}\) and that, in addition, one of the three following conditions holds:
(a) \( \sigma(X_{44} + X_{44}^*) \subset \mathbb{R} \),
(b) \( \sigma(X_{44} - X_{44}^*) \subset i\mathbb{R} \),
(c) \( X_{44} \) is \( H_4 \)-normal.

Then \( \mathcal{M} \) can be extended to a maximal \( H \)-nonpositive subspace \( \mathcal{M} \) that is invariant under both \( X \) and \( X^* \).

The conditions (a)–(c) are independent of the particular choice of a nondegenerate subspace \( \mathcal{M}_{nd} \) subject to (6).

**Proof.** A decomposition similar to (1) will be used. Since \( \mathcal{M}_0 = \mathcal{M} \cap \mathcal{M}^{[\perp]} \). Let \( \mathcal{M}_{sl} \) be a subspace skewly linked to \( \mathcal{M}_0 \), let \( \mathcal{M}_2 \) be a nondegenerate subspace of \( \mathcal{M} \) which is \( H \)-orthogonal to both \( \mathcal{M}_0 \) and \( \mathcal{M}_{sl} \), and finally, let \( \mathcal{M}_4 \) be the \( H \)-orthogonal complement of \( \mathcal{M}_0 + \mathcal{M}_2 + \mathcal{M}_{sl} \). Observe that \( \mathcal{M}_2 \) is a \( H \)-negative subspace in \( \mathcal{M} \) while \( \mathcal{M}_4 \) is a nondegenerate subspace in \( \mathcal{M}^{[\perp]} \). With respect to the decomposition

\[
\mathbb{C}^n = (\mathcal{M}_0 + \mathcal{M}_2 + \mathcal{M}_{sl})[\perp] \mathcal{M}_4,
\]

where \([\perp]\) stands for an \( H \)-orthogonal sum, and with respect to an appropriate choice of basis in each of the components we write

\[
H = \begin{bmatrix}
0 & 0 & I & 0 \\
0 & -I & 0 & 0 \\
I & 0 & 0 & 0 \\
0 & 0 & 0 & H_4
\end{bmatrix}, \quad X = \begin{bmatrix}
X_{11} & X_{12} & X_{13} & X_{14} \\
X_{21} & X_{22} & X_{23} & X_{24} \\
0 & 0 & X_{33} & X_{34} \\
0 & 0 & X_{43} & X_{44}
\end{bmatrix}.
\]

Using this we easily see that \( X^* \) is given by

\[
X^* = \begin{bmatrix}
X_{33}^* & -X_{23}^* & X_{13}^* & X_{43}^*H_4 \\
0 & X_{22}^* & -X_{12}^* & 0 \\
0 & -X_{31}^* & X_{11}^* & 0 \\
H_4^{-1}X_{34}^* & -H_4^{-1}X_{24}^* & H_4^{-1}X_{14}^* & H_4^{-1}X_{44}^*H_4
\end{bmatrix}.
\]

Partitioning \( Y := H(X^*X - XX^*) \) conformably with respect to the decomposition (7), we obtain that the \((4,4)\)-block \( Y_{44} \) takes the form

\[
Y_{44} = X_{34}^*X_{14} - X_{24}^*X_{24} + X_{14}^*X_{34} + H_4 \left( X_{44}^*X_{44} - X_{44}X_{44}^* \right),
\]

where \( X_{44}^* \) denotes the \( H_{44} \)-adjoint \( H_{44}^{-1}X_{44}^*H_{44} \) of \( X_{44} \). By assumption, the isotropic part \( \mathcal{M}_0 \) of \( \mathcal{M} \) is invariant under \( X^* \) which implies \( X_{34} = 0 \). But then, we obtain that \( X_{44} \) is \( H_4 \)-hyponormal, because we get from (8) that

\[
H_4 \left( X_{44}^*X_{44} - X_{44}X_{44}^* \right) = Y_{44} + X_{24}^*X_{24} \geq Y_{44} \geq 0,
\]

since \( X \) is \( H \)-hyponormal and, therefore, \( Y \) and \( Y_{44} \) are positive semidefinite.

Next, we show that the conditions (a)–(c) are independent of the particular choice of a nondegenerate subspace \( \mathcal{M}_{nd} \) subject to (6), i.e., we may assume without loss of generality that \( \mathcal{M}_{nd} = \mathcal{M}_4 \). Indeed, choosing another nondegenerate subspace \( \mathcal{M}_{nd} \) in \( \mathcal{M}^{[\perp]} \) in place of \( \mathcal{M}_4 \) amounts to a change of basis in \( \mathcal{M}^{[\perp]} \) given by a matrix of the form

\[
S = \begin{bmatrix}
I & 0 & 0 & S_{14} \\
0 & I & 0 & S_{24} \\
0 & 0 & I & S_{34} \\
0 & 0 & 0 & S_{44}
\end{bmatrix}
\]
with $S_{44}$ invertible. Thus, we obtain that with respect to the new decomposition
\[ C^n = (\mathcal{H}_0 + \mathcal{H}_2 + \mathcal{H}_3) + \mathcal{H}_{nd} \]
and the new basis, $X$ and $H$ take the forms
\[
\tilde{X} = S^{-1}XS = \begin{bmatrix}
X_{11} & X_{12} & * & * \\
X_{21} & X_{22} & * & * \\
0 & 0 & * & * \\
0 & 0 & S^{-1}_{44}X_{44}S_{44} + S^{-1}_{44}X_{43}S_{34}
\end{bmatrix},
\]
\[
\tilde{H} = S^*HS = \begin{bmatrix}
0 & 0 & I & S_{34} \\
0 & -I & 0 & -S_{24} \\
I & 0 & 0 & S_{14} \\
S_{34} & -S^*_{24} & S^*_{14} & (S_{44} - S^*_{14}S_{14} + S^*_{24}S_{24} - S^*_{24}S_{24})
\end{bmatrix}.
\]
Since $\mathcal{H}_{nd}$ is assumed to be a subspace in $\mathcal{H}^{[\perp]}$, we must have
\[
0 = \begin{bmatrix} I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{bmatrix} (S^* - 1)(S^*HS) \begin{bmatrix} 0 \\
0 \\
0 \\
I
\end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{bmatrix} \begin{bmatrix} S_{14} \\
S_{24} \\
S_{34} \\
S_{44}
\end{bmatrix} = \begin{bmatrix} S_{34} \\
-S_{24}
\end{bmatrix}
\]
which implies $S_{24} = 0$ and $S_{34} = 0$. Thus, the compressions $\tilde{X}_{44}$ and $\tilde{H}_{44}$ of $\tilde{X}$ resp. $\tilde{H}$ to $\mathcal{H}_{nd}$ are
\[
\tilde{X}_{44} = S^{-1}_{44}X_{44}S_{44}, \quad \tilde{H}_{44} = S^*_{44}X_{44}S_{44}.
\]
Clearly it follows from this that if each of the three conditions (a)–(c) holds for $\tilde{X}_{44}$ and $\tilde{H}_{44}$, then it holds also for $X_{44}$ and $H_{44}$. In particular, the conditions (a)–(c) are independent of the choice of $\mathcal{H}_{nd}$.

Consequently, assuming $\mathcal{H}_{nd} = \mathcal{H}_4$ and that we have either $\sigma\left(X_{44} + X_{44}^{[*]}\right) \subset \mathbb{R}$ or $\sigma\left(X_{44} - X_{44}^{[*]}\right) \subset i\mathbb{R}$ or that $X_{44}$ is $H_4$-normal, we obtain from Theorems 1 and 2 and Proposition 4 that there exists an $X_{44}$-invariant maximal $H_4$-nonpositive subspace $\mathcal{N}_4$ that is also invariant under $X_{44}^{[*]}$. In that case $\mathcal{H}_+ := \mathcal{H}_+ + \mathcal{N}_4$ is maximal $H$-nonpositive, $X$-invariant, and thus, by Proposition 4 also $X^{[*]}$-invariant. $\Box$

The following example, adapted from [6], shows that the conditions (a)–(c) are essential in Theorem 6.

**Example 7.** Let
\[
H = \begin{bmatrix} 0 & 1 \\
1 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} i & -i \\
i & -i \end{bmatrix}.
\]
Then one easily calculates
\[
X^{[*]} = \begin{bmatrix} i & i \\
-i & -i \end{bmatrix}, \quad A := \frac{1}{2} (X + X^{[*]}) = \begin{bmatrix} i & 0 \\
0 & -i \end{bmatrix}, \quad S := \frac{1}{2} (X - X^{[*]}) = \begin{bmatrix} 0 & -i \\
i & 0 \end{bmatrix}
\]
and $H(X^{[*]}X - XX^{[*]}) = 4 \cdot I$. Hence $X$ is $H$-hyponormal but not $H$-normal. Moreover, the spectrum of $A$ is not real, and neither is the spectrum of $S$ purely imaginary. Clearly, the zero space $\{0\}$ is $H$-neutral, invariant both under $X$ and $X^{[*]}$, and coincides with its isotropic subspace. Now the only nontrivial invariant subspace for $X$ is
\[ \mathcal{M}_+ = \text{Span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \]

which is easily seen to be maximal \( H \)-nonnegative, but it is not invariant under \( X^{[*]} \), because otherwise it would also be invariant for \( A \) and \( S \) which is obviously not the case. Thus, \( \{0\} \) cannot be extended neither to a maximal \( H \)-nonnegative nor to a maximal \( H \)-nonpositive subspace that is invariant for both \( X \) and \( X^{[*]} \).

On the other hand, Example 5 shows that also the hypothesis in Theorem 6 that the isotropic subspace \( \mathcal{M}_0 \) of \( \mathcal{M} \) is \( X^{[*]} \)-invariant is essential. Thus, the question arises under which conditions the isotropic subspace \( \mathcal{M}_0 \) of an \( X \)-invariant \( H \)-nonpositive subspace \( \mathcal{M} \) (where \( X \) is an \( H \)-hyponormal matrix) is \( X^{[*]} \)-invariant. One immediate answer is given in the following remark that can be verified in a straightforward manner.

**Remark 8.** If \( X \) is \( H \)-hyponormal and \( \mathcal{M} \) is a maximal \( H \)-nonpositive subspace that is invariant under both \( X \) and \( X^{[*]} \), then its isotropic part \( \mathcal{M}_0 = \mathcal{M} \cap \mathcal{M}^{[\perp]} \) is also invariant under both \( X \) and \( X^{[*]} \).

**Remark 9.** Theorem 6 contains Theorem 3 as a special case, because clearly, the isotropic part of an \( H \)-negative subspace is the zero space which is always invariant under \( X^{[*]} \).

We conclude the note with an observation that Proposition 4 and Theorem 6 are valid also for Pontryagin space operators, where \( H \) is an invertible self-adjoint operator on a Hilbert space with only finite dimensional invariant subspace corresponding to the positive part of the spectrum of \( H \). In the case of Theorem 6 an additional hypothesis that the codimension of \( \mathcal{M} \) is finite has to be imposed; this hypothesis would guarantee that \( \mathcal{M}_{nd} \) is finite dimensional. The proofs remain essentially the same.

**References**