ELSEVIER
Topology and its Applications 105 (2000) 65–88
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

The epimorphic hull of $C(X)$

Robert M. Raphael [a,1], R. Grant Woods [b,*]

[a] *Department of Mathematics and Statistics, Concordia University, Montreal, Quebec H4B 1R6, Canada*
[b] *Department of Mathematics, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada*

Received 29 June 1998; received in revised form 15 December 1998

Abstract

The epimorphic hull $H(A)$ of a commutative semiprime ring A is defined to be the smallest von Neumann regular ring of quotients of A.

Let X denote a Tychonoff space. In this paper the structure of $H(C(X))$ is investigated, where $C(X)$ denotes the ring of continuous real-valued functions with domain X. Spaces X that have a regular ring of quotients of the form $C(Y)$ are characterized, and a "minimum" such Y is found. Necessary conditions for $H(C(X))$ to equal $C(Y)$ for some Y are obtained. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Epimorphic hull of a ring; $C(X)$; Rings of quotients

AMS classification: Primary 46E25; 54C30, Secondary 54G10; 54G12

1. Introduction

If R is a commutative ring with identity, there exists a well-developed notion of a generalized ring of quotients of R (defined in 2.1(b) below). In particular, each such R possesses a "complete ring of quotients" denoted $Q(R)$ (see 2.1(c) below). Also, if R is semiprime then $Q(R)$ is regular in the sense of von Neumann (see 2.1(a) below). Closely related to rings of quotients and regularity are the notions of an epimorphism of rings, and the epimorphic hull of a ring. This latter object was defined and studied by Storrer [19] (see 2.1(e), (f) below). Our goal in this paper is to study these notions in the case of rings whose origins are topological. Our principal tool in this study will be Storrer's theorem that the epimorphic hull of R is the (unique) smallest regular ring that lies between R and $Q(R)$.

* Corresponding author. E-mail: rgwoods@cc.umanitoba.ca.
[1] E-mail: raphael@alcor.concordia.ca.

0166-8641/00/$ – see front matter
PII: S0166-8641(99)00036-X

The ring $C(X)$ of all continuous real-valued functions on a topological space X is commutative and semiprime. Its complete ring of quotients, $Q(X)$, is the subject of a seminal work by Fine, Gillman, and Lambek [6]. We are interested in the study of the epimorphic hull of $C(X)$, which we will denote by $H(X)$. A notable result of [6] is that $Q(X)$ may be represented as the ring of all continuous real-valued functions on the dense open subsets of X (modulo identification of functions that agree on the intersections of their domains), but we know of no corresponding representation for $H(X)$. As well, Hager [8] proved that $Q(X)$ is isomorphic to some $C(Y)$ if and only if the isolated points of X form a dense subset of X (provided no measurable cardinals exist nearby). It was the quest for an analogous result for $H(X)$ that prompted our study. Since $H(X)$ is always a Φ-algebra in the sense of Henriksen and Johnson, one is asking that a particular Φ-algebra be a ring of functions (see [9, Section 5]).

This problem factors into two parts. First we characterize those realcompact spaces X for which $C(X)$ has a regular ring of quotients of the form $C(Y)$. In this case we show that there is a space that yields a "smallest" such $C(Y)$, and we characterize it in terms of X. We then investigate when $C(Y)$ is isomorphic to $H(X)$. Although we do not solve this problem completely we obtain useful partial results.

A secondary goal, which we do achieve, is to characterize those spaces X for which the classical ring of quotients, $Q_{cl}(X)$, is isomorphic to a $C(Y)$. This complements work by Hager and Martinez [10], who studied Tychonoff spaces X for which $Q(X) = Q_{cl}(X)$.

By [7, 3.9] we may without loss of generality assume that all spaces are Tychonoff, i.e., completely regular and Hausdorff. As well, since $C(X)$ is ring-isomorphic to $C(\upsilon X)$, where υX denotes the Hewitt realcompactification of X [7, 8.1], it will sometimes be appropriate to assume that X is realcompact. Undefined notation and terminology will be as given in [7].

This work is the product of a collaboration between the two authors and Ruth Macoosh, who has declined to be a co-author. Her enthusiasm and insights were fundamental to the article, particularly to Sections 2–4 and 7.

2. Preliminaries on the epimorphic hull of a ring

We begin with a summary of some topics in commutative algebra that are needed for our study. Throughout all hypothesized rings will be assumed to be commutative and semiprime with identity. The reader is referred to [12] for general algebraic notions.

2.1. (a) *Semiprime and regular rings*

A ring is *semiprime* if it has no nilpotent element except 0. Clearly any family of real-valued functions (with a common domain) that is closed under the natural operations of addition and multiplication forms a semiprime ring.

The ring R is (von Neumann) *regular* if for each $r \in R$ there exists an $s \in R$ such that $r = r^2 s$. The element rs is idempotent, hence so is $1 - rs$. Since $r(1 - rs) = 0$, each element of R is either a zero-divisor or a unit. Each proper prime ideal of R is maximal. If the elements r and s of R satisfy $r = r^2 s$, then the element $s^2 r$ is the unique element

r^* satisfying simultaneously $r = r^2 r^*$ and $r^* = (r^*)^2 r$ [6, 10.1]. We shall call this unique element r^* the *quasi-inverse* of r.

(b) *Rings of quotients*

A ring T containing a ring R is a *ring of quotients* of R if and only if for each $0 \neq t \in T$ there exists an $r \in R$ such that $0 \neq tr \in R$ (see [12, Example 5, p. 46]). One verifies easily that if $R \subseteq S \subseteq T$ then T is a ring of quotients of R if and only if T is a ring of quotients of S and S is a ring of quotients of R.

(c) *The complete ring of quotients*

The complete ring of quotients of a ring R can be constructed from equivalence classes of module homomorphisms from dense ideals of R into R. Details appear in [12, Section 2.3]. The definition of addition and multiplication is natural, and the resulting ring, denoted $Q(R)$, is regular when R is semiprime [12, p. 42]. Thus each $r \in R$ has a quasi-inverse r^* in $Q(R)$. Furthermore, each non-zero-divisor of R is invertible in $Q(R)$. If T is a ring of quotients of R then [12, Proposition 6, p. 40] there is a monomorphism of T into $Q(R)$ that induces the canonical morphism of R into $Q(R)$. Less formally, we have:

if T is a ring of quotients of R then $R \subseteq T \subseteq Q(R)$.

(d) *The classical ring of quotients*

The *classical ring of quotients* of a ring R, denoted $Q_{cl}(R)$, is the subring of $Q(R)$ consisting of all elements of the form rs^{-1}, where $r, s \in R$, s is a non-zero-divisor of R, and s^{-1} is the inverse of s in $Q(R)$. All non-zero-divisors of R are units in $Q_{cl}(R)$, and hence $R = Q_{cl}(R)$ if and only if each non-unit of R is a zero-divisor of R.

(e) *Essential and epimorphic extensions*

An overring S of a ring R is called an *essential extension* of R if each non-zero ideal of S intersects R in a non-zero ideal [19, introduction to §9]. Clearly the rings of quotients of R are essential extensions of R.

A homomorphism of rings $f : R \to S$ is called an *epimorphism* if for any ring T and any pair of homomorphisms $g : S \to T$ and $h : s \to T$, we have that $g = h$ whenever $g \circ f = h \circ f$. Clearly the composition of two epimorphisms is an epimorphism. We shall use the following facts.

(i) If $f : R \to S$ and $g : S \to T$ are two homomorphisms such that $g \circ f$ is an epimorphism, then g is an epimorphism.

(ii) If R is regular then any ring epimorphism with domain R is surjective [19, 6.1].

An overring S of a ring R is called an *epimorphic extension* of R if the inclusion map is an epimorphism. An immediate consequence of (ii) is:

(iii) A regular ring has no proper epimorphic extensions.

(f) *The epimorphic hull of a ring*

The *epimorphic hull* of a ring R, denoted $H(R)$, is a canonical overring of R defined and studied by Storrer [19]. It can be characterized in each of the following ways:

(i) $H(R)$ is the unique (up to isomorphism over R) maximal essential epimorphic extension of R; in other words, S is an essential epimorphic extension of R if and only if $R \subseteq S \subseteq H(R)$ (where we denote monomorphisms by inclusions) [19, 8.3]. Thus $R \subseteq Q_{cl}(R) \subseteq H(R) \subseteq Q(R)$ [19, 11.3].

(ii) $H(R)$ is the (unique) smallest regular ring lying between R and $Q(R)$.
(iii) $H(R)$ is the unique ring of quotients of R that is both regular and an epimorphic extension of R.

The following result is part of [18, 1.6].

Lemma 2.2. *If the annihilator of r is principal for each $r \in R$, then $Q_{cl}(R)$ is regular.*

Lemma 2.3 [19]. *If $Q_{cl}(R)$ is regular then $H(R) = Q_{cl}(R)$.*

Lemma 2.4. *Let R be a ring, let B denote the set of all idempotents of $Q(R)$ of the form r^*r for $r \in R$, and let S denote the subring $R(B)$ of $Q(R)$ generated by R and B. Then $Q_{cl}(S) = H(S)$.*

Proof. An element of S has the form $s = \sum_{i=1}^{n} r_i e_i$ where $r_i \in R$ and $e_i^2 = e_i \in B$. By expanding the product $\prod_{i=1}^{n}(e_i + (1 - e_i)) = 1$ we can express 1 as a sum of 2^n orthogonal idempotents f_j in S (some possibly zero). Clearly $e_i f_j = f_j$ or $e_i f_j = 0$ for all i, j. A simple calculation will now show that

$$s = s1 = \sum_{i=1}^{n}\left(\sum_{j=1}^{2^n} r_i e_i f_j\right) = \sum_{j=1}^{2^n} r'_j f_j,$$

with the f_j orthogonal and the r'_j a subsum of the r_i. Since the f_j are orthogonal $\sum_{j=1}^{2^n}(r'_j)^* f_j$ is the quasi-inverse s^* for s in $Q(R)$. Therefore

$$ss^* = \sum_{j=1}^{2^n}(r'_j)^* r'_j f_j$$

which belongs to S, as does $1 - ss^*$.

Now consider the ideal $Ann(s) = \{t \in S\colon\ ts = 0\}$, the annihilator of s in S. Since $s = s^2 s^*$, $1 - ss^* \in Ann(s)$. Conversely, if $ts = 0$ then $tss^* = 0$ and $t = t(1 - ss^*)$. Thus $Ann(s)$ is principal and Lemmas 2.2 and 2.3 apply to S. □

The following is an independent proof of a result due to Olivier [16] and Storrer (unpublished).

Proposition 2.5. *The epimorphic hull of R is the subring T of $Q(R)$ generated by R and the quasi-inverses of the elements of R in $Q(R)$, i.e.,*

$$H(R) = \left\{\sum_{i=1}^{n} r_i s_i^*\colon\ r_i, s_i \in R,\ n \text{ a positive integer}\right\}.$$

Proof. Let S be as in Lemma 2.4. Then $S \subseteq T \subseteq H(R)$ and the proof of Lemma 2.4 shows that $s^* \in T$ for each $s \in S$. It follows that each non-zero-divisor of S has an inverse in T, for if s is a non-zero-divisor then $ss^* = 1$. Therefore $R \subseteq Q_{cl}(S) = H(S) \subseteq T \subseteq H(R)$, and by the remarks in 2.1(f)(ii) we have $H(S) = T = H(R)$. Thus each element of $H(R)$ has the form $\sum_{i=1}^{n} r_i(s_i^*)$ for some $r_i, s_i \in R$ and $n \in \mathbb{N}$. □

Corollary 2.6. *If R is a ring then R and $H(R)$ have the same cardinality.*

Lemma 2.7 (Isbell, Storrer; see [19, 3.3]). *Let R be a subring of a ring S. If for each $s \in S$ there exist $t \in S$ and $a, b \in R$ with $s = at$ and $a = tb$, then S is an epimorphic extension of R.*

We note that the quasi-inverses in $H(R)$ of the elements of R satisfy the conditions in Lemma 2.7 since $r^* = r(r^*)^2$ and $r = (r^*)^2(r^3)$ for each $r \in R$.

3. The epimorphic hull of $C(X)$

3.1. Definitions and preliminary remarks

The zero-set of a function $f \in C(X)$, denoted $Z(f)$, is the set $\{x \in X\colon f(x) = 0\}$. The cozero-set of f, $X \setminus Z(f)$, is denoted $\operatorname{coz} f$. We write $\mathbb{Z}(X)$ to designate the family of all zero-sets of X. A point p of X is called a P-point of X if p is in the interior of each zero-set containing it [7, 4L]. The set of all P-points of X will be denoted $P(X)$. The ring $C(X)$ is regular if and only if X is a P-space, i.e., if $P(X) = X$ [7, 4J and 4L].

If S is a dense subspace of X then the homomorphism $f \to f|S$ from $C(X)$ into $C(S)$ is a monomorphism. As in [6, 2.1] we sometimes abuse notation by identifying $\{f|S\colon f \in C(X)\}$ with $C(X)$ and writing $C(X) \subseteq C(S)$. The subring of all bounded functions in $C(X)$ is denoted $C^*(X)$. A subspace S of X is said to be C^*-embedded in X if every function in $C^*(S)$ can be extended to a function in $C^*(X)$. The Stone–Čech compactification of X, denoted βX, is the unique (up to homeomorphism fixing X pointwise) compact space in which X is dense and C^*-embedded.

As in [6], we denote the complete ring of quotients of $C(X)$ by $Q(X)$. Elements of $Q(X)$ can be represented by continuous real-valued functions whose domains are dense open subsets of X (see [6, 2.6 et seq.]). More precisely, let

$$\mathcal{D}(X) = \{V\colon V \text{ is a dense open subset of } X\}$$

and let

$$L(X) = \bigcup \{C(V)\colon V \in \mathcal{D}(X)\}.$$

Define a relation $\sim$ on $L(X)$ as follows: let $V, W \in \mathcal{D}(X)$, let $f \in C(V)$, and $g \in C(W)$. We say that $f \sim g$ if $f|V \cap W = g|V \cap W$. Then $\sim$ is an equivalence relation on $L(X)$ and $Q(X)$ is the set of equivalence classes associated with $\sim$, i.e., $Q(X) = \{[f]\colon f \in L(X)\}$. If $f \in C(V)$ and $g \in C(W)$, where $V, W \in \mathcal{D}(X)$, we define $[f] + [g]$ to be $[(f|V \cap W) + (g|V \cap W)]$. (Observe that $V \cap W \in \mathcal{D}(X)$ and so $(f|V \cap W) + (g|V \cap W) \in L(X)$.) The product $[f][g]$ is defined similarly. See [6, 2.6 et seq.] for details.

If $f \in C(X)$ we will often represent the element $[f]$ of $Q(X)$ by f. In other words, since $X \in \mathcal{D}(X)$ we will often identify the ring $C(X)$ with its isomorph, the subring $\{[f]\colon f \in C(X)\}$ of $Q(X)$. In this sense we have $C(X) \subseteq Q(X)$. Since $C(X) \subseteq H(X) \subseteq Q(X)$, members of $H(X)$ can be represented as equivalence classes of certain members of $L(X)$. In detail, we have the following:

Theorem 3.2. *Let X be a Tychonoff space. If $g \in C(X)$, denote $(\operatorname{int} Z(G)) \cup \operatorname{coz}(g)$ by $S(g)$. Define $g^\wedge : S(g) \to \mathbb{R}$ by*:

$$g^\wedge(x) = \frac{1}{g(x)} \quad \text{if } x \in \operatorname{coz}(g),$$
$$g^\wedge(x) = 0 \quad \text{if } x \in Z(g).$$

Then:
(a) $S(g) \in \mathcal{D}(X)$, $g^\wedge \in C(S(g))$, *and* $[g^\wedge] \in Q(X)$.
(b) $[g^\wedge]$ *is the quasi-inverse* $[g]^*$ *of* $[g]$ *in* $Q(X)$.
(c) $H(X) = \{\sum_{i=1}^n [f_i][g_i^\wedge]\colon$ *n a positive integer and* $f_i, g_i \in C(X)\}$.
(d) *Members of $H(X)$ can be represented as continuous real-valued functions whose domains are dense open sets of the form $\bigcap_{i=1}^n S(g_i)$, where $\{g_i\colon i, \ldots, n\}$ is a finite subset of $C(X)$.*

Proof. (a) This is straightforward.

(b) A routine computation shows that

$$\big(g^\wedge|S(g)\big)^2\big(g|S(g)\big) = g^\wedge|S(g) \quad \text{and} \quad \big(g|S(g)\big)^2\big(g^\wedge|S(g)\big) = g|S(g).$$

Consequently $[g^\wedge]^2[g] = [g^\wedge]$ and $[g]^2[g^\wedge] = [g]$ and so $[g^\wedge]$ is the quasi-inverse of $[g]$ in $Q(X)$.

(c) This follows immediately from (b) and Proposition 2.5.

(d) As indicated in the paragraph preceding Theorem 3.2, we can identify $\sum_{i=1}^n [f_i][g_i^\wedge]$ with the restriction of the real-valued function $\sum_{i=1}^n f_i g_i^\wedge$ to the element $\bigcap_{i=1}^n S(g_i)$ of $\mathcal{D}(X)$; clearly

$$\left(\sum_{i=1}^n f_i g_i^\wedge\right)\Bigg|\bigcap_{i=1}^n S(g_i) \in C\left(\bigcap_{i=1}^n S(g_i)\right). \qquad \square$$

Observe that $\bigcap\{S(g)\colon g \in C(X)\} = P(X)$.

By [6, 2.3 and 3.1] $Q(X) = Q(\beta X)$, $Q_{cl}(X) = Q_{cl}(\beta X)$, and we now show that a similar result holds for $H(X)$.

Proposition 3.3. *If Y is dense and C^*-embedded in X then $C(Y)$ is an epimorphic extension of $C(X)$ and consequently $H(Y) = H(X)$.*

Proof. By the "abuse of notation" mentioned earlier, we identify $C(X)$ with the subring $\{f|Y\colon f \in C(X)\}$ of $C(Y)$. As Y is C^*-embedded in X, if $g \in C^*(Y)$ then $g \in \{f|Y\colon f \in C(X)\}$ and so we have

$$C^*(Y) \subseteq \big\{f|Y\colon f \in C(X)\big\} \subseteq C(Y).$$

If $f \in C(Y)$ then $f/(f^2+1) = a \in C^*(Y)$, $f/(f^2+1)^2 = b \in C^*(Y)$, and $f^2+1 = t \in C(Y)$. Thus $a, b \in C(X)$. Furthermore $f = at$ and $a = tb$.

Therefore $C(Y)$ is an epimorphic extension of $C(X)$ by Lemma 2.7, and evidently it is a ring of quotients of $C(X)$. It then follows from 2.1(e) and 2.1(f)(i) that $C(Y) \subseteq H(X)$. Thus $H(Y) \subseteq H(X)$ and the opposite inclusion is obvious. □

Corollary 3.4. *Let X be a Tychonoff space. Then $H(X) = H(\beta X)$, and if X is a P-space then $C(X) = H(X) = H(\beta X)$.*

Corollary 3.5. *If X is an extremally disconnected space then $H(X) = Q(X)$.*

Proof. Let $f \in Q(X)$. Then $f \in C(V)$ for some dense open subset V of X. Since X is extremally disconnected, V is C^*-embedded in X (see [7, 1H.6]). The argument used in the proof of Proposition 3.3 above then shows that $f \in H(X)$. Thus $Q(X) \subseteq H(X)$, and the reverse inclusion always holds. □

Corollary 3.6 [8]. *If X is extremally disconnected and of non-measurable cardinal, then $H(X)$ is isomorphic to some $C(Y)$ if and only if the isolated points of X form a dense subset of X.*

Definition 3.7 [9]. A Φ-algebra A is an Archimedean lattice-ordered algebra over the field of real numbers which has an identity element 1 that is a weak order unit.

Each lattice-ordered algebra of real-valued functions that contains the constant functions is a Φ-algebra. In particular $Q(X)$ is a Φ-algebra [8, p. 9]. If A is a Φ-algebra then $\mathcal{M}(A)$ will denote the space of maximal ℓ-ideals (absolutely convex ideals) of A with the Stone topology (which has $\{M \in \mathcal{M}(A)\colon a \in M\}$ for $a \in A$ as a base for its closed sets; see [9, p. 79]). The subspace of real maximal ideals $\mathcal{R}(A)$ consists of those M in $\mathcal{M}(A)$ for which A/M is isomorphic to the real field $\mathbb{R}$. Note that A is called a Φ-algebra of real-valued functions if $\mathcal{R}(A)$ is dense in $\mathcal{M}(A)$.

Proposition 3.8. *$H(X)$ is a Φ-algebra.*

Proof. Let R be $C(X)$ and let S be as in Lemma 2.4. Each $s \in S$ may be written as $s = \sum_{i=1}^{n} f_i e_i$ where $f_i \in C(X)$ and $e_i = e_i^2 \in Q(X)$, so that the idempotents are pairwise orthogonal and the cozero sets of the $f_i e_i$ are therefore pairwise disjoint. Since $Q(X)$ is a Φ-algebra, we need only show that $H(X)$ is a lattice, and this is equivalent to showing that the elements of $H(X)$ have absolute values in $H(X)$ [7, p. 11]. Since $C(X)$ is a Φ-algebra and since the e_i are pairwise orthogonal, we have

$$\left|\sum_{i=1}^{n} f_i e_i\right| = \sum_{i=1}^{n} |f_i||e_i| = \sum_{i=1}^{n} |f_i| e_i \in S.$$

Thus S is a Φ-algebra.

As observed in Lemma 2.4, $H(X) = Q_{cl}(S)$. Thus if $f \in H(X)$, then $f = gh^{-1}$ where $g, h \in S$. Evidently $|g|, |h| \in S$ as S is a Φ-algebra. Since h is a non-zero divisor in S, it is invertible in the Φ-algebra $Q(X)$. Thus $|h|$ is invertible in $Q(X)$, and therefore it is a non-zero-divisor in S, whose inverse lies in $H(X)$. Hence $|f| = |g||h|^{-1}$, which is an element of $H(X)$. Thus $H(X)$ is a Φ-algebra. □

The space of maximal ideals of $C(X)$ and that of $H(X)$ are denoted respectively by $\mathcal{M}(X)$ and $\mathcal{M}(H(X))$; their subspaces of real maximal ideals by $\mathcal{R}(X)$ and $\mathcal{R}(H(X))$.

Lemma 3.9. *If $M \in \mathcal{M}(X)$ then $MH(X) \in \mathcal{M}(H(X))$ or $\mathcal{M}(H(X)) = H(X)$.*

Proof. Assume $MH(X) = I \neq H(X)$. Then $I \cap C(X) = M$ by the maximality of M. We now claim that the induced embedding $C(X)/M \to H(X)/I$ is an isomorphism. First observe that the map $C(X) \to H(X) \to H(X)/I$ is an epimorphism because it is the composition of two epimorphisms (see 2.1(e)). This composition can also be written as $C(X) \to C(X)/M \to H(X)/I$ and since it is an epimorphism so is $C(X)/M \to H(X)/I$ by 2.1(e)(i). But $C(X)/M$, being a field, and hence regular, has no proper epimorphic extensions (see 2.1(e)(iii)). □

Definition 3.10 [14]. A space X is called an almost P-space if every non-empty zero-set has non-empty interior.

By [14, 1.1] X is an almost P-space if and only if every zero-set is the closure (in X) of its interior. Evidently X is an almost P-space if and only if no cozero-set is dense, or equivalently if each non-unit in $C(X)$ is a zero-divisor. Thus X is an almost P-space if and only if $C(X) = Q_{cl}(X)$. Almost P-spaces were studied systematically in [14], and in [5] it is shown that if X is a locally compact realcompact Tychonoff space then $\beta X \backslash X$ is a compact almost P-space. Since P-spaces are characterized by the fact that their zero-sets are clopen (i.e., open-and-closed), obviously P-spaces are almost P-spaces.

It is pointed out in [14, p. 285], and follows immediately from the algebraic characterization above, that X is an almost P-space if and only if its Hewitt realcompactification υX is.

Corollary 3.11. *If X is an almost P-space then $MH(X) \in \mathcal{M}(H(X))$ for each $M \in \mathcal{M}(X)$.*

Proof. Let $\sum_{i=1}^{n} f_i m_i \in MH(X)$ where $m_i \in M$ and $f_i \in H(X)$. Then $|m_i| \in M$ for each i because M is absolutely convex. Since each non-unit of $C(X)$ is a zero-divisor, there exists a non-zero $g \in C(X)$ such that $g(\sum_{i=1}^{n} |m_i|) = 0$. Then $g|m_i| = 0$ for each i because all $|m_i|$ are non-negative and therefore $g(\sum_{i=1}^{n} f_i m_i) = 0$. Thus $MH(X)$ consists of zero-divisors of $H(X)$ and hence is a proper maximal ideal in $H(X)$. □

Corollary 3.12. *If X is an almost P-space then $H(X)$ is a Φ-algebra of real-valued functions.*

Proof. It follows from the proof of Lemma 3.9 that $MH(X) \in \mathcal{R}(H(X))$ for each $M \in \mathcal{R}(X)$. If $\mathbf{0} \neq f \in \bigcap \mathcal{R}(H(X))$ then there exists $g \in C(X)$ such that $\mathbf{0} \neq gf \in \bigcap \mathcal{R}(X)$, which is a contradiction since $C(X)$ is an algebra of real functions [9]. □

Remark 3.13. In view of the importance of almost P-spaces in connection with this work we note here that:

(1) A basically disconnected almost P-space is a P-space.
(2) An extremally disconnected almost P-space of non-measurable cardinal is discrete.

The first statement follows from Definition 3.10 and the fact that the interiors of zero-sets in X are closed [7, 1H]. Since an extremally disconnected space is basically disconnected, the second statement follows from [7, 12H].

4. Regular rings of quotients of $C(X)$ of the form $C(Y)$

Definitions and preliminary remarks 4.1. An ideal I in $C(X)$ is said to be fixed if $\bigcap\{Z(f)\colon f \in I\}$ is non-empty. The space X is called realcompact if every real maximal ideal in $C(X)$ is fixed. Realcompact spaces are discussed in detail in [7, Chapter 8]. We say a subspace S of X is C-embedded in X if every function in $C(S)$ can be extended to a function in $C(X)$. For each X there exists a unique realcompact space υX in which X is dense and C-embedded; thus υX is a realcompact space such that $C(\upsilon X)$ is isomorphic to $C(X)$. Consequently we shall henceforth assume that all hypothesized spaces are realcompact Tychonoff spaces. Of course, when we consider specific examples, we shall have to check that they are realcompact and Tychonoff.

The following is a modification of [7, 10.8 and 10.9(a)].

Theorem 4.2. *Let X and Y be realcompact Tychonoff spaces.*

(a) *If $t : C(X) \to C(Y)$ is a ring embedding for which $t(\mathbf{1}) = \mathbf{1}$, then there is a continuous function $t^\star : Y \to X$ such that*:
   (i) *$t^\star[Y]$ is dense in X.*
   (ii) *If $f \in C(X)$ then $t(f) = (f|t^\star[Y]) \circ t^\star$.*
(b) *If $\sigma : Y \to X$ is a continuous function such that $\sigma[Y]$ is dense in X, then the map $\sigma' : C(X) \to C(Y)$ defined by*
$$\sigma'(f) = \big(f|\sigma[Y]\big) \circ \sigma$$
is a ring embedding of $C(X)$ into $C(Y)$ for which $\sigma'(\mathbf{1}) = \mathbf{1}$.
(c) *$(t^\star)' = t$ and $(\sigma')^\star = \sigma$.*

Theorem 4.2 tells us that any ring embedding $C(X) \to C(Y)$ that preserves 1 can be viewed as arising from a continuous mapping σ from Y onto a dense subspace of X. In such situations we can (and will) identify $C(X)$ with its isomorphic copy $\{(f|\sigma[Y])\circ\sigma\colon\ f\in C(X)\}$, which is a subring of $C(Y)$.

Lemma 4.3. *Let X and Y be spaces and suppose that $C(Y)$ is a ring of quotients of $C(X)$. If $\sigma: Y\to X$ is the induced continuous map from Y onto a dense subset T of X, and if D is a dense subset of T, then $\sigma^{\leftarrow}[D]$ is dense in Y.*

Proof. As above, we identify $C(X)$ with the subring $\{(f|T)\circ\sigma\colon\ f\in C(X)\}$ of $C(Y)$. Suppose the lemma fails, and that D is a dense subset of T for which $\sigma^{\leftarrow}[D]$ is not dense in Y. As Y is Tychonoff and $Y\setminus cl_Y(\sigma^{\leftarrow}[D])\neq\emptyset$, there is an $f\in C(Y)\setminus\{\mathbf{0}\}$ such that $f[\sigma^{\leftarrow}[D]]=\{0\}$. By hypothesis there exist $g,h\in C(X)$ such that

$$f\cdot\big((g|T)\circ\sigma\big)=(h|T)\circ\sigma\neq\mathbf{0}.$$

As $\sigma[Y]=T$, if $d\in D$ then $\emptyset\neq\sigma^{\leftarrow}(d)$. Let $p\in\sigma^{\leftarrow}(d)$. Then

$$0=f(p)=f(p)\cdot g\big(\sigma(p)\big)=h\big(\sigma(p)\big)=h(d)$$

and so $h[D]=\{0\}$. As D is dense in T, and hence in X, it follows that $h=\mathbf{0}$, a contradiction. The lemma follows. □

Recall that P-spaces were defined in Definition 3.1.

Theorem 4.4 [17, 1W]. *Let X be a space. Then*:

(1) *$\mathbb{Z}(X)$ is an open base for a Tychonoff P-space topology on X. The G_δ-sets of X form an open base for the same topology.*
(2) *Let X_δ denote X equipped with this P-space topology and let $j: X_\delta\to X$ denote the identity map on the underlying set of X. Then if Y is a Tychonoff P-space and if $f: Y\to X$ is any continuous map, then there is a continuous map $k: Y\to X_\delta$ such that $j\circ k=f$.*

The space X_δ is called the P-space coreflection of X. By [17, 5F(7)], X_δ is realcompact if X is.

Recall that almost P-spaces were defined in Definition 3.10.

Theorem 4.5. *The following are equivalent for a realcompact Tychonoff space X*:

(1) *$C(X_\delta)$ is a ring of quotients of $C(X)$.*
(2) *X is an almost P-space.*
(3) *If D is dense in X then $j^{\leftarrow}[D]$ (see Theorem 4.4) is dense in X_δ.*

Proof. (1) ⇒ (3) This follows from Lemma 4.3.

(3) ⇒ (2) Let $\emptyset\neq Z\in\mathcal{Z}(X)$. Then $(\mathrm{int}_X Z)\cup(X\setminus Z)$ is dense in X, and hence by hypothesis dense in X_δ. But Z is a nonempty open subset of X_δ, and hence

$$Z\cap\big((\mathrm{int}_X Z)\cup(X\setminus Z)\big)\neq\emptyset.$$

Thus $\mathrm{int}_X Z\neq\emptyset$, and X is an almost P-space.

(2) ⇒ (1) Suppose $\mathbf{0} \neq f \in C(X_\delta)$. Then there exists $r \in \mathbb{R}\backslash\{0\}$ such that $f^{\leftarrow}(r) \neq \emptyset$. As $f^{\leftarrow}(r) \in \mathbb{Z}(X_\delta)$, there exists $S \in \mathbb{Z}(X)$ such that $\emptyset \neq S \subseteq f^{\leftarrow}(r)$. As X is an almost P-space, there exists $g \in C(X)$ and $p \in \operatorname{int}_X S$ such that $g(p) = 1$ and $g[X \setminus \operatorname{int}_X S] = \{0\}$. A straightforward calculation shows that

$$0 \neq f \cdot (g \circ j) = (\boldsymbol{r} g) \circ j.$$

Since $C(X)$ is identified with the subring $\{k \circ j \colon k \in C(X)\}$ of $C(X_\delta)$, it follows that $C(X_\delta)$ is a ring of quotients of $C(X)$. □

Corollary 4.6. *If $C(Y)$ is a regular ring of quotients of $C(X)$ and if $T = t^\star[Y]$ is the image of Y under the induced mapping $Y \to X$, then $C(Y)$ is a ring of quotients of $C(T_\delta)$, and $C(T_\delta)$ is a regular ring of quotients of $C(T)$.*

Proof. By Theorem 4.4 there is a mapping $\kappa : Y \to T_\delta$ which is onto because $Y \to T$ is onto (4.2). The induced homomorphisms $C(T) \to C(T_\delta) \to C(Y)$ are then embeddings and rings of quotients of their domains because $C(T) \to C(Y)$ is. (See 2.1(b).) □

Lemma 4.7. *Let T be a dense subspace of X. If T is an almost P-space then $C(T)$ is a ring of quotients of $C(X)$.*

Proof. As before, we identify $C(X)$ with $\{f|T\colon f \in C(X)\}$. Let $\mathbf{0} \neq f \in C(T)$. Then $f^{\leftarrow}(r)$ is a nonempty zero-set of T for some $r \neq 0$. By assumption there is an open subset V of X such that $\emptyset \neq V \cap T \subseteq f^{\leftarrow}(r)$. Choose $p \in V \cap T$ and $g \in C(X)$ such that $g(p) = 1$ and $g[X \backslash V] = \{0\}$. Then $\mathbf{0} \neq f \cdot (g|T) = (\boldsymbol{r} g)|T \in C(X)$. □

[We remark that it is not always true that if T is dense in X, then $C(T)$ is a ring of quotients of $\{f|T\colon f \in C(X)\}$. For example, let $X = [0,1]$ and let T be the irrational points of X. It is well known that there exists $f \in C(T)$ such that f cannot be continuously extended to any point of $X \backslash T$. A modification of [6, 3.12] (which is the source of this example) shows that there do not exist $g, h \in C(X)$ such that $\mathbf{0} \neq f \cdot (g|T) = h|T$.]

Definition 4.8. The intersection of all dense cozero sets of X will be denoted by gX. By [7, 8.9, 8.14], gX will be realcompact if X is. We note as well that gX contains the set of all P-points of X.

Lemma 4.9. *The following are equivalent for a space X:*

(1) *gX is dense in X.*
(2) *X has a dense subspace that is an almost P-space.*
(3) *gX is an almost P-space, it is dense in X, and it contains every almost P-space that is dense in X.*

Proof. (1) ⇒ (2) Let $Z \in Z(gX)$ and let $p \in Z$. As Z is a G_δ-set of gX [7, 1.10] there is a countable family $\{V(n)\colon n \in \mathbb{N}\}$ of open subsets of X such that $Z = \bigcap\{V(n) \cap gX\colon n \in \mathbb{N}\}$. By [7, 3.11(b)] there exists $F \in \mathbb{Z}(X)$ such that $p \in F \subseteq \bigcap\{V(n)\colon n \in \mathbb{N}\}$. Since

$F \cap gX \neq \emptyset$, it follows from the definition of gX that $\mathrm{int}_X F \neq \emptyset$. As gX is dense in X and $F \cap X \subseteq Z$, it follows that $\mathrm{int}_{gX} Z \neq \emptyset$. Hence gX is an almost P-space.

(2) $\Rightarrow$ (3) Let T be a dense subspace of X and an almost P-space. If V is any dense cozero-set of X then $V \cap T$ is a dense cozero-set of T and thus $V \cap T = T \subseteq V$. Consequently $T \subseteq gX$. Thus gX is dense in X and is therefore an almost P-space by the preceding argument.

(3) $\Rightarrow$ (1) This is obvious. □

Corollary 4.10. *If X has a dense almost-P subspace, then it has a largest dense almost-P subspace, namely gX.*

Theorem 4.11.

(1) *The following are equivalent for a space X:*
  (a) *$C(X)$ has a regular ring of quotients of the form $C(Y)$.*
  (b) *gX is dense in X.*

(2) *If the equivalent conditions in* (1) *hold then there is a canonical copy of $C((gX)_\delta)$ between $C(X)$ and $C(Y)$.*

Proof. (a) $\Rightarrow$ (b) Let $\sigma : Y \to X$ be the continuous mapping from Y onto a dense subset T of X induced by the embedding of $C(X)$ into $C(Y)$ (see Theorem 4.2). By Corollary 4.6 we have the ring embeddings

$$C(X) \to C(T) \to C(T_\delta) \to C(Y),$$

where $f \in C(X)$ is taken to $(f|T) \circ \sigma \in C(Y)$. By 2.1(b), $C(T_\delta)$ is a ring of quotients of $C(T)$ because $C(Y)$ is a ring of quotients of $C(X)$. It follows from Theorem 4.5 that T is an almost P-space. Since T is dense in X, it then follows that gX is dense in X.

(b) $\Rightarrow$ (a) By Lemma 4.9 gX is an almost P-space. By Lemma 4.7 $C(gX)$ is a ring of quotients of $C(X)$ (via the embedding $f \mapsto f|gX$). By Theorem 4.5 $C((gX)_\delta)$ is a ring of quotients of $C(gX)$, and is regular as $(gX)_\delta$ is a P-space. Thus by 2.1(b), $C((gX)_\delta)$ is a ring of quotients of $C(X)$.

(2) By Corollary 4.6 and the above we have embeddings

$$C(X) \xrightarrow{k} C(gX) \xrightarrow{i} C\big((gX)_\delta\big) \xrightarrow{j} C(Y),$$

where $f \in C(X)$ is taken to $f|gX \in C((gX)_\delta)$ by $i \circ k$, and to $(f|gX) \circ \kappa \in C(Y)$ by $j \circ i \circ k$ (where κ is as in Corollary 4.6). □

Corollary 4.12. *If $H(X)$ is isomorphic to a $C(Y)$ then $Y = (gX)_\delta$ and furthermore $H(X) = H(gX)$.*

Proof. In what follows the maps i, j, and k are as in the proof of Theorem 4.11(2) above. By 2.1(f)(ii) $H(X)$ is the smallest regular ring of quotients of $C(X)$. If $H(X)$ is isomorphic to a $C(Y)$, then as $C((gX)_\delta)$ is regular (as $(gX)_\delta$ is a P-space) and a ring of quotients of $C(X)$ (by 2.1(b)), it follows that j must be an isomorphism. By the realcompactness of

$(gX)_\delta$ and Y, this implies that Y is homeomorphic to $(gX)_\delta$. But as the embedding $i \circ k$ is then the canonical embedding of $C(X)$ in $H(X)$ and as this canonical embedding is an epimorphism (see 2.1(f)(i)), then i is an epimorphism (see 2.1(e)(i)). But $H(X)$ is a ring of quotients of $C(gX)$ (by 2.1(b)) and an epimorphic extension of $C(gX)$ (as noted above), so by 2.1(f)(iii), $C((gX)_\delta) = H(g(X))$. □

Corollary 4.13. *If X has a dense subspace that is an almost P-space then $H(X)$ is a Φ-algebra of real-valued functions.*

Proof. By 2.1(b) and Theorem 4.11(2), $C((gX)_\delta)$ is a regular ring of quotients of $C(X)$ (as noted in the preceding proof). Hence by 2.1(f)(ii) there is an embedding of $H(X)$ into $C((gX)_\delta)$. It is standard that any fixed maximal ideal of $C((gX)_\delta)$ contracts to a real maximal ideal of the subring $H(X)$. Furthermore, the intersection of these contractions to $H(X)$ is clearly the zero ideal. Since $H(X)$ is also a Φ-algebra (see Proposition 3.8), the conclusion follows. □

5. When is $H(X)$ a $C(Y)$?

As yet we do not have a complete answer to this question. However, we do have considerable information about several classes of special cases. We begin by recapitulating the relationship among $C(X)$, $C(gX)$, $C((gX)_\delta)$ and $H(X)$.

Suppose that there is a (realcompact Tychonoff) space Y such that $C(Y)$ is a regular ring of quotients of (an embedded copy of) $C(X)$. Then by Theorem 4.11 $C((gX)_\delta)$ is such a ring, gX is dense in X, and we have

$$C(X) \cong \{f|gX\colon\ f \in C(X)\} \subseteq C(gX) \subseteq C\big((gX)_\delta\big).$$

Since $H(X)$ is the smallest regular ring of quotients of $C(X)$, it follows that

$$C(X) \cong \{f|gX\colon\ f \in C(X)\} \subseteq H(X) \subseteq C\big((gX)_\delta\big). \qquad (*)$$

By Corollary 4.12 we know that $H(X)$ is a $C(Y)$ for some Y iff gX is dense in X and $H(X) = C((gX)_\delta)$, i.e., iff the second inclusion above is in fact an equality. In general it is not immediately obvious under what conditions we would have $C(gX) \subseteq H(X)$. The minimality of $H(X)$ among regular rings of quotients of $C(X)$ does mean that $H(X)$ is naturally embedded in $H(gX)$. Note that Example 6.4 shows that $C(gX)$ need not be included in $H(X)$ and that $H(X)$ can be a proper subring of $H(gX)$. However, $H(X)$ and $H(gX)$ can coincide without being a $C(Y)$. This will happen if X is a compact, non-scattered almost-P space (such as $\beta\mathbb{N}\setminus\mathbb{N}$).

One consequence of $(*)$ is that since gX and $(gX)_\delta$ have the same underlying set, each member of $H(X)$ would be representable as a real-valued function with domain gX. In particular, if $f \in C(X)$ then $(f|gX)^*$ would be representable in this fashion. A straightforward computation shows that $S(f|gX)$ is a dense subset of $(gX)_\delta$, and hence by Theorem 3.2 there is a unique member $f^\wedge \in C((gX)_\delta)$ such that $f^\wedge|S(f|gX) =$

$(f|gX)^*$, namely: $f^\wedge(x) = 1/f(x)$ if $x \in (\operatorname{coz} f) \cap gX$, and $f^\wedge(x) = 0$ if $x \in Z(f) \cap gX$. We will (again) abuse notation slightly and identify $(f|gX)^*$ with $f^\wedge$. Thus we have:

Lemma 5.1. *Let gX be dense in the Tychonoff space X.*

(a) *If $f \in C(X)$ then $(f|gX)^*(x) = 1/f(x)$ if $x \in gX \cap \operatorname{coz} f$ and $(f|gX)^*(x) = 0$ if $x \in gX \cap Z(f)$.*

(b) *The ring $H(X) = \{\sum_{i=1}^n (f_i|gX)(h_i|gX)^*\colon\ f_i, h_i \in C(X)$, and $n \in \mathbb{N}\}$.*

Proof. Part (a) follows from the previous discussion. Part (b) follows from (a) and Theorem 3.2. □

We briefly digress from our consideration of $H(X)$ to consider a more general algebra of real-valued functions. Let X be a Tychonoff space. If $g \in C(X)$, let us *define* $g^\wedge : X \to \mathbb{R}$ by

$$g^\wedge(x) = 1/g(x) \quad \text{if } x \in \operatorname{coz}(g),$$
$$g^\wedge(x) = 0 \quad \text{if } x \in Z(g).$$

We then define $G(X)$ to be the subring of the ring of all real-valued functions with domain X generated by $C(X) \cup \{g^\wedge\colon\ g \in C(X)\}$. Since $(fg)^\wedge = f^\wedge g^\wedge$, it is immediate that

$$G(X) = \left\{ \sum_{n=1}^{n} f_i(g_i^\wedge)\colon\ n \in \mathbb{N},\ f_i, g_i \in C(X) \right\}.$$

By Lemma 5.1(b), if X is an almost P-space (i.e., if $gX = X$), then $G(X)$ is the epimorphic hull $H(X)$ of $C(X)$. However, we warn the reader that in general this is not the case. More specifically, let X be a Tychonoff space and consider the map λ from the set $C(X) \cup \{f^\wedge\colon\ f \in C(X)\}$ onto $C(X) \cup \{f^*\colon\ f \in C(X)\}$ defined by: $\lambda(g) = g$ if $g \in C(X)$, and $\lambda(f^\wedge) = f^*$ if $f \in C(X)$. This map cannot, in general, be "extended by linearity" to a ring isomorphism from $G(X)$ onto $H(X)$. If X is an almost P-space, however, the obvious linear extension of λ is indeed a ring isomorphism.

Denote the subring of bounded members of $G(X)$ by $G^*(X)$. Then $G^*(X)$ carries the usual "sup norm" defined by:

$$\|k\| = \sup\bigl\{|k(x)|\colon\ x \in X\bigr\} \quad \text{if } k \in G^*(X),$$

and this norm induces a metric on $G^*(X)$. As usual, we say that $G^*(X)$ is *uniformly closed* if it is complete with respect to this "sup norm" metric.

If X is an almost P-space and if $H(X) = C(X_\delta)$, then $G^*(X) = H^*(X) = C^*(X_\delta)$, and so $G^*(X)$ would be uniformly closed as $C^*(X_\delta)$ is. Consequently it becomes of interest to know what conditions X must satisfy if $G^*(X)$ is to be uniformly closed.

Lemma 5.2. *Let X be a Tychonoff space. If $\sum_{i=1}^n f_i(g_i^\wedge) = h \in G(X)$, then $Z(h)$ has the form $\bigcup_{i=1}^k S_i \cap V_i$, where each $S_i \in \mathbb{Z}(X)$ and each $V_i \in \operatorname{coz}(X)$.*

Proof. For each subset A of $\{1,\dots,n\}$, let

$$T(A) = \Bigl[\bigcap\bigl\{\operatorname{coz}(g_i)\colon\ i \in A\bigr\}\Bigr] \cap \Bigl[\bigcap\bigl\{Z(g_i)\colon\ i \in \{1,\dots,n\}\backslash A\bigr\}\Bigr].$$

Clearly $\{T(A)\colon\ A \subseteq \{1,\ldots,n\}\}$ partitions X and each $T(A)$ has the form $S \cap V$, where $S \in \mathbb{Z}(X)$ and $V \in \mathrm{coz}(X)$. If $x \in T(A)$ then $h(x) = 0$ iff $\sum_{i\in A} f_i(x)/g_i(x) = 0$ iff $\sum_{i\in A} f_i(x)[\prod_{j\in A\setminus\{i\}} g_j(x)] = 0$. But $\sum_{i\in A} f_i[\prod_{j\in A\setminus\{i\}} g_j] = k_A \in C(X)$, so evidently $Z(h) = \bigcup\{T(A) \cap Z(k_A)\colon\ A \subseteq \{1,\ldots,n\}\}$ which is the required form. □

Corollary 5.3. *If $h \in G(X)$ then $Z(h)$ is an F_σ-set of X.*

Proof. Each cozero-set of X is a union of countably many closed subsets of X (i.e., is an F_σ-set of X), and the intersection of a closed set and an F_σ-set is clearly an F_σ-set. The result now follows from Lemma 5.2. □

Theorem 5.4. *Let X be a Tychonoff space for which $G^*(X)$ is uniformly closed. Then the union of any countable family of zero-sets of X is a G_δ-set of X.*

Proof. Let $\{Z(i)\colon\ i \in \mathbb{N}\}$ be a countable family of zero-sets of X. Let g_i be the characteristic function of $Z(i)$. Then $g_i = 1 - f_i f_i^\wedge$, where $Z(i) = Z(f_i)$, and so $g_i \in G^*(X)$ for each i. Let $h_n = \sum_{i=1}^{n} (\frac{1}{2})^i g_i$. Then each $h_n \in G^*(X)$. Clearly (h_n) is a Cauchy sequence of functions in $G^*(X)$ with respect to the "sup norm" metric. Thus as $G^*(X)$ is assumed to be uniformly closed, the limit of this sequence, which is $\sum_{i=1}^{\infty} (\frac{1}{2})^i g_i$ (henceforth denoted by h) is in $G^*(X)$. Clearly $\mathrm{coz}(h) = \bigcup_{i\in\mathbb{N}} Z(f_i)$. By Corollary 5.3 $Z(h)$ is an F_σ-set so its complement $\mathrm{coz}(h)$ is a G_δ-set of X. The theorem follows. □

Corollary 5.5. *Let X be an almost P-space. If $H(X) = C(X_\delta)$ then the union of countably many zero-sets of X must be a G_δ-set of X.*

Definition 5.6. A Tychonoff topological space X is called *scattered* if every subspace of X contains isolated points.

Spaces of ordinal numbers are examples of scattered spaces, as are one-point compactifications of discrete spaces. The property of being scattered is preserved by subspaces (obviously) and the formation of products with finitely many factors (but not infinitely many). It is easy to verify that a space is scattered space if and only if each of its subspaces contains a dense set of isolated points. See [11, §9 VI] for more information.

Theorem 5.7. *If $G^*(X)$ is uniformly closed then every compact subspace of X is scattered.*

Proof. By Theorem 5.4 it suffices to prove that if X has a compact subspace T that is not scattered, then there is a countable family $\{Z_n\colon\ n \in \mathbb{N}\}$ of zero-sets such that $\bigcup\{Z_n\colon\ n \in \mathbb{N}\}$ is not a G_δ-set of X.

In the first part of the proof of [15, 3.1] it is shown that if L is a non-scattered compact space, then there is a compact subspace B of L and a continuous surjection from B onto the Cantor set C. So, there is a compact subset F of T and a continuous surjection $g: F \to C$. As g is a perfect surjection there is a compact subspace K of F such that the restriction

$g|K$ of g is a perfect continuous irreducible surjection (see [17, 6.1(b) and 6.5(c)]. By [17, 6.1(b)], K is separable and has no isolated points.

Let $\{p_n\colon\ n \in \mathbb{N}\}$ be a faithfully indexed countable dense subset of K. Fix $n \in \mathbb{N}$. Because X is Tychonoff, for each $j \in \mathbb{N}\backslash\{n\}$ there exists a zero-set $Z(n, j)$ of X such that $p_n \in Z(n, j)$ and $p_j \notin Z(n, j)$. Let $Z(n) = \bigcap\{Z(n, j)\colon\ j \in \mathbb{N}\backslash\{n\}\}$. Then $Z(n)$ is a zero-set of X. Since K has no isolated points, the set $\{p_j\colon\ j \in \mathbb{N}\backslash\{n\}\}$ is dense in K and so $Z(n) \cap K$ is a closed nowhere dense zero-set of K.

Now suppose that $\bigcup\{Z(n)\colon\ n \in \mathbb{N}\}$ is a G_δ-set of X, i.e., is of the form $\bigcap\{W(j)\colon\ j \in \mathbb{N}\}$ where each $W(j)$ is open in X. Then $[X\backslash W(j)] \cap K$, which we denote by $M(j)$, is a closed subset of K, and as it is disjoint from the dense subset $\{p_n\colon\ n \in \mathbb{N}\}$ of K, it is nowhere dense in K. Thus

$$K = \bigcup\big\{Z(n) \cap K\colon\ n \in \mathbb{N}\big\} \cup \Big[\bigcup\big\{M(j)\colon\ j \in \mathbb{N}\big\}\Big],$$

and so we have expressed the compact space K as the union of countably many closed nowhere dense subsets, which contradicts the Baire category theorem. Thus $\bigcup\{Z(n)\colon\ n \in \mathbb{N}\}$ is not a G_δ-set of X, and our theorem follows. □

Theorem 5.8. *Let X be Tychonoff. If there is a space Y such that $H(X) = C(Y)$ then gX is dense in X and each compact subspace of gX is scattered.*

Proof. By Corollary 4.12 our hypotheses imply that gX is dense in X, that $Y = (gX)_\delta$, and that $H(X) = H(gX)$. As $H(X) = C((gX)_\delta)$, it follows from the representation of $H(X)$ given in Lemma 5.1 that $C(gX) \subseteq H(X) = H(gX) = C((gX)_\delta)$. By the discussion following the proof of Lemma 5.2, it follows that $G^*(gX) = H^*(gX) = C^*((gX)_\delta)$. As $C^*((gX)_\delta)$ is uniformly closed, it follows that $G^*(gX)$ is, and so by Theorem 5.7 every compact subset of gX is scattered. □

For example, if D is an infinite discrete space then its Stone–Čech outgrowth $\beta D\backslash D$ is a compact almost-P space without isolated points (see [5]) and hence by Theorem 5.8, $H(\beta D\backslash D)$ is not a $C(Y)$; it is not even uniformly closed.

Theorem 5.9. *Let X be a realcompact almost-P space. If $H(X)$ and $C(X_\delta)$ have the same idempotents and if $H(X)$ is uniformly closed then $H(X) = C(X_\delta)$ and consequently $H(X)$ is a ring of continuous functions.*

Proof. One has the monomorphism $C(X) \to H(X) \to C(X_\delta)$. Since both $H(X)$ and $C(X_\delta)$ are regular and have the same idempotents, contraction and extension of ideals establish a homeomorphism between Spec $H(X)$ and Spec$(C(X_\delta)) = \beta(X_\delta)$. By [9] (see p. 89 lines before 5.1, and p. 90 lines before 5.2), the following are equivalent for the Φ-algebra $H(X)$.

(i) $H(X)$ is isomorphic to a ring of continuous functions,

(ii) $H(X)$ is isomorphic to $C(R(H(X)))$,

(iii) $\mathcal{M}(H(X)) = \beta(R(H(X)))$,
(iv) $H(X) = C(X_\delta)$.

Thus it suffices to show that $\mathcal{M}(H(X)) = \beta(R(H(X)))$, which we now do. It is clear that contraction of maximal ideals from $\beta(R(H(X)))$ to $\mathcal{M}(H(X))$ when restricted to real maximal ideals, defines a one-to-one continuous map from X_δ to $\mathcal{R}(H(X))$. If this map is onto, it will be a homeomorphism because $\beta(X_\delta)$ and Spec $H(X)$ are homeomorphic. Let M be a real maximal ideal in $H(X)$. Then $M \cap C(X)$ is real maximal in $C(X)$; say it is N_p for some point $p \in X$.

Let $N_p^\wedge$ denote the corresponding real maximal ideal in $C(X_\delta)$. Both $N_p^\wedge \cap H(X)$ and M are real maximal ideals of $H(X)$ that contract to N_p in $C(X)$. But contraction defines maps

$$\mathrm{Spec}\big(C(X_\delta)\big) \to \mathrm{Spec}\big(H(X)\big) \to \mathrm{Spec}(X).$$

The first is one-to-one and onto as noted above, and the second is one-to-one because $H(X)$ is an epimorphic extension of $C(X)$ (see [13, Proposition 1.6]). Thus $M = N_p^\wedge \cap H(X)$, and the contraction map from the real maximal ideals of $C(X_\delta)$ is onto. □

Definition 5.10 [15]. A Tychonoff topological space is called *functionally countable* if $|f[X]| \leqslant \aleph_0$ for each $f \in C(X)$.

Lemma 5.11. *Let X be a Tychonoff space that either contains a compact subspace with no isolated points, or else is not totally disconnected. Then X is not functionally countable.*

Proof. Suppose K is a compact subspace of X without isolated points. Then as demonstrated in the proof of 3.1 of [15], there exists a continuous surjection g from K onto the Cantor set (viewed as a subspace of $\mathbb{R}$). By 3.11(c) of [7] there exists $f \in C(X)$ such that $f|K = g$. Clearly $|f[X]| \geqslant |g[K]| = \mathfrak{c}$ so X is not functionally countable.

Secondly, let C be a connected component of X with more than one point. As X is Tychonoff there exists $f \in C(X)$ such that $|f[C]| > 1$. As $f[C]$ is a connected subset of $\mathbb{R}$, it follows that $|f[X]| \geqslant |f[C]| = c$. Thus X is not functionally countable. □

Theorem 5.12. *Let X be a realcompact Tychonoff space for which $|C(gX)| = \mathfrak{c}$. If gX is not functionally countable then $H(X) \neq C((gX)_\delta)$, and hence $H(X)$ is not a $C(Y)$. In particular if either gX is not totally disconnected, or else gX contains a compact subspace without isolated points, then $H(X)$ is not a $C(Y)$.*

Proof. If $H(X) = C((gX)_\delta)$ then by Theorem 4.11 gX is dense in X. Thus by Corollary 2.6 $|H(X)| = |C(X)| = |C(gX)| = \mathfrak{c}$. Since gX is not functionally countable, by the proof of 3.1 of [15] there exists an $f \in C(gX)$ for which $|f[gX]| = \mathfrak{c}$. Denote $f[gX]$ by S. Thus $\{f^{\leftarrow}(r)\colon r \in S\}$ partitions $(gX)_\delta$ into $\mathfrak{c}$ pairwise disjoint clopen subsets. If $A \subseteq S$ then the characteristic function of the set $\bigcup\{f^{\leftarrow}(r)\colon r \in A\}$ belongs to $C((gX)_\delta)$, and so $|C((gX)_\delta)| \geqslant 2^{\mathfrak{c}}$. Thus $H(X) \neq C((gX)_\delta)$. The rest of the theorem follows from Lemma 5.11 and Corollary 4.12. □

We now consider some situations in which $H(X)$ is a $C(Y)$. We begin by analyzing what gX is in several situations.

Lemma 5.13. *Let T be a realcompact almost-P space, and let T be a dense subspace of a space X. If T is C^*-embedded in X then $gX = T$.*

Proof. By [7, 6.7] we know that $T \subseteq X \subseteq \beta T$. Let $p \in X \backslash T$. By [17, 5.11(c)] there exists $C(p) \in \operatorname{coz} \beta X$ such that $T \subseteq C(p) \subseteq \beta T \backslash \{p\}$. As $X \cap C(p) \in \operatorname{coz} X$, it follows that $gX \subseteq T$. By Lemma 4.9 $gX = T$. □

Recall (see problem 3P of [17], for example) that a Tychonoff space X is called *weakly Lindelöf* if each open cover of X has a countable subfamily whose union is dense in X. Lindelöf spaces and spaces of countable cellularity are weakly Lindelöf; see the cited reference.

Lemma 5.14. *The following are equivalent for a realcompact almost-P Tychonoff space T:*

(1) *T is weakly Lindelöf.*
(2) *T is Lindelöf.*
(3) *If T is dense in the realcompact space X then $gX = T$.*

Proof. Clearly (2) implies (1). For the converse, suppose T is a weakly Lindelöf space and let $\boldsymbol{C}$ be an open cover of T. Each member of $\boldsymbol{C}$ can be written as a union of cozero sets of T as T is Tychonoff. By hypothesis there are countably many of these cozero sets whose union is dense in T. As T is almost P, this union must in fact be all of T. So, the countable subcollection of $\boldsymbol{C}$ whose members contain the countably many cozero sets in question must be a countable subcover of $\boldsymbol{C}$. Thus T is Lindelöf. Hence (1) implies (2).

Suppose (2) holds. By Lemma 4.9 in order to prove (3) it suffices to prove that $gX \subseteq T$. So, let $p \in X \backslash T$. As X is Tychonoff there exists a family $\boldsymbol{C(p)}$ of cozero sets of X such that $\bigcup \boldsymbol{C(p)} = X \backslash \{p\}$. As T is Lindelöf, there is a countable subfamily $\{C(i)\colon i \in \mathbb{N}\}$ of $\boldsymbol{C(p)}$ such that $T \subseteq \bigcup \{C(i)\colon i \in \mathbb{N}\} = V(p)$. Clearly $V(p)$ is a dense cozero set of X that contains T and not p, so $p \notin gX$. Consequently $T = gX$.

To prove that (3) implies (2), suppose that (2) fails. As βT is Tychonoff, there is a family $\boldsymbol{C}$ of cozero sets of βT such that $T \subseteq \bigcup \boldsymbol{C}$, but no countable subfamily of $\boldsymbol{C}$ covers T. Let K denote the one-point compactification of the subspace $\bigcup \boldsymbol{C}$ of βT, with p being the "point at infinity". Then T is a dense subspace of K. If W were a dense cozero set of K then as T is almost P, $T \subseteq W$. If $p \notin W$ then $W \subseteq \bigcup \boldsymbol{C}$, and as W is Lindelöf (being a cozero set of a compact space and hence σ-compact), there would be a countable subfamily of $\boldsymbol{C}$ that covers W and hence T, in contradiction to the choice of $\boldsymbol{C}$. Thus $p \in gK \backslash T$. Hence (3) fails and (2) implies (3). □

Lemma 5.15. *Let X be a realcompact space and let $\{K(n)\colon n \in \mathbb{N}\}$ be a sequence of compact open almost-P subspaces whose direct sum $\bigoplus \{K(n)\colon n \in \mathbb{N}\} = T$ is a dense subspace of X. Then $gX = T$ and $C(T) \subseteq H(X)$ (and so $H(T) = H(X)$).*

Proof. As T is a σ-compact free union of compact almost-P spaces, it is Lindelöf and almost-P. Hence by Lemma 5.14 $gX = T$. By the remarks preceding Lemma 5.1,

$$C(X) \cong \{f|T\colon\ f \in C(X)\} \subseteq H(X) \subseteq C(T_\delta).$$

So by Lemma 5.1(b) it suffices to show that if $k \in C(T)$ there exist $f, h \in C(X)$ such that $k = (f|T)(h|T)^*$. Define f and h as follows:

$$\begin{aligned} f(x) &= [k(x)][n(|k(x)|+1)]^{-1} && \text{if } x \in K(n),\\ f(x) &= 0 && \text{if } x \in X\backslash T;\\ h(x) &= [n(|k(x)|+1)]^{-1} && \text{if } x \in K(n),\\ h(x) &= 0 && \text{if } x \in X\backslash T. \end{aligned}$$

Since, for each $a \geqslant 0$, $f^{\leftarrow}[\mathbb{R}\backslash(-a,a)]$ and $h^{\leftarrow}[\mathbb{R}\backslash(-a,a)]$ are compact, one easily shows that f and h belong to $C(X)$. A routine computation shows that $k = (f|T)(h|T)^*$. □

Theorem 5.16. *Let X be a realcompact space with a countable dense set $D = \{d(i)\colon i \in \mathbb{N}\}$ of isolated points. Then $H(X) \cong C(\mathbb{N})$.*

Proof. By Lemma 5.14 $gX = D = D_\delta$ so by the remarks preceding Lemma 5.1,

$$C(X) \cong \{f|D\colon\ f \in C(X)\} \subseteq H(X) \subseteq C(D) \cong C(\mathbb{N}).$$

But by the preceding lemma (with D in the role of T) $C(D) \subseteq H(X)$. Thus $H(X) \cong C(\mathbb{N})$ as claimed. □

Theorem 5.17. *If T is a realcompact P-space and if T is dense and C^*-embedded in the realcompact space X then $H(X) \cong C(T)$.*

Proof. By Lemma 5.13 $gX = T = T_\delta$ so arguing as in the proof of Lemma 5.15, we need only show that if $k \in C(D)$ there exist $f, h \in C(X)$ such that $k = (f|P)(h|P)^*$. Clearly the functions $k/(|k| + \mathbf{1})$ and $\mathbf{1}/(|k| + \mathbf{1})$ belong to $C^*(T)$ and hence by hypothesis can be continuously extended to f and h in $C(X)$, respectively. A routine computation shows that $k = (f|D)(h|D)^*$. □

Example 5.18. The C^*-embedding hypothesis cannot be dropped from Theorem 5.17; in many models of set theory $\beta\mathbb{N}\backslash\mathbb{N}$ has a dense set of P-points, and this set is realcompact if the continuum hypothesis is assumed, yet as noted after Theorem 5.8, $H(\beta\mathbb{N}\backslash\mathbb{N})$ cannot be a $C(Y)$.

6. Some further examples

In the previous section all our examples of spaces X for which $H(X)$ is a $C(Y)$ had the property that gX was a P-space. We now present some examples where this is not the case. The first example is mentioned in [19], but no details are provided there.

Example 6.1. Let D be an uncountable discrete space and let $K = D \cup \{p\}$ be its one-point compactification. We claim that $H(K) = C(K_\delta)$. First observe that K is an almost P-space since any dense cozero set C of K will be Lindelöf (as it is a union of countably many closed sets of K) and contain D; consequently $p \in C$ as otherwise $\{\{d\}\colon d \in D\}$ would be an open cover of C with no countable subcover. Thus $gK = K$ and so by the remarks preceding Lemma 5.1 we have

$$C(K) \subseteq H(K) \subseteq C(K_\delta).$$

Hence we must show that $H(K) = C(K_\delta)$. By Lemma 5.1(b) it suffices to show that if $k \in C(K_\delta)$ then there exist f and h in $C(K)$, and $s \in \mathbb{R}$, such that $k = (f)(h)^* + \boldsymbol{s}$.

Denote K_δ by L. It is routine to verify that each point of D is isolated in L, and if $p \in S \subseteq L$ then S is open in L iff $L \backslash S$ is countable. (This immediately implies that L is Lindelöf.) Suppose $k(p) = r$. Then $L \backslash k^{\leftarrow}[r - 1/n, r + 1/n)]$ is countable for each $n \in \mathbb{N}$, so $L \backslash k^{\leftarrow}(r)$ is a countable subset C of L. Clearly C is clopen in L; let $C = \{a(n)\colon n \in \mathbb{N}\}$. Let $g = k - \boldsymbol{r}$ and observe that $g(x) = 0$ if $x \in K \backslash C$. Define functions f and h with domain K as follows.

$$\begin{aligned}
&f\big(a(n)\big) = g\big(a(n)\big)/n\big[|g(a(n))| + 1\big] && \text{if } n \in \mathbb{N},\\
&f(x) = 0 && \text{if } x \in K \backslash C;\\
&h\big(a(n)\big) = 1/n\big[|g(a(n))| + 1\big] && \text{if } n \in \mathbb{N},\\
&h(x) = 0 && \text{if } x \in K \backslash C.
\end{aligned}$$

It is easy to verify that f and h belong to $C(K)$, and a routine computation shows that $k = (f)(h)^* + \boldsymbol{r}$. Consequently $H(K) = C(K_\delta)$ and we are done.

Example 6.2. Let $\bigoplus\{K(n)\colon n \in \mathbb{N}\}$ be denoted by T, where each $K(n)$ is a copy of the space K of the previous example. Let X be any realcompact space containing T as a dense subspace. By Lemma 5.15 $gX = T$ and $C(T) \subseteq H(X) \subseteq C(T_\delta)$; consequently $H(X) = H(T)$ (see Corollary 4.12). It is well-known (and easily proved) that if a Tychonoff space S can be written as a direct sum $\bigoplus\{Y(n)\colon n \in \mathbb{N}\}$ then $C(S) \cong \prod\{C(Y(n))\colon n \in \mathbb{N}\}$. Although formation of the epimorphic hull may not commute with direct products of rings, because of the simplicity of the expression for $H(K)$ derived in Example 6.1 above, and because each summand of T is K, it follows that

$$H\Big(\prod\big\{C(K(n))\colon n \in \mathbb{N}\big\}\Big) \cong \prod\big\{H(K(n))\colon n \in \mathbb{N}\big\}.$$

Hence by Lemma 5.15

$$\begin{aligned}
H(X) = H(T) &\cong H\Big(\prod\big\{C(K(n))\colon n \in \mathbb{N}\big\}\Big) \cong \prod\big\{H(K(n))\colon n \in \mathbb{N}\big\}\\
&\cong \prod\big\{C(L(n))\colon n \in \mathbb{N}\big\} \cong C\Big(\bigoplus\{L(n)\colon n \in \mathbb{N}\}\Big) = C(T_\delta),
\end{aligned}$$

where each $L(n)$ is homeomorphic to the space L described in the previous example. Thus if X contains T as a dense subspace (for example, if $X = \beta T$) then $H(X)$ is a $C(Y)$, even though $gX \neq X$ and gX is not a P-space.

Example 6.3. Let W denote the space of countable ordinals with the order topology, and let W^* denote its one-point compactification (see Chapter 5 of [7]). It is a routine exercise to show, with the aid of Lemma 5.14, that $g(W^*) = W^* \backslash \{\alpha\colon \alpha$ is a countable limit ordinal$\}$, which is homeomorphic to (and hence will be identified with) the space L described in Example 6.1. As L is a P-space, if we can show that $C(L) \subseteq H(W^*)$ then it will follow that $C(L) = H(W^*)$. As before, by Lemma 5.1(b) it suffices to show that if $k \in C(L)$ then there exist $r \in \mathbb{R}$ and $f, h \in C(W^*)$ such that $k = (f|L)(h|L)^* + \boldsymbol{r}$. As seen above, there exists $\alpha \in W$ such that $k[L \backslash [0, \alpha]] = \{r\}$. By applying Theorem 5.16 to the compact open subspace $[0, \alpha]$ of W^*, we see that $H([0, \alpha]) = C(L \cap [0, \alpha]) \cong C(\mathbb{N})$ and so there exist functions j and m in $C([0, \alpha])$ such that $(k - \boldsymbol{r})|L \cap [0, \alpha] = (j|L \cap [0, \alpha])(m|L \cap [0, \alpha])^*$. Extend j to $f \in C(W^*)$ by decreeing that $f[W^* \backslash [0, \alpha]] = \{0\}$, and extend m to $h \in C(W^*)$ similarly. One then easily checks that $k = (f|L)(h|L)^* + \boldsymbol{r}$, and so $k \in H(W^*)$. Thus $H(W^*) \cong C(L)$ as claimed.

Example 6.4. Let M denote a compact metric space without isolated points. Its *Alexandroff double*, denoted $A(M)$, is constructed as follows. If $x \in M$ and $j \in \mathbb{N}$ let $S(x, j)$ denote the open sphere in M with center x and radius $1/j$. The underlying set of $A(M)$ is $M \times \{0, 1\}$, and $A(M)$ is topologized as follows. Each point of the form $(x, 1)$ is isolated, and a neighborhood base at $(x, 0)$ is $\{T(x, j)\colon j \in \mathbb{N}\}$ where $T(x, j) = [S(x, j) \times \{0, 1\}] \backslash \{(x, 1)\}$. Then $A(M)$ is a first countable compact Hausdorff space of weight $\mathfrak{c}$ and the set $M \times \{1\}$ is a dense subset D of isolated points of $A(M)$ of cardinality $\mathfrak{c}$. (The reader is referred to [4, 3.1.26] for a detailed analysis in the case where M is the circle.) It follows from [2, 2.2] that $|C(A(M))| = \mathfrak{c}$. Since $A(M)$ is compact and first countable, each singleton subset is a zero-set (see [7, 3.11(b)]), so $A(M) \backslash \{p\}$ is a dense cozero-set of $A(M)$ for each $p \in M \times \{0\}$. It immediately follows that $[g(A(M))]_\delta = g(A(M)) = D$. Thus $|H(A(M))| = |C(A(M))| = \mathfrak{c}$ while $|C(g(A(M)))| = 2^{\mathfrak{c}}$; consequently, in contrast to our earlier examples, $H(X)$ is a proper subring of $C(gX)$ when X is taken to be $A(M)$, even though gX is a P-space and it is dense in X. Hence $H(X)$ is not a $C(Y)$ in this case. This example also illustrates that $\{k|gX\colon k \in H(X)\} \neq H(gX)$ in general.

7. Observations on $Q_{cl}(X)$

We begin by asking under what conditions $Q_{cl}(X)$ will be isomorphic to a $C(Y)$. This question is more easily answered than the analogous question about $H(X)$ because we have at our disposal the following

Theorem 7.1 [6, 2.6]. *$Q_{cl}(X)$ is the ring of all continuous functions on the dense cozero-sets of X, modulo the relation which identifies functions that agree on the intersections of their domains.*

We note here that if $Q_{cl}(X) = C(Y)$, then $Q_{cl}(X)$ is uniformly closed [9, 3.1, 5.2], and is therefore isomorphic to the ring of all continuous functions on the countable intersections of dense cozero-sets of βX [3, 3.4]. As well, Y is then an almost-P-space since non-zero-divisors in $Q_{cl}(X)$ are units.

In view of the Remarks 4.1 we assume in what follows that X is realcompact.

Theorem 7.2. *The following are equivalent for a space X:*

(1) $Q_{cl}(X) = C(Y)$ *for some Tychonoff space* Y.

(2) $Q_{cl}(X) = C(gX)$.

(3) (a) gX *is dense in* X, *and*

(b) *Every function in* $C(gX)$ *has a continuous extension to a dense cozero-set of* X.

Proof. (1) ⇒ (2) As in Theorem 4.2 and Lemma 4.3 there is a mapping $\sigma : Y \to X$ such that $\sigma[Y] = T$ is dense in X and preimages of dense subsets of T are dense in Y. Since $Q_{cl}(X) = C(Y)$, Y is an almost P-space. Since $C(Y)$ is a ring of quotients of $C(X)$, Lemma 4.3 is applicable and shows that the inverse image of a proper dense cozero-set of T is a proper dense cozero-set of Y. But Y, being almost-P, has no such subset. Hence neither does T, and so T is an almost P-space. It follows that $T \subseteq gX$. Thus $Q_{cl}(X) \subseteq C(gX)$ and we have

$$Q_{cl}(X) \subseteq C(gX) \subseteq C(T) \subseteq C(Y) \cong Q_{cl}(X).$$

(2) ⇒ (3) Evidently gX is dense in X. We then have a monomorphism of $Q_{cl}(X)$ onto $C(gX)$ defined by $f \to f|gX$, where $f \in C(V)$ for some dense cozero-set V of X.

The third implication being obvious, the result follows. □

In general we know that for any Tychonoff space X, one has the natural inclusions $C(X) \subseteq Q_{cl}(X) \subseteq H(X) \subseteq Q(X)$. We next discuss circumstances under which some or all of these inclusions are equalities. We also consider conditions on X under which some of $Q_{cl}(X)$, $H(X)$, and $Q(X)$ are isomorphic to rings of the form $C(Y)$ while at the same time others are not.

Remark 7.3 (*Equalities among rings of quotients*).

(a) It is possible for $Q(X)$ to be isomorphic to a $C(Y)$ without either $Q_{cl}(X)$ or $H(X)$ being isomorphic to a $C(T)$. A class of spaces of this sort is described in Example 6.4. Note that these spaces satisfy 3(a) but not 3(b) of Theorem 7.2.

(b) It is possible for $Q_{cl}(X)$ and $H(X)$ to coincide; this happens if X is basically disconnected (see [7, 1H, 6M.1] and Lemma 2.2 above). If X is extremally disconnected then $Q_{cl}(X) = H(X) = Q(X)$ (see Corollary 3.5). If X is a P-space then $C(X) = Q_{cl}(X) = H(X)$ (Corollary 3.4). Thus if X is an extremally disconnected P-space then $C(X)$ has no proper ring of quotients and hence $C(X) = Q_{cl}(\beta X) = H(\beta X) = Q(\beta X)$ (Corollary 3.5). It is known that an extremally disconnected P-space of non-measurable cardinal is discrete [7, 12H].

(c) There are spaces X and T such that $H(X)$ and $Q_{cl}(X)$ coincide and are isomorphic to $C(T)$. By Theorem 7.2, Lemma 5.13 and the remarks preceding Lemma 5.1, this will happen if T is a realcompact P-space and $X = \beta T$.

(d) It is possible for $C(X)$, $H(X)$ and $Q_{cl}(X)$ to be distinct, and for each to be isomorphic to a $C(Y)$. For example, if X and T are as defined in Example 6.2, then it follows from Theorem 7.2 that $Q_{cl}(X) \cong C(T)$ and from Example 6.2 that $H(X) \cong C(T_\delta)$.

(e) It is possible for $Q_{cl}(X)$ to be isomorphic to a $C(T)$ but for $H(X)$ not to be isomorphic to a $C(Y)$. As an example, let $T = \bigoplus\{B(n)\colon n \in \mathbb{N}\}$, where each $B(n)$ is a copy of $\beta\mathbb{N}\backslash\mathbb{N}$, and let X be any realcompact Tychonoff space that contains T properly as a dense subspace (e.g., $X = \beta T$). By Theorem 7.2 it follows that $Q_{cl}(X) \cong C(T)$ but as $gX = T$ and not every compact subset of T is scattered (as $\beta\mathbb{N}\backslash\mathbb{N}$ is not), then it follows from Theorem 5.8 that $H(X)$ is not a $C(Y)$.

8. Open questions

Several interesting and obvious questions remain unresolved. The most fundamental is:

Question 8.1. Characterize those realcompact Tychonoff spaces X for which $H(X) = C((gX)_\delta)$.

Questions pertaining to special cases of Question 8.1 are the following:

Question 8.2. If $H(X) = C((gX)_\delta)$ does it follow that X has a dense set of P-points?

Despite persistent efforts we have been unable to answer Question 8.2 in the general case. However, observe that the answer is "yes" if X is basically disconnected, for in that case $gX = P(X)$; we leave verification of this as an exercise for the reader.

Question 8.3. If X is a compact scattered almost-P space, does it follow that $H(X) = C((X)_\delta)$? (cf. Theorem 5.8).

Note added in proof

The authors have recently answered this question in the negative.

Acknowledgements

The authors gratefully acknowledge the support of the Natural Sciences and Engineering Research Council of Canada.

The authors wish to thank the referee for many valuable suggestions.

References

[1] R.L. Blair, A.W. Hager, Extensions of zero-sets and of real-valued functions, Math. Z. 136 (1974) 41–52.

[2] W.W. Comfort, A.W. Hager, Estimates for the number of real-valued continuous functions, Trans. Amer. Math. Soc. 150 (1970) 619–631.
[3] F. Dashiell, A. Hager, M. Henriksen, Order-Cauchy completions of rings and vector lattices of continuous functions, Canad. J. Math. 32 (1980) 657–685.
[4] R. Engelking, General Topology (revised and completed edition), Heldermann, Berlin, 1989.
[5] N. Fine, L. Gillman, Extensions of continuous functions in βN, Bull. Amer. Math. Soc. 66 (1960) 376–381.
[6] N.J. Fine, L. Gillman, J. Lambek, Rings of Quotients of Rings of Functions, McGill University Press, Montreal, Quebec, 1966.
[7] L. Gillman, M. Jerison, Rings of Continuous Functions, Van Nostrand, Princeton, NJ, 1960.
[8] A.W. Hager, Isomorphism with a $C(Y)$ of the maximal ring of quotients of $C(X)$, Fund. Math. 66 (1969) 7–13.
[9] M. Henriksen, D.G. Johnson, On the structure of a class of archimedian lattice-ordered algebras, Fund. Math. 50 (1961) 73–94.
[10] A. Hager, J. Martinez, Fraction-dense algebras and spaces, Canad. J. Math. 45 (1993) 977–996.
[11] K. Kuratowski, Topology, Vol. I, Academic Press, New York, 1966.
[12] J. Lambek, Lectures on Rings and Modules, Blaisdell, Toronto, 1966.
[13] D. Lazard, Epimorphismes plats d'anneaux, C. R. Acad. Sci. Paris Sér. A 266 (1968) 314–316.
[14] R. Levy, Almost P-spaces, Canad. J. Math. 29 (2) (1977) 284–288.
[15] R. Levy, M. Rice, Normal spaces and the G_δ-topology, Colloq. Math. 44 (1981) 227–240.
[16] J.P. Olivier, Anneaux absolument plats universels et epimorphismes d'anneaux, C. R. Acad. Sci. Paris Sér. A 266 (1968) 317–318.
[17] J.R. Porter, R.G. Woods, Extensions and Absolutes of Hausdorff Spaces, Springer, New York, 1988.
[18] R. Raphael, Injective rings, Comm. Algebra 1 (5) (1974) 403–414.
[19] H.H. Storrer, Epimorphismen von kommutativen Ringen, Comment. Math. Helv. 43 (1968) 378–401.